9. Surface Integrals - Projection Method

Surfaces in $\mathbb{R}^3$

In $\mathbb{R}^3$ a surface can be represented by a vector parametric equation

$$\vec{\mathbf{r}} = x(u,v)\,\hat{\mathbf{i}} + y(u,v)\,\hat{\mathbf{j}} + z(u,v)\,\hat{\mathbf{k}}$$

where u, v are **parameters**.

Example 9.01

The unit sphere, centre O, can be represented by

$$\vec{\mathbf{r}}(\theta,\phi) = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \qquad \underset{\uparrow}{0 \le \theta \le \pi} \quad \text{and} \quad \underset{\uparrow}{0 \le \phi < 2\pi}$$

If every vertical line (parallel to the *z*-axis) in $\mathbb{R}^3$ meets the surface no more than once, then the surface can also be parameterized as

$$\vec{\mathbf{r}}(x,y) = \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix} \qquad \text{or as} \quad z = f(x,y)$$

Example 9.02

$$z = \sqrt{4 - x^2 - y^2}\,, \qquad \{(x,y) \mid x^2 + y^2 \le 4\} \quad \text{is a}$$

A **simple surface** does not cross itself.
If the following condition is true:

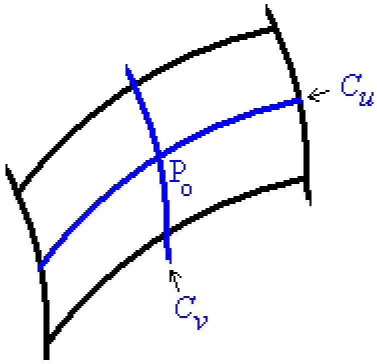
$$\{\,\vec{\mathbf{r}}(u_1,v_1) = \vec{\mathbf{r}}(u_2,v_2) \;\Rightarrow\; (u_1,v_1) = (u_2,v_2) \text{ for all pairs of points in the domain}\}$$

then the surface is simple.

The converse of this statement is not true.
This condition is sufficient, but it is not necessary for a surface to be simple.
The condition may fail on a simple surface at coordinate singularities. For example, one of the angular parameters of the polar coordinate systems is undefined everywhere on the *z*-axis, so that spherical polar (2, 0, 0) and (2, 0, π) both represent the same Cartesian point (0, 0, 2). Yet a sphere remains simple at its *z*-intercepts.

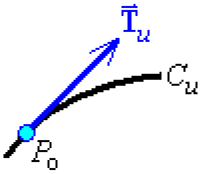
Tangent and Normal Vectors to Surfaces

A surface S is represented by $\vec{\mathbf{r}}(u,v)$. Examine the neighbourhood of a point P_0 at $\vec{\mathbf{r}}(u_0,v_0)$. Hold parameter v constant at v_0 (its value at P_0) and allow the other parameter u to vary. This generates a slice through the two-dimensional surface, namely a one-dimensional curve C_u containing P_0 and represented by a vector parametric equation $\vec{\mathbf{r}} = \vec{\mathbf{r}}(u, v_o)$ with only one freely-varying parameter (u).



If, instead, u is held constant at u_0 and v is allowed to vary, we obtain a different slice containing P_0, the curve C_v: $\vec{\mathbf{r}}(u_o, v)$.

On each curve a unique tangent vector can be defined.



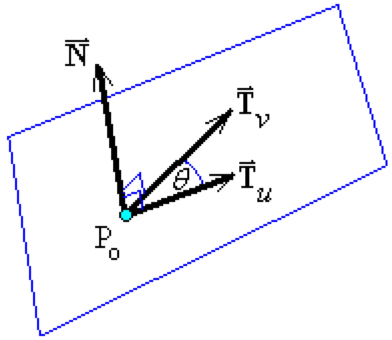
At all points along C_u, a tangent vector is defined by $\vec{\mathbf{T}}_u = \dfrac{\partial}{\partial u}\left(\vec{\mathbf{r}}(u,v_o)\right)$.

[Note that this is not necessarily a *unit* tangent vector.]

At P_0 the tangent vector becomes $\vec{\mathbf{T}}_u\Big|_{P_o} = \dfrac{\partial}{\partial u}\left(\vec{\mathbf{r}}(u,v)\right)\Big|_{P_o} = \dfrac{\partial}{\partial u}\left(\vec{\mathbf{r}}(u_o,v_o)\right)$.

Similarly, along the other curve C_v, the tangent vector at P_o is $\vec{\mathbf{T}}_v\Big|_{P_o} = \dfrac{\partial}{\partial v}\left(\vec{\mathbf{r}}(u_o,v_o)\right)$.

If the two tangent vectors are not parallel and neither of these tangent vectors is the zero vector, then they define the orientation of tangent plane to the surface at P_0.



A normal vector to the tangent plane is

$$= \left[\frac{\partial(y,z)}{\partial(u,v)} \;\; \frac{\partial(z,x)}{\partial(u,v)} \;\; \frac{\partial(x,y)}{\partial(u,v)} \right]^T \Bigg|_{(u_o,v_o)},$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the **Jacobian** $\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$.

Cartesian parameters

With $u = x,\ v = y,\ z = f(x,y)$, the components of the normal vector $\vec{\mathbf{N}} = N_1\hat{\mathbf{i}} + N_2\hat{\mathbf{j}} + N_3\hat{\mathbf{k}}$ are:

$$N_1 = \frac{\partial(y,z)}{\partial(x,y)} =$$

$$N_2 = \frac{\partial(z,x)}{\partial(x,y)} = \qquad N_3 = \frac{\partial(x,y)}{\partial(x,y)} =$$

$\Rightarrow$ a normal vector to the surface $z = f(x,y)$ at (x_0, y_0, z_0) is

$$\boxed{\vec{\mathbf{N}} = \left[-\frac{\partial f}{\partial x} \;\; -\frac{\partial f}{\partial y} \;\; +1 \right]^T \Bigg|_{(x_o,y_o)}}$$

If the normal vector $\vec{\mathbf{N}}$ is continuous and non-zero over all of the surface S, then the surface is said to be **smooth**.

Example 9.03

A sphere is smooth.

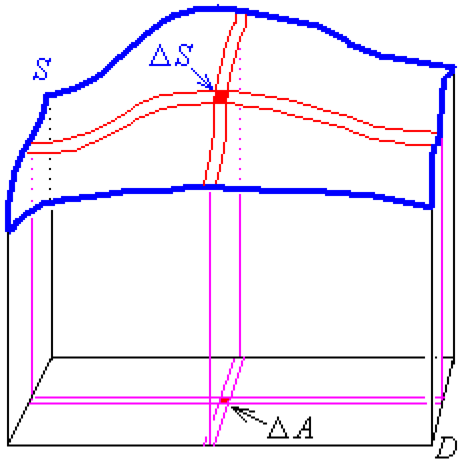
A cube is

A cone is

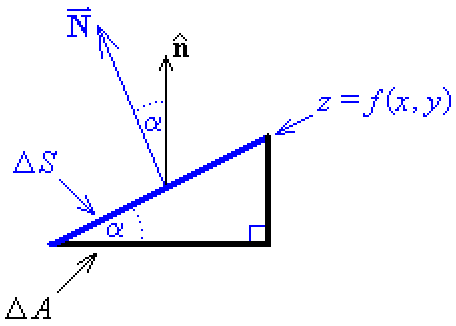
Surface Integrals (Projection Method)

This method is suitable mostly for surfaces which can be expressed easily in the Cartesian form $z = f(x, y)$.

The plane region D is the projection (or shadow) of the surface $S: f(\vec{\mathbf{r}}) = c$ onto a plane (usually the xy-plane) in a 1:1 manner.



The plane containing D has a constant unit normal $\hat{\mathbf{n}}$.
$\vec{\mathbf{N}}$ is any non-zero normal vector to the surface S.



Surface Integrals (Projection Method) (continued)

For $z = f\left(x, y\right)$ and D = a region of the *xy*-plane,

$$\vec{\mathbf{N}} = \begin{bmatrix} -\dfrac{\partial z}{\partial x} & -\dfrac{\partial z}{\partial y} & 1 \end{bmatrix}^{\mathrm{T}} \quad \text{and} \quad \hat{\mathbf{n}} = \hat{\mathbf{k}}$$

$\Rightarrow \quad \left|\vec{\mathbf{N}} \bullet \hat{\mathbf{n}}\right| = 1$ and

$$\boxed{\iint_S g\left(\vec{\mathbf{r}}\right) dS = \iint_D g\left(\vec{\mathbf{r}}\right) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA}$$

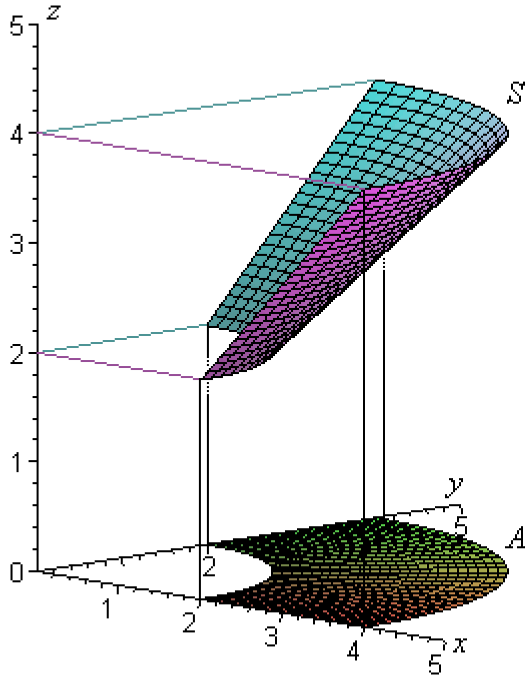
which is the projection method of integration of $g\left(x, y, z\right)$ over the surface $z = f\left(x, y\right)$.

Advantage:

Disadvantage:

Example 9.04

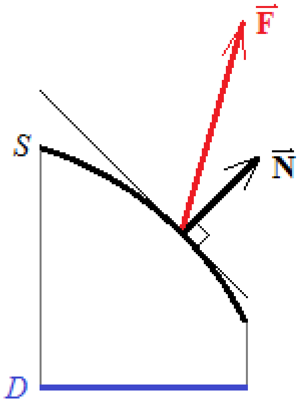
Evaluate $\iint_S z\, dS$, where the surface S is the section of the cone $z^2 = x^2 + y^2$ in the first octant, between $z = 2$ and $z = 4$.



Example 9.04 (continued)

Flux through a Surface (Projection Method)

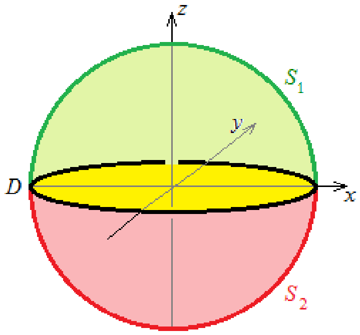
Set $g(\vec{\mathbf{r}}) = F_N$ (the normal component of vector field $\vec{\mathbf{F}}$, that is, $\vec{\mathbf{F}}$ resolved in the direction of the normal $\vec{\mathbf{N}}$ to the surface *S*), then proceed as before:



where $\vec{\mathbf{N}} = \begin{bmatrix} -\dfrac{\partial z}{\partial x} & -\dfrac{\partial z}{\partial y} & 1 \end{bmatrix}^{\mathrm{T}}$

Example 9.05

Find the flux due to the vector field $\vec{\mathbf{F}} = r\,\vec{\mathbf{r}}$ through the sphere S, radius 2, centre the origin.

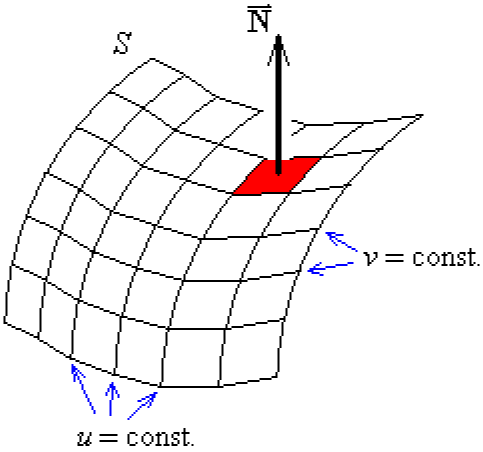


Example 9.05 (continued)

Surface Integrals - Surface Method

When a surface S is defined in a vector parametric form $\vec{\mathbf{r}} = \vec{\mathbf{r}}(u, v)$, one can lay a coordinate grid (u, v) down on the surface S.

A normal vector everywhere on S is $\vec{\mathbf{N}} = \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}$.



$$\boxed{\iint_S g(\vec{\mathbf{r}})\, dS = \iint_S g(\vec{\mathbf{r}}) \left| \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right| du\, dv}$$

Advantage:

- only one integral to evaluate

Disadvantage:

- it is often difficult to find optimal parameters (u, v).

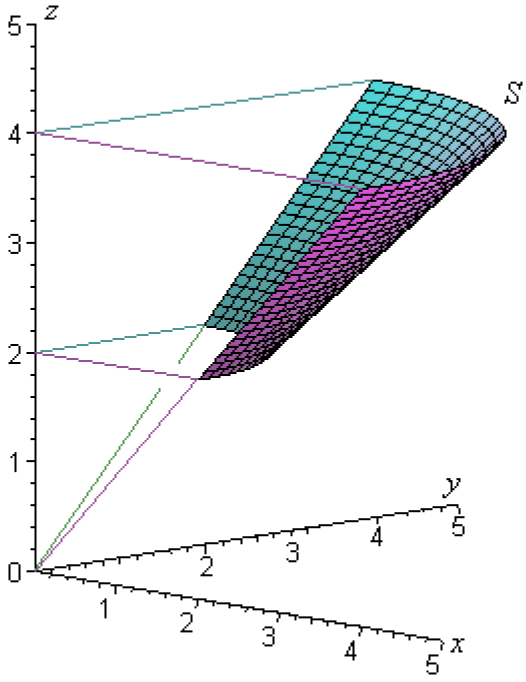
The total flux of a vector field $\vec{\mathbf{F}}$ through a surface S is

$$\Phi = \iint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\, dS = \iint_S \vec{\mathbf{F}} \bullet \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\, du\, dv$$

(which involves the scalar triple product $\vec{\mathbf{F}} \bullet \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}$).

Example 9.06: (same as Example 9.04, but using the surface method).

Evaluate $\iint_S z\, dS$, where the surface S is the section of the cone $z^2 = x^2 + y^2$ in the first octant, between $z = 2$ and $z = 4$.

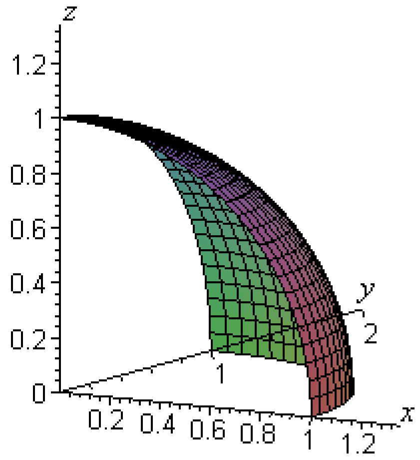


Choose a convenient parametric net:

Just as we used line integrals to find the mass and centre of mass of [one dimensional] wires, so we can use surface integrals to find the mass and centre of mass of [two dimensional] sheets.

Example 9.07

Find the centre of mass of the part of the unit sphere (of constant surface density) that lies in the first octant.



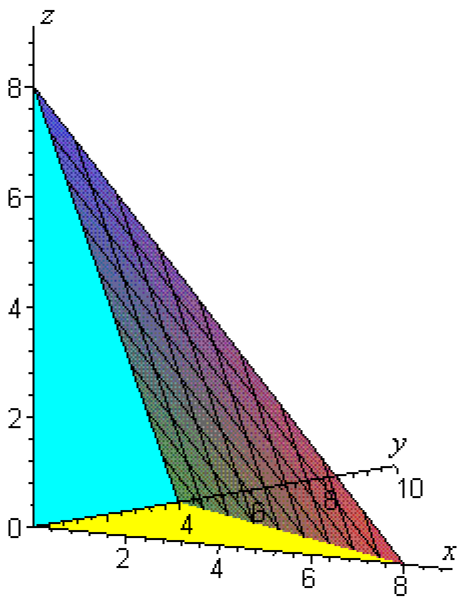
Example 9.07 (continued)

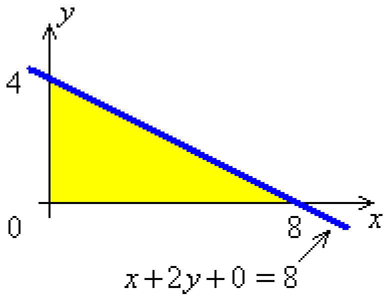
Example 9.08 (same as Example 9.05, but using the surface method)

Find the flux due to the vector field $\vec{\mathbf{F}} = r\,\vec{\mathbf{r}}$ through the sphere S, radius 2, centre the origin.

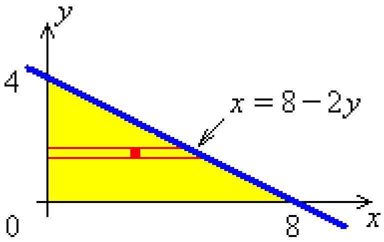
Example 9.09

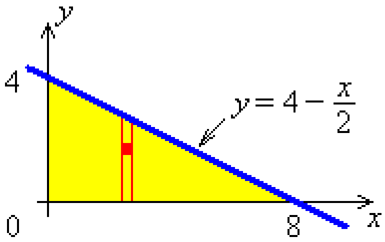
Find the flux of the field $\vec{\mathbf{F}} = \left[\, x \;\; y \;\; -z \,\right]^{\mathrm{T}}$ across that part of $x + 2y + z = 8$ that lies in the first octant.





Example 9.09 (continued)





Example 9.10

Find the total flux Φ of the vector field $\vec{\mathbf{F}} = z\,\hat{\mathbf{k}}$ through the simple closed surface S

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Use the parametric grid (θ, ϕ), such that the displacement vector to any point on the ellipsoid is

$$\vec{\mathbf{r}}(\theta, \phi) = \begin{bmatrix} a \sin\theta \cos\phi \\ b \sin\theta \sin\phi \\ c \cos\theta \end{bmatrix}$$

This grid is a generalisation of the spherical polar coordinate grid and covers the entire surface of the ellipsoid for $0 \le \theta \le \pi,\ 0 \le \phi < 2\pi$.

One can verify that $x = a \sin\theta \cos\phi,\ y = b \sin\theta \sin\phi,\ z = c \cos\theta$ does lie on the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ for all values of (θ, ϕ):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2 \sin^2\theta \cos^2\phi}{a^2} + \frac{b^2 \sin^2\theta \sin^2\phi}{b^2} + \frac{c^2 \cos^2\theta}{c^2}$$

$$= \sin^2\theta \cos^2\phi + \sin^2\theta \sin^2\phi + \cos^2\theta = \sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta = 1 \qquad \forall\theta \text{ and } \forall\phi$$

The tangent vectors along the coordinate curves ϕ = constant and θ = constant are

Example 9.10 (continued)

For vector fields $\vec{\mathbf{F}}(\vec{\mathbf{r}})$,

Line integral: $\int_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}}$

Surface integral:

$$\iint_S \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet \mathbf{d}\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}}(\vec{\mathbf{r}}) \bullet \mathbf{N}\, dS = \iint_S \vec{\mathbf{F}} \bullet \vec{\mathbf{N}}\, du\, dv = \pm \iint_S \vec{\mathbf{F}} \bullet \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\, du\, dv$$

On a closed surface, take the sign such that $\vec{\mathbf{N}}$ points **outward**.

Some Common Parametric Nets

1) The circular plate $(x - x_o)^2 + (y - y_o)^2 \le a^2$ in the plane $z = z_o$.

Let the parameters be r, θ where $0 < r \le a,\ 0 \le \theta < 2\pi$

$x = x_o + r\cos\theta,\ y = y_o + r\sin\theta,\ z = z_o$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial r} \times \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\right) = \pm \begin{vmatrix} \hat{\mathbf{i}} & \cos\theta & -r\sin\theta \\ \hat{\mathbf{j}} & \sin\theta & r\cos\theta \\ \hat{\mathbf{k}} & 0 & 0 \end{vmatrix} = \pm r\,\hat{\mathbf{k}}$$

2) The circular cylinder $(x - x_o)^2 + (y - y_o)^2 = a^2$ with $z_0 \le z \le z_1$.

Let the parameters be z, θ where $z_0 \le z \le z_1,\ 0 \le \theta < 2\pi$

$x = a\cos\theta,\ y = a\sin\theta,\ z = z$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial z} \times \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\right) = \pm \begin{vmatrix} \hat{\mathbf{i}} & 0 & -a\sin\theta \\ \hat{\mathbf{j}} & 0 & a\cos\theta \\ \hat{\mathbf{k}} & 1 & 0 \end{vmatrix} = \pm\left(-a\cos\theta\,\hat{\mathbf{i}} - a\sin\theta\,\hat{\mathbf{j}}\right)$$

Outward normal: $\vec{\mathbf{N}} = a\cos\theta\,\hat{\mathbf{i}} + a\sin\theta\,\hat{\mathbf{j}}$

3) The frustum of the circular cone $w - w_o = a\sqrt{(u - u_o)^2 + (v - v_o)^2}$ where $w_1 \le w \le w_2$ and $w_o \le w_1$. Let the parameters here be r, θ where

$$\frac{w_1 - w_o}{a} \le r \le \frac{w_2 - w_o}{a}, \qquad 0 \le \theta < 2\pi$$

$$x = u = u_o + r\cos\theta, \quad y = v = v_o + r\sin\theta, \quad z = w = w_o + ar$$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial r} \times \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\right) = \pm \begin{vmatrix} \hat{\mathbf{i}} & \cos\theta & -r\sin\theta \\ \hat{\mathbf{j}} & \sin\theta & r\cos\theta \\ \hat{\mathbf{k}} & a & 0 \end{vmatrix}$$

$$= \pm\left[(-ar\cos\theta)\,\hat{\mathbf{i}} + (-ar\sin\theta)\,\hat{\mathbf{j}} + r\,\hat{\mathbf{k}}\right]$$

Outward normal: $\vec{\mathbf{N}} = ar\cos\theta\,\hat{\mathbf{i}} + ar\sin\theta\,\hat{\mathbf{j}} - r\,\hat{\mathbf{k}}$

4) The portion of the elliptic paraboloid

$$z - z_{\text{o}} \;=\; a^2\left(x - x_{\text{o}}\right)^2 \;+\; b^2\left(y - y_{\text{o}}\right)^2 \quad \text{with} \quad z_{\text{o}} \le z_1 \le z \le z_2$$

Let the parameters here be r, θ where

$$\sqrt{\frac{z_1 - z_{\text{o}}}{a^2\cos^2\theta + b^2\sin^2\theta}} \;\le\; r \;\le\; \sqrt{\frac{z_2 - z_{\text{o}}}{a^2\cos^2\theta + b^2\sin^2\theta}} \;, \quad 0 \le \theta < 2\pi$$

$$x = x_{\text{o}} + r\cos\theta,\;\; y = y_{\text{o}} + r\sin\theta,\;\; z = z_{\text{o}} + r^2\left(a^2\cos^2\theta + b^2\sin^2\theta\right)$$

$$\vec{\mathbf{N}} \;=\; \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial r}\times\frac{\partial\vec{\mathbf{r}}}{\partial\theta}\right) \;=\; \pm\begin{vmatrix} \hat{\mathbf{i}} & \cos\theta & -r\sin\theta \\ \hat{\mathbf{j}} & \sin\theta & r\cos\theta \\ \hat{\mathbf{k}} & 2r\left(a^2\cos^2\theta + b^2\sin^2\theta\right) & 2r^2\left(b^2 - a^2\right)\sin\theta\cos\theta \end{vmatrix}$$

$$= \;\pm\left[\left(-2a^2r^2\cos\theta\right)\hat{\mathbf{i}} \;+\; \left(-2b^2r^2\sin\theta\right)\hat{\mathbf{j}} \;+\; r\,\hat{\mathbf{k}}\right]$$

Outward normal: $\vec{\mathbf{N}} \;=\; \left(2a^2r^2\cos\theta\right)\hat{\mathbf{i}} \;+\; \left(2b^2r^2\sin\theta\right)\hat{\mathbf{j}} \;-\; r\,\hat{\mathbf{k}}$

5) The surface of the sphere $(x - x_\circ)^2 + (y - y_\circ)^2 + (z - z_\circ)^2 = a^2$.

Let the parameters here be θ, ϕ where $0 \le \theta \le \pi, \quad 0 \le \phi < 2\pi$

$$x = x_{\text{o}} + a\sin\theta\cos\phi,\;\; y = y_{\text{o}} + a\sin\theta\sin\phi,\;\; z = z_{\text{o}} + a\cos\theta$$

$$\vec{\mathbf{N}} \;=\; \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial\theta}\times\frac{\partial\vec{\mathbf{r}}}{\partial\phi}\right) \;=\; \pm\begin{vmatrix} \hat{\mathbf{i}} & a\cos\theta\cos\phi & -a\sin\theta\sin\phi \\ \hat{\mathbf{j}} & a\cos\theta\sin\phi & a\sin\theta\cos\phi \\ \hat{\mathbf{k}} & -a\sin\theta & 0 \end{vmatrix}$$

$$= \;\pm\, a^2\sin\theta\,\left[\left(\sin\theta\cos\phi\right)\hat{\mathbf{i}} \;+\; \left(\sin\theta\sin\phi\right)\hat{\mathbf{j}} \;+\; \left(\cos\theta\right)\hat{\mathbf{k}}\right]$$

Outward normal: $\vec{\mathbf{N}} \;=\; a^2\sin\theta\,\left[\left(\sin\theta\cos\phi\right)\hat{\mathbf{i}} \;+\; \left(\sin\theta\sin\phi\right)\hat{\mathbf{j}} \;+\; \left(\cos\theta\right)\hat{\mathbf{k}}\right]$

OR $\vec{\mathbf{N}} \;=\; \dfrac{\partial\vec{\mathbf{r}}}{\partial\theta}\times\dfrac{\partial\vec{\mathbf{r}}}{\partial\phi} \;=\; \left(a\,\hat{\boldsymbol{\theta}}\right)\times\left(\,a\sin\theta\,\hat{\boldsymbol{\phi}}\right) \;=\; a^2\sin\theta\,\hat{\mathbf{r}} \;=\; a\sin\theta\,\vec{\mathbf{r}}$

6) The part of the plane $A(x - x_\circ) + B(y - y_\circ) + C(z - z_\circ) = 0$ in the first octant with $A, B, C > 0$ and $Ax_{\text{o}} + By_{\text{o}} + Cz_{\text{o}} > 0$.

Let the parameters be x, y where

$$0 \;\le\; x \;\le\; \frac{Ax_\circ + By_\circ + Cz_\circ - By}{A}\;; \qquad 0 \;\le\; y \;\le\; \frac{Ax_\circ + By_\circ + Cz_\circ}{B}$$

$$\vec{\mathbf{N}} \;=\; \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial x}\times\frac{\partial\vec{\mathbf{r}}}{\partial y}\right) \;=\; \pm\begin{vmatrix} \hat{\mathbf{i}} & 1 & 0 \\ \hat{\mathbf{j}} & 0 & 1 \\ \hat{\mathbf{k}} & -A/C & -B/C \end{vmatrix} \;=\; \pm\left[\frac{A}{C}\,\hat{\mathbf{i}} \;+\; \frac{B}{C}\,\hat{\mathbf{j}} \;+\; \hat{\mathbf{k}}\right]$$

[End of Chapter 9]