11. Partial Differential Equations

Partial differential equations (PDEs) are equations involving functions of more than one variable and their partial derivatives with respect to those variables.

Most (but not all) physical models in engineering that result in partial differential equations are of at most second order and are often linear. (Some problems such as elastic stresses and bending moments of a beam can be of fourth order). In this course we shall have time to look at only a very small subset of second order linear partial differential equations.

Major Classifications of Common PDEs

A general second order linear partial differential equation in two Cartesian variables can be written as

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$$

Three main types arise, based on the value of $D = B^2 - 4AC$ (a discriminant):
Hyperbolic, wherever (x, y) is such that $D > 0$;
Parabolic, wherever (x, y) is such that $D = 0$;
Elliptic, wherever (x, y) is such that $D < 0$.

Among the most important partial differential equations in engineering are:

The **wave equation**: $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\nabla^2 u$

or its one-dimensional special case $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\dfrac{\partial^2 u}{\partial x^2}$ [which is hyperbolic everywhere]

(where u is the displacement and c is the speed of the wave);

The **heat** (or **diffusion**) **equation**: $\mu\rho\dfrac{\partial u}{\partial t} = K\,\nabla^2 u + \vec{\nabla}K \bullet \vec{\nabla}u$

a one-dimensional special case of which is

$$\frac{\partial u}{\partial t} = \frac{K}{\mu\rho}\frac{\partial^2 u}{\partial x^2} \qquad \text{[which is parabolic everywhere]}$$

(where u is the temperature, μ is the specific heat of the medium, ρ is the density and K is the thermal conductivity);

The potential (or **Laplace's**) **equation**: $\nabla^2 u = 0$

a special case of which is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ [which is elliptic everywhere]

The complete solution of a PDE requires additional information, in the form of initial conditions (values of the dependent variable and its first partial derivatives at $t = 0$), boundary conditions (values of the dependent variable on the boundary of the domain) or some combination of these conditions.

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d'Alembert Solution

Example 11.01

In Term 3 (example 4.8.5 in ENGI 3424) we saw that

$$y(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$

is a solution to the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$$

This solution also satisfies the initial conditions $y(x,0) = f(x)$ and $\left.\dfrac{\partial}{\partial t} y(x,t)\right|_{t=0} = 0$

for ***any*** twice differentiable function $f(x)$.

A more general d'Alembert solution to the wave equation for an infinitely long string is

$$\boxed{y(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du}$$

This satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for } -\infty < x < \infty \quad \text{and} \quad t > 0$$

and

Initial configuration of string: $y(x,0) = f(x)$ for $x \in \mathbb{R}$

and

Initial speed of string: $\left.\dfrac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$ for $x \in \mathbb{R}$

for *any* twice differentiable functions $f(x)$ and $g(x)$.

Physically, this represents two identical waves, moving with speed c in opposite directions along the string.

Proof that $y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$ satisfies both initial conditions:

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du \quad \Rightarrow \quad y(x,0) = \frac{1}{2c}\int_{x}^{x} g(u)\,du = 0 \qquad \checkmark$$

For the other condition, we require Leibnitz differentiation of a definite integral:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = f\big(x,b(x)\big)\frac{db}{dx} - f\big(x,a(x)\big)\frac{da}{dx} + \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}\,dt$$

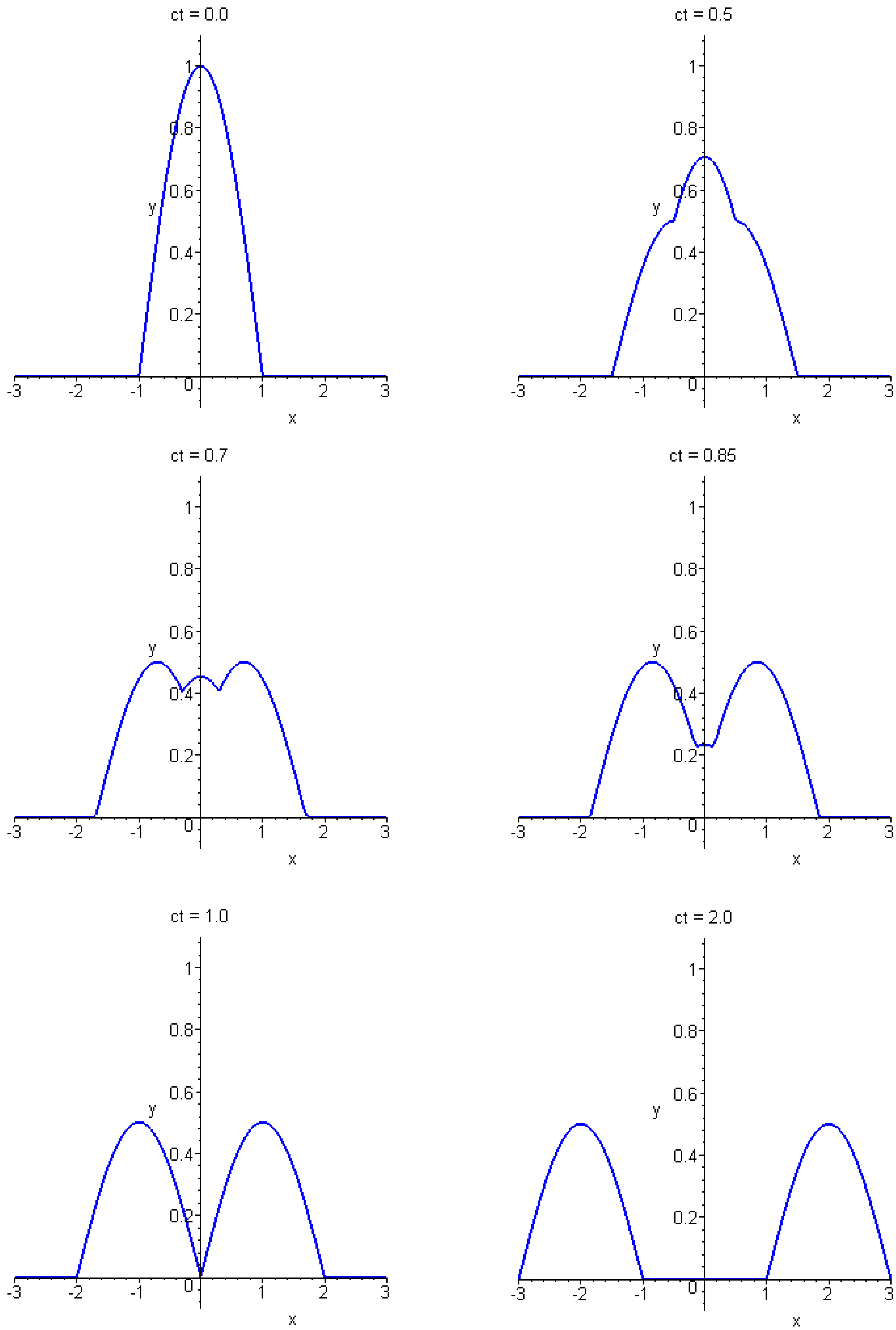
Example 11.02

An elastic string of infinite length is displaced into the form $y = \cos\left(\dfrac{\pi x}{2}\right)$ on $[-1, 1]$ only (and $y = 0$ elsewhere) and is released from rest. Find the displacement $y(x, t)$ at all locations on the string $x \in \mathbb{R}$ and at all subsequent times ($t > 0$).

See the web page "www.engr.mun.ca/~ggeorge/4430/demos/ex1102.html" for an animation of this solution.

Example 11.02 (continued)

Some snapshots of the solution are shown here:



A more general case of a d'Alembert solution arises for the homogeneous PDE with constant coefficients

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

The characteristic (or auxiliary) equation for this PDE is

$$A\lambda^2 + B\lambda + C = 0$$

This leads to the complementary function (which is also the general solution for this homogeneous PDE)

$$\boxed{u(x, y) = f_1(y+\lambda_1 x) + f_2(y+\lambda_2 x),}$$

where

$$\boxed{\lambda_1 = \frac{-B-\sqrt{D}}{2A} \quad \text{and} \quad \lambda_2 = \frac{-B+\sqrt{D}}{2A}}$$

and $\quad D = B^2 - 4AC$

and f_1, f_2 are **arbitrary** twice-differentiable functions of their arguments.

λ_1 and λ_2 are the roots (or eigenvalues) of the characteristic equation.

In the event of equal roots, the solution changes to

$$u(x,y) = f_1(y+\lambda x) + h(x,y)\, f_2(y+\lambda x)$$

where $h(x,y)$ is any non-trivial linear function of x and/or y

(except any multiple of $y+\lambda x$).

The wave equation is a special case with $y=t$, $A=1$, $B=0$, $C=-\frac{1}{c^2}$ and $\lambda=\pm\frac{1}{c}$.

Note how this method has some similarities with a method for solving second order linear ordinary differential equations with constant coefficients (in ENGI 3424 or equivalent).

Example 11.03

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$
$$u(x,0) = -x^2$$
$$u_y(x,0) = 0$$

(a) Classify the partial differential equation.
(b) Find the value of u at $(x, y) = (0, 1)$.

(a) Compare this PDE to the standard form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

Example 11.03 (continued)

Example 11.04

Find the complete solution to

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 14,$$

$$u(x,0) = 2x+1,$$

$$u_y(x,0) = 4-6x.$$

This PDE is non-homogeneous.

For the particular solution, we require a function such that the combination of second partial derivatives resolves to the constant 14. It is reasonable to try a quadratic function of *x* and *y* as our particular solution.

Try $u_P = ax^2 + bxy + cy^2$

Example 11.04 (continued)

Example 11.04 - Alternative Treatment of the Particular Solution

Find the complete solution to

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 14,$$

$$u(x,0) = 2x+1,$$

$$u_y(x,0) = 4-6x.$$

This PDE is non-homogeneous.

For the particular solution, we require a function such that the combination of second partial derivatives resolves to the constant 14. It is reasonable to try a quadratic function of x and y as our particular solution.

Try $\quad u_P = ax^2 + bxy + cy^2$

$$\Rightarrow \quad \frac{\partial u_P}{\partial x} = 2ax + by \quad \text{and} \quad \frac{\partial u_P}{\partial y} = bx + 2cy$$

$$\Rightarrow \quad \frac{\partial^2 u_P}{\partial x^2} = 2a, \quad \frac{\partial^2 u_P}{\partial x\,\partial y} = b \quad \text{and} \quad \frac{\partial^2 u_P}{\partial y^2} = 2c$$

$$\Rightarrow \quad 6\frac{\partial^2 u_P}{\partial x^2} - 5\frac{\partial^2 u_P}{\partial x\,\partial y} + \frac{\partial^2 u_P}{\partial y^2} = 12a - 5b + 2c = 14$$

We have one condition on three constants, two of which are therefore a free choice.

Let us leave the free choice unresolved for now.

Complementary function:

$$\lambda = \frac{+5 \pm \sqrt{1}}{12} = \frac{1}{3} \text{ or } \frac{1}{2}$$

The complementary function is

$$u_C(x,y) = f\left(y + \tfrac{1}{3}x\right) + g\left(y + \tfrac{1}{2}x\right)$$

and the general solution is

Example 11.04 (continued)

Example 11.04 (continued)

Example 11.05

Find the complete solution to

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$u = 0$ on $x = 0$,

$u = x^2$ on $y = 1$.

Example 11.05 - Alternative Treatment of the Complementary Function

Find the complete solution to

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \, \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$u = 0$ on $x = 0$,
$u = x^2$ on $y = 1$.

$A = 1,\ B = 2,\ C = 1 \quad \Rightarrow \quad D = 4 - 4 \times 1 = 0$

$$\lambda = \frac{-2 \pm \sqrt{0}}{2} = -1 \text{ or } -1$$

The complementary function (and general solution) is

$$u(x, y) = f(y - x) + h(x, y)\, g(y - x)$$

where $h(x, y)$ is any convenient non-trivial linear function of (x, y) except a multiple of $(y - x)$. The most general choice possible is

Example 11.05 Extension (continued)

Two-dimensional Laplace Equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

A function $f(x, y)$ is **harmonic** if and only if $\nabla^2 f = 0$ everywhere inside a domain Ω.

<u>Example 11.06</u>

Is $u = e^x \sin y$ harmonic on $\mathbb{R}^2$?

Example 11.07

Find the complete solution $u(x,y)$ to the partial differential equation $\nabla^2 u = 0$, given the additional information

$$u(0,y) = y^3 \quad \text{and} \quad \left.\frac{\partial u}{\partial x}\right|_{x=0} = 0$$

The PDE is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(which means that the solution $u(x,y)$ is an harmonic function).

$\Rightarrow \quad A = C = 1, \quad B = 0 \qquad \Rightarrow \quad D = B^2 - 4AC = -4 < 0$

The PDE is **elliptic** everywhere.

A.E.: $\lambda^2 + 1 = 0 \qquad \Rightarrow \quad \lambda = \pm j$

C.F.: $u_C(x,y) = f(y - jx) + g(y + jx)$

The PDE is homogeneous $\Rightarrow$

P.S.: $u_P(x,y) = 0$

G.S.: $u(x,y) = f(y - jx) + g(y + jx)$

$\Rightarrow \quad u_x(x,y) = -j\,f'(y - jx) + j\,g'(y + jx)$

Using the additional information,

$u(0,y) = f(y) + g(y) = y^3 \qquad \Rightarrow \quad g(y) = y^3 - f(y) \quad \Rightarrow \quad g'(y) = 3y^2 - f'(y)$

and

$u_x(0,y) = 0 = -j\,f'(y) + j\,g'(y) = j\left(-f'(y) + 3y^2 - f'(y)\right)$

$\Rightarrow \quad 2f'(y) = 3y^2 \qquad \Rightarrow \quad 2f(y) = y^3 \quad \Rightarrow \quad f(y) = \tfrac{1}{2}y^3$

$\Rightarrow \quad g(y) = y^3 - \tfrac{1}{2}y^3 = \tfrac{1}{2}y^3$

Therefore the complete solution is

$u(x,y) = \tfrac{1}{2}(y - jx)^3 + \tfrac{1}{2}(y + jx)^3$

$= \tfrac{1}{2}\left(y^3 - \cancel{3(jx)y^2} + 3(jx)^2 y - \cancel{(jx)^3} + y^3 + \cancel{3(jx)y^2} + 3(jx)^2 y + \cancel{(jx)^3}\right)$

$= y^3 + 3j^2x^2y \quad \Rightarrow$

$$\boxed{u(x,y) = y^3 - 3x^2y}$$

Note that the solution is completely real, even though the eigenvalues are not real.

Summary:

For the partial differential equation $A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = r(x,y)$

the general solution is the sum of a complementary function and a particular solution

$$u(x,y) = u_C(x,y) + u_P(x,y)$$

Complementary function:

Find $D = B^2 - 4AC$ then $\lambda = \frac{-B \pm \sqrt{D}}{2A}$

$u_C(x,y) = f(y+\lambda_1 x) + g(y+\lambda_2 x)$

unless $D = 0$, in which case $u_C(x,y) = f(y+\lambda x) + h(x,y)\cdot g(y+\lambda x)$ where $h(x,y)$ is any convenient linear function that is not a constant multiple of $y+\lambda x$.

Particular solution:

If $r(x,y) = 0$ (homogeneous PDE) then trivially $u_P(x,y) = 0$.

If $r(x,y)$ is a non-zero constant, then try $u_P(x,y) = ax^2 + bxy + cy^2$

If $r(x,y)$ is a linear function $r(x,y) = lx + my$, then

try $u_P(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$

In this course no other forms of $r(x,y)$ will be examinable.

Apply initial / boundary conditions only **after** finding the general solution, in order to find the arbitrary functions $f(y+\lambda_1 x)$, $g(y+\lambda_2 x)$ and therefore the complete solution.

[Space for Additional Notes]

[End of Chapter 11]