12. The Wave Equation – Vibrating Finite String

The wave equation is

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u}$$

If $u(x,t)$ is the vertical displacement of a point at location x on a vibrating string at time t, then the governing PDE is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

If $u(x,y,t)$ is the vertical displacement of a point at location (x,y) on a vibrating membrane at time t, then the governing PDE is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

or, in plane polar coordinates (r,θ), (appropriate for a circular drum),

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

<u>Example 12.01</u>

An elastic string of length L is fixed at both ends $(x=0$ and $x=L)$. The string is displaced into the form $y = f(x)$ and is released from rest. Find the displacement $y(x,t)$ at all locations on the string $(0<x<L)$ and at all subsequent times $(t>0)$.

The boundary value problem for the displacement function $y(x,t)$ is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 < x < L \quad \text{and} \quad t > 0$$

Both ends fixed for all time:

Initial configuration of string:

String released from rest:

Example 12.01 (continued)

Separation of Variables (or Fourier Method)

Attempt a solution of the form $y(x,t) = X(x)T(t)$

Substitute $y(x,t) = X(x)T(t)$ into the PDE:

$$\frac{\partial^2}{\partial t^2}\big(X(x)T(t)\big) = c^2 \frac{\partial^2}{\partial x^2}\big(X(x)T(t)\big) \quad \Rightarrow \quad X\frac{d^2T}{dt^2} = c^2 \frac{d^2X}{dx^2}T$$

$$\Rightarrow \quad \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$

Example 12.01 (continued)

Therefore our complete solution is

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$

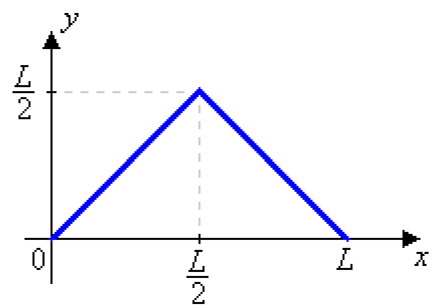
Example 12.01 (continued)

This solution is valid for any initial displacement function $f(x)$ that is continuous with a piece-wise continuous derivative on $[0, L]$ with $f(0) = f(L) = 0$.

If the initial displacement is itself sinusoidal $\left(f(x) = a \sin\left(\frac{n\pi x}{L}\right) \quad \text{for some } n \in \mathbb{N} \right)$, then the complete solution is a single term from the infinite series,

$$y(x,t) = a \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right).$$

Suppose that the initial configuration is triangular:



$$y(x,0) = f(x) = \begin{cases} x & \left(0 \le x \le \frac{1}{2}L\right) \\ L - x & \left(\frac{1}{2}L < x \le L\right) \end{cases}$$

Then the Fourier sine coefficients are

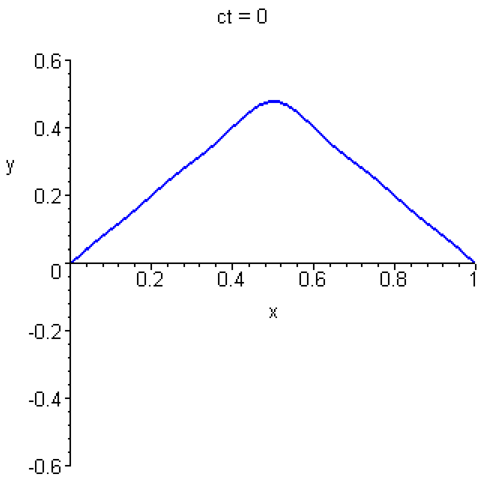
$$C_n = \frac{2}{L}\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du$$

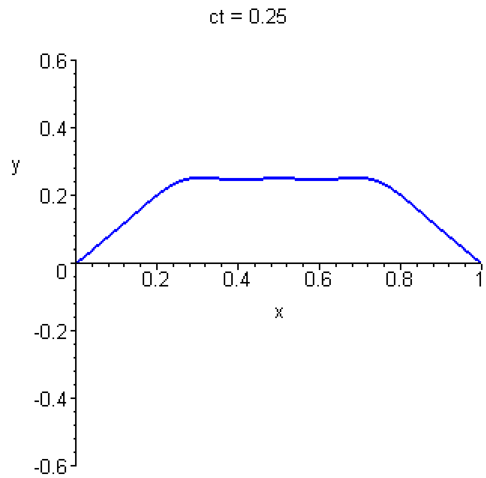
Example 12.01 (continued)

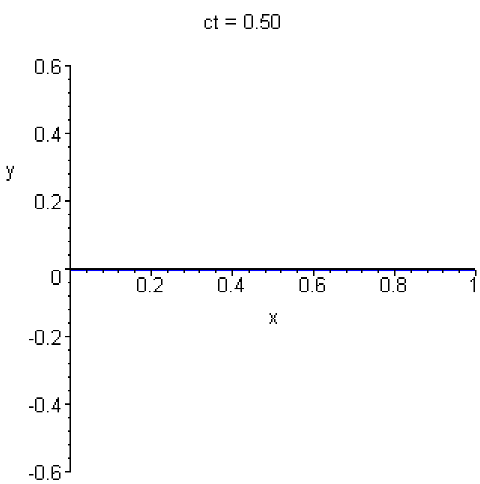
See the web page "www.engr.mun.ca/~ggeorge/4430/demos/ex1201.html" for an animation of this solution.

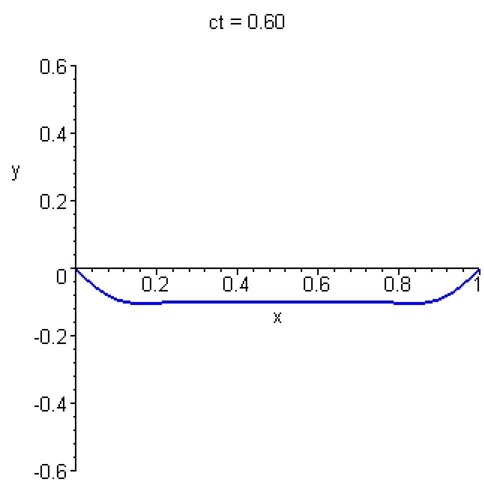
Example 12.01 (continued)

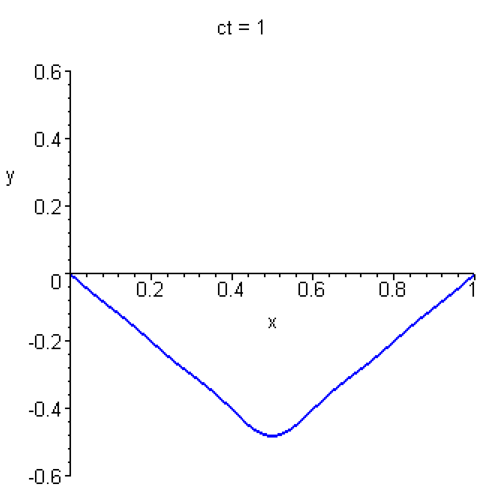
Some snapshots of the solution are shown here:











These graphs were generated from the Fourier series, truncated after the fifth non-zero term.

Example 12.02

An elastic string of length L is fixed at both ends $(x=0$ and $x=L)$. The string is initially in its equilibrium state [$y(x,0)=0$ for all x] and is released with the initial velocity $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$. Find the displacement $y(x,t)$ at all locations on the string $(0<x<L)$ and at all subsequent times $(t>0)$.

The boundary value problem for the displacement function $y(x,t)$ is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 < x < L \quad \text{and} \quad t > 0$$

Both ends fixed for all time: $y(0,t)=y(L,t)=0$ for $t \geq 0$

Initial configuration of string: $y(x,0)=0$ for $0 \leq x \leq L$

String released with initial velocity: $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = g(x) \qquad \text{for } 0 \leq x \leq L$

As before, attempt a solution by the method of the **separation of variables**.

Substitute $y(x,t)=X(x)T(t)$ into the PDE:

$$\frac{\partial^2}{\partial t^2}\big(X(x)T(t)\big) = c^2 \frac{\partial^2}{\partial x^2}\big(X(x)T(t)\big) \qquad \Rightarrow \quad X\frac{d^2T}{dt^2} = c^2 \frac{d^2X}{dx^2}T$$

Again, each side must be a negative constant.

$$\Rightarrow \quad \frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

We now have the pair of ODEs

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + \lambda^2 c^2 T = 0$$

The general solutions are

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad \text{and} \quad T(t) = C\cos(\lambda ct) + D\sin(\lambda ct)$$

respectively, where A, B, C and D are arbitrary constants.

Example 12.02 (continued)

Consider the boundary conditions:

$y(0,t) = X(0)T(t) = 0 \quad \forall t \geq 0$

For a non-trivial solution, this requires $X(0)=0 \quad \Rightarrow \quad A=0$.

$y(L,t) = X(L)T(t) = 0 \quad \forall t \geq 0 \qquad \Rightarrow \quad X(L)=0$

$\Rightarrow \quad B\sin(\lambda L)=0 \quad \Rightarrow \quad \lambda_n = \dfrac{n\pi}{L}, \quad (n \in \mathbb{Z})$

We now have a solution only for a discrete set of eigenvalues λ_n, with corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

and

$$y_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{L}\right)T_n(t), \quad (n = 1, 2, 3, \ldots)$$

So far, the solution has been identical to Example 12.01.

Consider the initial condition $y(x,0)=0$:

$$y(x,0) = 0 \quad \Rightarrow \quad X(x)T(0) = 0 \quad \forall x \quad \Rightarrow \quad T(0) = 0$$

The initial value problem for $T(t)$ is now

$$T'' + \lambda^2c^2T = 0, \quad T(0)=0, \qquad \text{where} \quad \lambda = \frac{n\pi}{L}$$

the solution to which is

$$T_n(t) = C_n\sin\left(\frac{n\pi c t}{L}\right), \qquad (n \in \mathbb{N})$$

Our eigenfunctions for y are now

$$y_n(x,t) = X_n(x)T_n(t) = C_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right), \qquad (n \in \mathbb{N})$$

Example 12.02 (continued)

Differentiate term by term and impose the initial velocity condition:

$$\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

which is just the Fourier sine series expansion for the function $g(x)$.
The coefficients of the expansion are

$$C_n \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du$$

which leads to the complete solution

$$y(x,t) = \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

This solution is valid for any initial velocity function $g(x)$ that is continuous with a piece-wise continuous derivative on [0, L] with $g(0) = g(L) = 0$.

The solutions for Examples 12.01 and 12.02 may be superposed.

Let $y_1(x,t)$ be the solution for initial displacement $f(x)$ and zero initial velocity.
Let $y_2(x,t)$ be the solution for zero initial displacement and initial velocity $g(x)$.
Then $y(x,t) = y_1(x,t) + y_2(x,t)$ satisfies the wave equation
(the sum of any two solutions of a linear homogeneous PDE is also a solution),
and satisfies the boundary conditions $y(0,t) = y(L,t) = 0$.

$y(x,0) = y_1(x,0) + y_2(x,0) = f(x) + 0$,
which satisfies the condition for initial displacement $f(x)$.

$y_t(x,0) = y_{1t}(x,0) + y_{2t}(x,0) = 0 + g(x)$,
which satisfies the condition for initial velocity $g(x)$.

Therefore the sum of the two solutions is the complete solution for initial displacement $f(x)$ **and** initial velocity $g(x)$:

$$\begin{aligned} y(x,t) = {} & \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right) \\ & + \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right) \end{aligned}$$

The Heat Equation

For a material of constant density ρ, constant specific heat μ and constant thermal conductivity *K*, the partial differential equation governing the temperature *u* at any location (x, y, z) and any time t is

$$\frac{\partial u}{\partial t} = k \nabla^2 u \ , \quad \text{where} \quad k = \frac{K}{\mu\rho}$$

Example 12.03

Heat is conducted along a thin homogeneous bar extending from $x = 0$ to $x = L$. There is no heat loss from the sides of the bar. The two ends of the bar are maintained at temperatures T_1 (at $x = 0$) and T_2 (at $x = L$). The initial temperature throughout the bar at the cross-section x is $f(x)$.
Find the temperature at any point in the bar at any subsequent time.

The partial differential equation governing the temperature $u(x,t)$ in the bar is

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

together with the boundary conditions

$u(0,t) = T_1$ and $u(L,t) = T_2$

and the initial condition

$u(x,0) = f(x)$

[Note that if an end of the bar is insulated, instead of being maintained at a constant temperature, then the boundary condition changes to $\frac{\partial u}{\partial t}(0,t) = 0$ or $\frac{\partial u}{\partial t}(L,t) = 0$.]

Attempt a solution by the method of separation of variables.

$$u(x,t) = X(x)T(t)$$

$$\Rightarrow \quad X T' = k X'' T \qquad \Rightarrow \quad \frac{T'}{T} = k \frac{X''}{X} = c$$

Again, when a function of *t* only equals a function of *x* only, both functions must equal the same absolute constant. Unfortunately, the two boundary conditions cannot both be satisfied unless $T_1 = T_2 = 0$. Therefore we need to treat this more general case as a perturbation of the simpler $(T_1 = T_2 = 0)$ case.

Example 12.03 (continued)

Let $u(x,t) = v(x,t) + g(x)$
Substitute this into the PDE:

$$\frac{\partial}{\partial t}\big(v(x,t)+g(x)\big) = k\,\frac{\partial^2}{\partial x^2}\big(v(x,t)+g(x)\big) \;\Rightarrow\; \frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial x^2} + g''(x)\right)$$

This is the standard heat PDE for v if we choose g such that $g''(x) \equiv 0$.
$g(x)$ must therefore be a linear function of x.

We want the perturbation function $g(x)$ to be such that

$$u(0,t) = T_1, \quad u(L,t) = T_2$$

and

$$v(0,t) = v(L,t) = 0$$

Therefore $g(x)$ must be the linear function for which $g(0) = T_1$ and $g(L) = T_2$. It follows that

$$g(x) = \left(\frac{T_2 - T_1}{L}\right)x + T_1$$

and we now have the simpler problem

$$\frac{\partial v}{\partial t} = k\,\frac{\partial^2 v}{\partial x^2}$$

together with the boundary conditions

$$v(0,t) = v(L,t) = 0$$

and the initial condition

$$v(x,0) = f(x) - g(x)$$

Now try separation of variables on $v(x,t)$:

$$v(x,t) = X(x)T(t) \quad\Rightarrow\quad X\,T' = k\,X''T \qquad\Rightarrow\quad \frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = c$$

But $v(0,t) = v(L,t) = 0 \;\Rightarrow\; X(0) = X(L) = 0$

This requires c to be a negative constant, say $-\lambda^2$.
The solution is very similar to that for the wave equation on a finite string with fixed ends (examples 12.01 and 12.02). The eigenvalues are $\lambda = \dfrac{n\pi}{L}$ and the corresponding eigenfunctions are any non-zero constant multiples of

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Example 12.03 (continued)

The ODE for $T(t)$ is first order:

$$T' + \left(\frac{n\pi}{L}\right)^2 kT = 0$$

whose general solution is

$$T_n(t) = c_n e^{-n^2\pi^2 kt/L^2}$$

Therefore

$$v_n(x,t) = X_n(x)T_n(t) = c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$$

If the initial temperature distribution $f(x) - g(x)$ is a simple multiple of $\sin\left(\frac{n\pi x}{L}\right)$ for some integer *n*, then the solution for v is just $v(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$.

Otherwise, we must attempt a superposition of solutions.

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$$

such that $v(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x) - g(x)$.

The Fourier sine series coefficients are $c_n = \frac{2}{L}\int_0^L \left(f(z) - g(z)\right)\sin\left(\frac{n\pi z}{L}\right) dz$

so that the complete solution for $v(x,t)$ is

$$v(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L \left(f(z) - \frac{T_2 - T_1}{L}z - T_1\right)\sin\left(\frac{n\pi z}{L}\right)dz\right)\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$$

and the complete solution for $u(x,t)$ is

$$u(x,t) = v(x,t) + \left(\frac{T_2 - T_1}{L}\right)x + T_1$$

Note how this solution can be partitioned into a transient part $v(x,t)$ (which decays to zero as *t* increases) and a linear steady-state part $g(x)$ which is the limiting value that the temperature distribution approaches.

Example 12.03 (continued)

As a specific example, let $k=9$, $T_1=100$, $T_2=200$, $L=2$ and $f(x)=145x^2-240x+100$, (for which $f(0)=100$, $f(2)=200$ and $f(x)>0 \ \forall x$).

Then $g(x) = \frac{200-100}{2} x + 100 = 50x + 100$

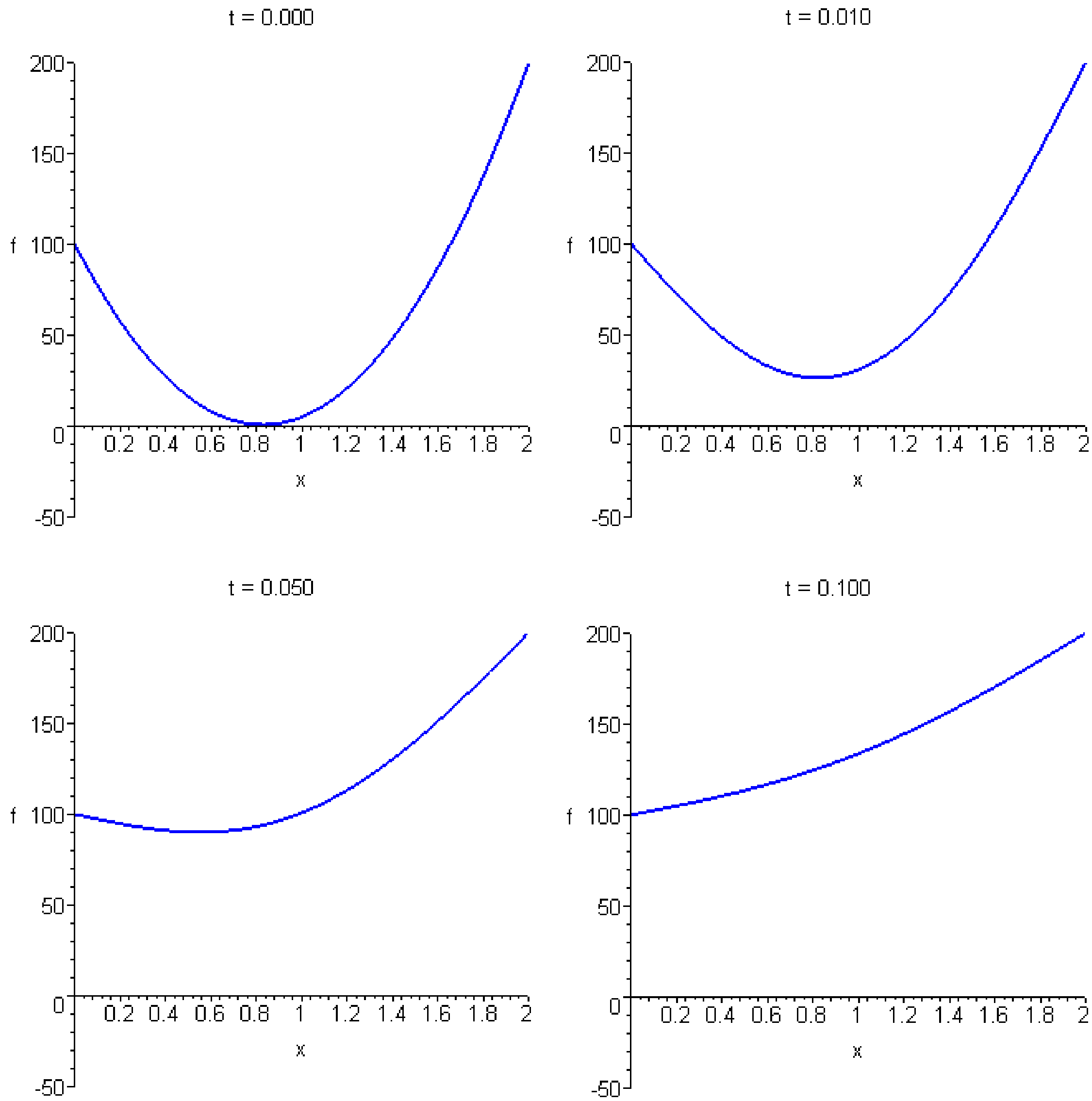
The Fourier sine series coefficients are

The complete solution is

$$u(x,t) = 50x + 100 - \frac{2320}{\pi^3} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n^3} \right) \sin\left(\frac{n\pi x}{2} \right) \exp\left(-\frac{9n^2\pi^2 t}{4} \right)$$

Some snapshots of the temperature distribution (from the tenth partial sum) from the Maple file at "www.engr.mun.ca/~ggeorge/4430/demos/ex1205.mws" are shown on the next page.

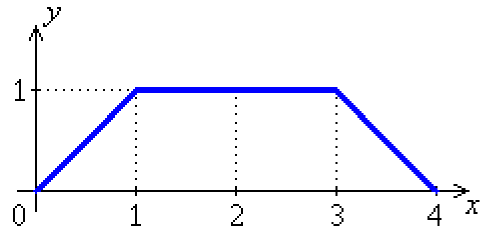
Example 12.03 (continued)



The steady state distribution is nearly attained in much less than a second!

Example 12.04

A perfectly elastic string of equilibrium length 4 metres is released from rest in a trapezoidal configuration shown here:



The string is clamped at both ends ($x = 0$ and $x = 4$).
Waves move on the string with speed c. There is no friction.
Determine the subsequent evolution of the displacement $y(x,t)$ of the string.

The initial displacement is

$$y(x,0) = f(x) = \begin{cases} x & (0 \le x \le 1) \\ 1 & (1 < x \le 3) \\ 4-x & (3 < x \le 4) \\ 0 & (\text{otherwise}) \end{cases}$$

The coefficients in the Fourier sine series for $y(x,t)$ are

$$c_n = \int_0^4 f(u) \sin\left(\frac{n\pi u}{4}\right) du$$

However the integrand changes its functional form abruptly at $x = 1$ and again at $x = 3$. The integral must therefore be split into three pieces:

$$c_n = \int_0^1 u \sin\left(\frac{n\pi u}{4}\right) du + \int_1^3 1 \sin\left(\frac{n\pi u}{4}\right) du + \int_3^4 (4-u) \sin\left(\frac{n\pi u}{4}\right) du$$

Using integrations by parts on the first and last integrals,

$$c_n = \left[-\frac{4}{n\pi} u \cos\left(\frac{n\pi u}{4}\right) + \left(\frac{4}{n\pi}\right)^2 \sin\left(\frac{n\pi u}{4}\right) \right]_0^1$$

$$+ \left[-\frac{4}{n\pi} \cos\left(\frac{n\pi u}{4}\right) \right]_1^3$$

$$+ \left[-\frac{4}{n\pi}(4-u) \cos\left(\frac{n\pi u}{4}\right) - \left(\frac{4}{n\pi}\right)^2 \sin\left(\frac{n\pi u}{4}\right) \right]_3^4$$

D1	**D3**		**I**
u	$4-u$		$\sin \frac{n\pi u}{4}$
		+	
1	-1		$-\frac{4}{n\pi} \cos \frac{n\pi u}{4}$
		−	
0	0		$-\left(\frac{4}{n\pi}\right)^2 \sin \frac{n\pi u}{4}$

Example 12.04 (continued)

$$\Rightarrow \quad c_n = \left(-\frac{4}{n\pi}\cos\left(\frac{n\pi}{4}\right) + \left(\frac{4}{n\pi}\right)^2 \sin\left(\frac{n\pi}{4}\right) + 0 - 0 \right)$$

$$+ \left(-\frac{4}{n\pi}\cos\left(\frac{3n\pi}{4}\right) + \frac{4}{n\pi}\cos\left(\frac{n\pi}{4}\right) \right) + \left(-0 - 0 + \frac{4}{n\pi}\cos\left(\frac{3n\pi}{4}\right) + \left(\frac{4}{n\pi}\right)^2 \sin\left(\frac{3n\pi}{4}\right) \right)$$

$$\Rightarrow \quad c_n = \left(\frac{4}{n\pi}\right)^2 \left(\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right) \right) = 2\left(\frac{4}{n\pi}\right)^2 \sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{4}\right)$$

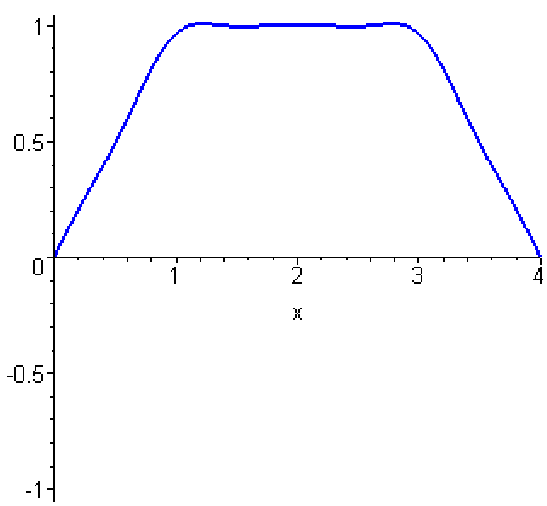
(using the trigonometric identity $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$)

This series has a fairly rapid convergence: it is of the form $\frac{1}{n^2}$ with $c_n = 0$ for all even *n*.
Substituting into the complete solution of the wave PDE, we have

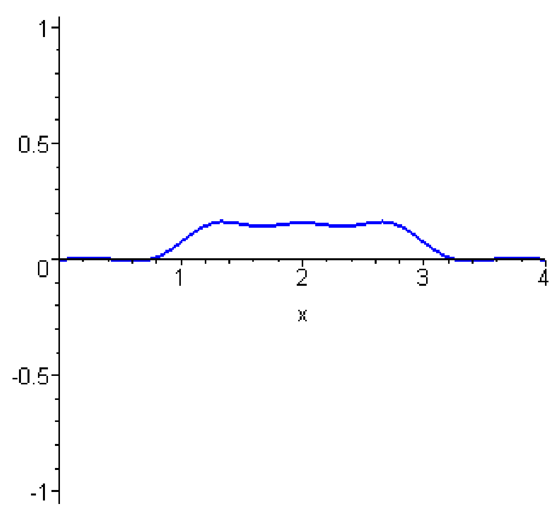
$$y(x,t) = \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi x}{4}\right)\cos\left(\frac{n\pi c t}{4}\right)$$

An animation of the solution and a Maple file are available from the course web site, at " www.engr.mun.ca/~ggeorge/4430/demos/ex1206.html ".
Two snapshots (from the sum of the first six non-trivial terms in the series) are shown here.

$t = 0$



$ct \approx 1.6$



Example 12.05

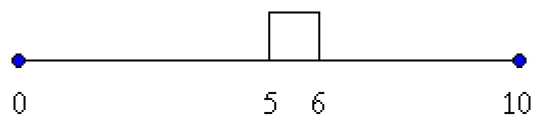
A perfectly elastic string of equilibrium length 10 metres is released from rest in a configuration very close to that of a square pulse of unit height, between $x = 5$ and $x = 6$, that is

$$y(x,0) = f(x) = \begin{cases} 1 & (5 \le x \le 6) \\ 0 & (\text{otherwise}) \end{cases}$$

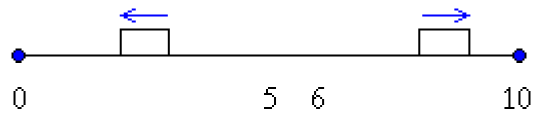
The string is clamped at both ends ($x = 0$ and $x = 10$).
Waves move on the string with speed c. There is no friction.
Determine the subsequent evolution of the displacement $y(x,t)$ of the string.

First let us anticipate the response.

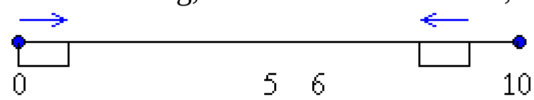
The initial configuration looks like this. The pulse is just to the right of the centre of the string.



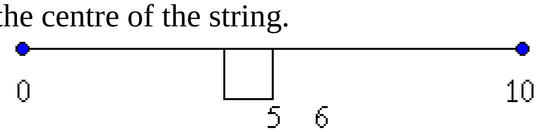
Two copies of the initial configuration, each half the height of the original, move off in opposite directions.



If the string were infinite, then the two waves would continue to recede from each other forever. At the fixed ends of the string, the waves bounce back, reflected.



They travel towards each other until they recombine into the negative of the original wave, just to the left of the centre of the string.



The second half of the cycle looks like a time-reversed version of the first half.

Example 12.05 (continued)

The general solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 < x < L \quad \text{and} \quad t > 0$$

with conditions

Both ends fixed for all time: $y(0,t) = y(L,t) = 0$ for $t \geq 0$

Initial configuration of string: $y(x,0) = f(x)$ for $0 \leq x \leq L$

String released from rest: $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = 0 \qquad \text{for } 0 \leq x \leq L$

is the Fourier series

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$

Here $L = 10$ and $y(x,0) = f(x) = \begin{cases} 1 & (5 \leq x \leq 6) \\ 0 & (\text{otherwise}) \end{cases}$

$$\int_0^{10} f(u)\sin\left(\frac{n\pi u}{10}\right)du = 0 + \int_5^6 1\sin\left(\frac{n\pi u}{10}\right)du + 0 = \left[-\frac{10}{n\pi}\cos\left(\frac{n\pi u}{10}\right)\right]_5^6$$

$$= \frac{10}{n\pi}\left(-\cos\left(\frac{n\pi 6}{10}\right) + \cos\left(\frac{n\pi 5}{10}\right)\right) = \frac{10}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{3n\pi}{5}\right)\right) \Rightarrow$$

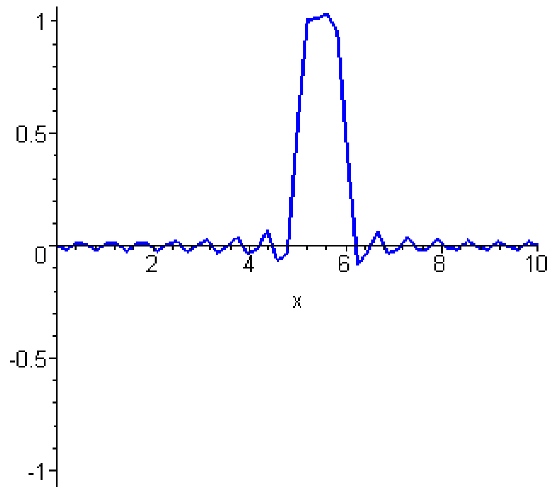
$$\boxed{y(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\left(\cos\left(\frac{n\pi}{2}\right) - \cos\left(\frac{3n\pi}{5}\right)\right)\right)\sin\left(\frac{n\pi x}{10}\right)\cos\left(\frac{n\pi c t}{10}\right)}$$

This series is very slow to converge (which is no surprise, given that $f(x)$ is not even continuous, let alone differentiable, in two places).

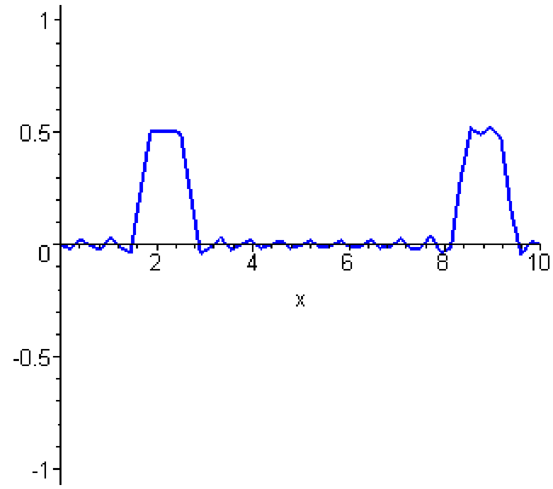
An animation of the solution and a Maple file are available from the course web site, at "www.engr.mun.ca/~ggeorge/4430/demos/ex1207.html".

The animation of the solution uses the partial sum of the first *thirty* terms of the Fourier series. Even with this large number of terms, the approximation is quite rough. Some snapshots are shown on the next page.

Example 12.05 (continued)

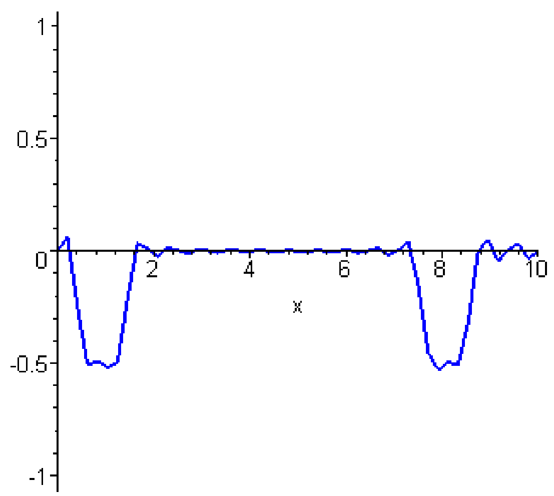


t = 0

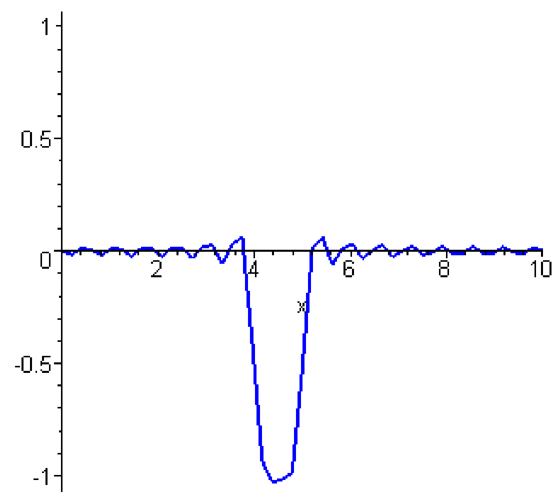


ct = 3.5

ct = 6.5



ct = 10.0



<u>Example 12.06</u>

A perfectly elastic string of equilibrium length L is released from the initial shape

$$y(x,0) = f(x) = \sin\left(\frac{2\pi x}{L}\right)$$

with an initial velocity profile

$$y_t(x,0) = g(x) = \sin\left(\frac{4\pi x}{L}\right)$$

The string is clamped at both ends ($x = 0$ and $x = L$).
Waves move on the string with speed c. There is no friction.
Determine the subsequent evolution of the displacement $y(x,t)$ of the string.

The trigonometric identity $-2\sin(A)\sin(B) = \cos(A+B) - \cos(A-B)$ can be used to verify the orthogonality property of the set of sine functions $\left\{\sin\left(\frac{n\pi x}{L}\right)\right\}$ on [0, L]:

$$\frac{2}{L}\int_0^L \sin\left(\frac{m\pi u}{L}\right)\sin\left(\frac{n\pi u}{L}\right)du = \begin{cases} 1 & (m = \pm n) \\ 0 & (\text{otherwise}) \end{cases}$$

Therefore in the complete solution of the wave equation

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$$
$$+ \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

only the $n = 2$ term is non-zero in the first series and
only the $n = 4$ term is non-zero in the second series.
The complete solution reduces to

$$\boxed{y(x,t) = \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi ct}{L}\right) + \frac{L}{4\pi c}\sin\left(\frac{4\pi x}{L}\right)\sin\left(\frac{4\pi ct}{L}\right)}$$

An animation of the solution and a Maple file are available from the course web site, at "www.engr.mun.ca/~ggeorge/4430/demos/ex1208.html".

[End of Chapter 12]
END OF ENGI 4430!