13. Suggestions for the Formula Sheets

Below are some suggestions for many more formulae than can be placed easily on both sides of the two standard 8½"×11" sheets of paper for the final examination. It is strongly recommended that students compose their own formula sheets, to suit each individual's needs.

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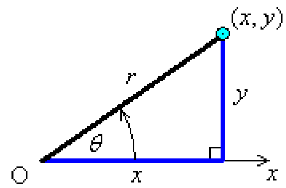
1. Parametric and Polar Curves

Distance $r(t) = |\vec{\mathbf{r}}(t)| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$

To sketch $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$: Find all values of t at which any of $x(t)$, $y(t)$, $x'(t)$, $y'(t)$ are zero, then construct a table for all four functions.



$\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0 \quad \Rightarrow \quad$ horizontal tangent

$\frac{dy}{dt} \neq 0$ and $\frac{dx}{dt} = 0 \quad \Rightarrow \quad$ vertical tangent

Polar coordinates: $x = r\cos\theta,\ y = r\sin\theta; \qquad r^2 = x^2 + y^2,\ \tan\theta = \frac{y}{x}$

$(r, \theta + 2n\pi)$ and $(-r, \theta + (2n+1)\pi)$ $(n \in \mathbb{Z})$ are the same point as (r, θ).

To sketch $r = f(\theta)$, sketch Cartesian $y = f(x)$ with $y = r,\ x = \theta$, then transfer onto a polar sketch.

$r(\theta_o) = 0$ and $r'(\theta_o) \neq 0 \quad \Rightarrow \quad \theta = \theta_o$ is a tangent at the pole.

2. Vectors

The **component** of vector $\vec{\mathbf{u}}$ in the direction of vector $\vec{\mathbf{v}}$ is $u_v = \vec{\mathbf{u}} \bullet \hat{\mathbf{v}} = u\cos\theta$

$$\frac{d}{dt}(\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}) = \frac{d\vec{\mathbf{u}}}{dt} \bullet \vec{\mathbf{v}} + \vec{\mathbf{u}} \bullet \frac{d\vec{\mathbf{v}}}{dt} \quad \text{and}$$

$$\frac{d}{dt}(\vec{\mathbf{u}} \times \vec{\mathbf{v}}) = \frac{d\vec{\mathbf{u}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \frac{d\vec{\mathbf{v}}}{dt} = -\vec{\mathbf{v}} \times \frac{d\vec{\mathbf{u}}}{dt} + \vec{\mathbf{u}} \times \frac{d\vec{\mathbf{v}}}{dt}$$

The **distance along a curve** between two points whose parameter values are t_0 and t_1 is

$$L = \int_{t=t_0}^{t=t_1} ds = \int_{t_0}^{t_1} \frac{ds}{dt}\, dt = \int_{t_0}^{t_1} \left| \frac{d\vec{\mathbf{r}}}{dt} \right| dt = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The **distance along a polar curve** $r = f(\theta)$ is $L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

The **unit tangent** at any point on a curve is

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \frac{d\vec{\mathbf{r}}}{ds}$$

The **unit principal normal** at any point on a curve is

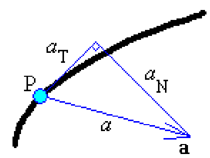
$$\hat{\mathbf{N}} = \rho \frac{d\hat{\mathbf{T}}}{ds} = \frac{d\hat{\mathbf{T}}}{dt} \div \left|\frac{d\hat{\mathbf{T}}}{dt}\right|, \quad \text{where } \rho = \text{ radius of curvature } = \frac{1}{\kappa}$$

The **unit binormal** is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$$

Velocity is $\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}}{dt}$, **speed** is $v(t) = |\vec{\mathbf{v}}(t)| = \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \frac{ds}{dt}$ and $\vec{\mathbf{v}} = v\hat{\mathbf{T}}$

The **acceleration** [vector] is



$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2} = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} + \frac{d^2z}{dt^2}\hat{\mathbf{k}} = a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}}$$

where $a_T = \frac{dv}{dt} = \vec{\mathbf{a}} \bullet \hat{\mathbf{T}} = \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{a}}}{v}$ and $a_N = \kappa v^2 = \sqrt{a^2 - a_T^{\,2}} = \vec{\mathbf{a}} \bullet \hat{\mathbf{N}} = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{v}$

The **surface of revolution** of $y = f(x)$ around $y = c$ is $(y-c)^2 + z^2 = (f(x) - c)^2$

The curved surface area from $x = a$ to $x = b$ is $A = 2\pi \int_a^b |f(x) - c| \sqrt{1 + (f'(x))^2}\, dx$

The **area** between a curve and the x axis is $A = \int_{t_a}^{t_b} |y(t)| \frac{dx}{dt}\, dt$

The area swept out by a polar curve $(\alpha < \theta < \beta < \alpha + 2\pi)$ is $A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$

Components of velocity: $v_{\text{radial}} = \dot{r}$, $v_{\text{transverse}} = r\dot{\theta}$, $v_T = v$, $v_N \equiv 0$

Components of acceleration:

$$a_{\text{radial}} = \ddot{r} - r(\dot{\theta})^2, \quad a_{\text{transverse}} = 2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}), \quad a_T = \frac{dv}{dt}, \quad a_N = \kappa v^2 = \sqrt{a^2 - a_T^{\,2}}$$

Line parallel to $[\,a \;\; b \;\; c\,]^{\mathrm{T}}$ through (x_o, y_o, z_o) is $\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c}$

(except where any of a, b, c is zero)

Plane normal to $[\,A \;\; B \;\; C\,]^{\mathrm{T}}$ containing (x_o, y_o, z_o) is $Ax + By + Cz + D = 0$, where $D = -(Ax_o + By_o + Cz_o)$

3. Multiple Integrals

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\,dx\,dy$$

or $\displaystyle\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,r\,dr\,d\theta$

Centre of mass: $\displaystyle\langle\vec{\mathbf{r}}\rangle = \frac{\vec{\mathbf{M}}}{m} = \left(\iint_D \sigma\vec{\mathbf{r}}\,dA\right) \div \left(\iint_D \sigma\,dA\right)$ or $\displaystyle\left(\iiint_V \rho\vec{\mathbf{r}}\,dV\right) \div \left(\iiint_V \rho\,dV\right)$

Moment of inertia $\displaystyle I_x = \iint_R y^2\sigma\,dA,\quad I_y = \iint_R x^2\sigma\,dA,\quad I = I_x + I_y = \iint_R r^2\sigma\,dA$

Parallel axis theorem, second moment $I_{x'}$ of mass m about axis $y = y_0$ a distance b from the axis $y = \bar{y}$ through the centre of mass: $I_{x'} = I_x + mb^2$

4. Streamlines (lines of force)

Streamlines to $\vec{\mathbf{F}} = [\, f_1 \;\; f_2 \;\; f_3 \,]^{\mathrm{T}}$ are the solutions of $\dfrac{d\vec{\mathbf{r}}}{ds} = k\,\vec{\mathbf{F}}$

$\Rightarrow \quad \dfrac{dx}{f_1} = \dfrac{dy}{f_2} = \dfrac{dz}{f_3}$ (except that $f_i = 0 \Rightarrow$ that component is constant).

5. Numerical Integration

$[a,b]$ divided into n equal intervals. $h = \dfrac{b-a}{n}$

Trapezoidal rule: $\displaystyle\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n\right)$

Simpson's rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left(f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \ldots + 2f_{n-2} + 4f_{n-1} + f_n\right)$$

Newton's method to solve $f(x) = 0$: $\displaystyle x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

6. The Gradient Vector

The **directional derivative** of f in the direction of the unit vector $\hat{\mathbf{u}}$ is $D_{\hat{\mathbf{u}}} f = \vec{\nabla} f \bullet \hat{\mathbf{u}}$

$$\frac{df}{dt} = \vec{\nabla} f \bullet \frac{d\vec{\mathbf{r}}}{dt}, \quad \text{where} \quad \vec{\nabla} f = \left[\frac{\partial f}{\partial x_1} \;\; \frac{\partial f}{\partial x_2} \;\; \cdots \;\; \frac{\partial f}{\partial x_n} \right]^{\mathrm{T}} \quad \text{and} \quad \frac{d\vec{\mathbf{r}}}{dt} = \left[\frac{dx_1}{dt} \;\; \frac{dx_2}{dt} \;\; \cdots \;\; \frac{dx_n}{dt} \right]^{\mathrm{T}}$$

Tangent plane to $f(x,y,z) = c$ at $P(x_o, y_o, z_o)$ is $\vec{\mathbf{n}} \bullet \vec{\mathbf{r}} = \vec{\mathbf{n}} \bullet \vec{\mathbf{r}}_o$, where $\vec{\mathbf{n}} = \vec{\nabla} f\big|_P$.

If $\vec{\mathbf{v}}(x,y) = u(x,y)\,\hat{\mathbf{i}} + v(x,y)\,\hat{\mathbf{j}}$ and $\operatorname{div}\vec{\mathbf{v}} = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$, then the **stream function** $\psi(x,y)$ exists such that $\dfrac{\partial \psi}{\partial x} = v$ and $\dfrac{\partial \psi}{\partial y} = -u$. **Streamlines** are $\psi(x,y) = c$.

$$\vec{\nabla}\times\vec{\nabla}f = \text{curl grad}\, f = \vec{\mathbf{0}}$$

$$\vec{\nabla}\bullet\vec{\nabla}\times\vec{\mathbf{F}} = \text{div curl}\,\vec{\mathbf{F}} = 0$$

$$\vec{\nabla}(f\,g) = \left(\vec{\nabla}f\right)g + f\left(\vec{\nabla}g\right)$$

$$\text{Laplacian of } V = \nabla^2 V = \vec{\nabla}\bullet\left(\vec{\nabla}V\right) = \text{div grad } V$$

7. <u>Conversions between Coordinate Systems</u>

To convert a vector expressed in Cartesian components $v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ into the equivalent vector expressed in **<u>cylindrical polar</u>** <u>coordinates</u> $v_\rho\hat{\boldsymbol{\rho}} + v_\phi\hat{\boldsymbol{\phi}} + v_z\hat{\mathbf{k}}$, express the Cartesian components v_x, v_y, v_z in terms of (ρ,ϕ,z) using
$x = \rho\cos\phi,\ \ y = \rho\sin\phi,\ \ z = z$; then evaluate

$$\begin{bmatrix} v_\rho \\ v_\phi \\ v_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Use the inverse matrix [= transpose] to transform back to Cartesian coordinates.
To convert a vector expressed in Cartesian components $v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$ into the equivalent vector expressed in **<u>spherical polar</u>** <u>coordinates</u> $v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\phi\hat{\boldsymbol{\phi}}$, express the Cartesian components v_x, v_y, v_z in terms of (r,θ,ϕ) using
$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi,\quad z = r\cos\theta$; then evaluate

$$\begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Use the inverse matrix [= transpose] to transform back to Cartesian coordinates.

Basis Vectors

Cylindrical Polar:

$$\frac{d}{dt}\hat{\boldsymbol{\rho}} = \frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d}{dt}\hat{\boldsymbol{\phi}} = -\frac{d\phi}{dt}\hat{\boldsymbol{\rho}}$$

$$\frac{d}{dt}\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{r}} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{k}}$$

$$\Rightarrow \quad \vec{\mathbf{v}} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\mathbf{k}}$$

Spherical Polar:

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \sin\theta\,\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}} + \cos\theta\,\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)\frac{d\phi}{dt}$$

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad \Rightarrow \quad \vec{\mathbf{v}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}$$

Gradient operator in any orthonormal coordinate system

Gradient operator
$$\vec{\nabla} = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3}$$

Gradient
$$\vec{\nabla}V = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial V}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial V}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial V}{\partial u_3}$$

Divergence
$$\vec{\nabla}\bullet\vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial\left(h_2 h_3 f_1\right)}{\partial u_1} + \frac{\partial\left(h_3 h_1 f_2\right)}{\partial u_2} + \frac{\partial\left(h_1 h_2 f_3\right)}{\partial u_3}\right)$$

Curl
$$\vec{\nabla}\times\vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\,\hat{\mathbf{e}}_1 & \dfrac{\partial}{\partial u_1} & h_1\,f_1 \\ h_2\,\hat{\mathbf{e}}_2 & \dfrac{\partial}{\partial u_2} & h_2\,f_2 \\ h_3\,\hat{\mathbf{e}}_3 & \dfrac{\partial}{\partial u_3} & h_3\,f_3 \end{vmatrix}$$

Laplacian
$$\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right)$$

$$dV = h_1 h_2 h_3 \; du_1 du_2 du_3 = \left| \frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} \right| du_1 du_2 du_3$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & \frac{\partial x}{\partial u_3} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} & \frac{\partial y}{\partial u_3} \\ \frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} & \frac{\partial z}{\partial u_3} \end{vmatrix} du_1 du_2 du_3$$

Cartesian: $h_x = h_y = h_z = 1$

Cylindrical polar: $h_\rho = h_z = 1, \quad h_\phi = \rho$

Spherical polar: $h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta$

8. **Line Integrals** Work done by $\vec{\mathbf{F}}$ along curve C is $W = \int_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt$

The location $\langle \vec{\mathbf{r}} \rangle$ of the centre of mass of a wire is $\langle \vec{\mathbf{r}} \rangle = \dfrac{\vec{\mathbf{M}}}{m}$, where

$$\vec{\mathbf{M}} = \int_{t_0}^{t_1} \rho \vec{\mathbf{r}} \frac{ds}{dt}\, dt\,, \quad m = \int_{t_0}^{t_1} \rho \frac{ds}{dt}\, dt \quad \text{and} \quad \frac{ds}{dt} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,.$$

If a potential function V for $\vec{\mathbf{F}}$ exists, then $W =$ (potential difference) $= \left[\, V \,\right]_{\text{start}}^{\text{end}}$

Green's Theorem

For a simple closed curve C enclosing a finite region D of $\mathbb{R}^2$ and for any vector function $\vec{\mathbf{F}} = [\, f_1 \;\; f_2 \,]^{\mathrm{T}}$ that is differentiable everywhere on C and everywhere in D,

$$\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

Path Independence

When a vector field **F** is defined on a simply connected domain Ω, these statements are all equivalent (that is, **all** of them are true or all of them are false):

- $\vec{\mathbf{F}} = \vec{\nabla}\phi$ for some scalar field ϕ that is differentiable everywhere in Ω;
- $\vec{\mathbf{F}}$ is conservative;
- $\int_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}}$ is path-independent (has the same value no matter which path within Ω is chosen between the two endpoints, for any two endpoints in Ω);
- $\int_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \phi_{\text{end}} - \phi_{\text{start}}$ (for any two endpoints in Ω);
- $\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = 0$ for all closed curves C lying entirely in Ω;
- $\dfrac{\partial f_2}{\partial x} = \dfrac{\partial f_1}{\partial y}$ everywhere in Ω; and
- $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ everywhere in Ω (so that the vector field $\vec{\mathbf{F}}$ is irrotational).

There must be no singularities anywhere in the domain Ω in order for the above set of equivalencies to be valid.

9. Surface Integrals - Projection Method

For surfaces $z = f(x, y)$, $\vec{\mathbf{N}} = \left[-\dfrac{\partial f}{\partial x} \quad -\dfrac{\partial f}{\partial y} \quad +1 \right]^{\mathrm{T}}$ and

$$\iint_S g(\vec{\mathbf{r}})\, dS = \iint_D g(\vec{\mathbf{r}}) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA \qquad (\text{where } dA = dx\, dy)$$

Surface Integrals - Surface Method

With a coordinate grid (u, v) on the surface S, $\displaystyle\iint_S g(\vec{\mathbf{r}})\, dS = \iint_S g(\vec{\mathbf{r}}) \left| \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right| du\, dv$

The total flux of a vector field $\vec{\mathbf{F}}$ through a surface S is

$$\Phi = \iint_S \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\, dS = \iint_S \vec{\mathbf{F}} \bullet \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\, du\, dv$$

Some common parametric nets are listed on pages 9.19 and 9.20.

10. Theorems of Gauss and Stokes; Potential Functions

Gauss' divergence theorem: $\oiint_S \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{S}} = \iiint_V \vec{\nabla} \bullet \vec{\mathbf{F}}\, dV$ on a simply-connected domain.

Gauss' law for the net flux through any smooth simple closed surface *S*, in the presence of a point charge *q*, is:

$$\oiint_S \vec{\mathbf{E}} \bullet \mathbf{d}\vec{\mathbf{S}} = \begin{cases} \dfrac{q}{\varepsilon} & \text{if } S \text{ encloses } O \\ 0 & \text{otherwise} \end{cases}$$

Stokes' theorem: $\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \iint_S \vec{\nabla} \times \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\, dS = \iint_S \left(\text{curl}\, \vec{\mathbf{F}}\right) \bullet \mathbf{d}\vec{\mathbf{S}}$

On a simply-connected domain Ω the following statements are either all true or all false:

- $\vec{\mathbf{F}}$ is conservative.
- $\vec{\mathbf{F}} \equiv \vec{\nabla}\phi$
- $\vec{\nabla} \times \vec{\mathbf{F}} \equiv \vec{\mathbf{0}}$
- $\int_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \phi\left(P_{\text{end}}\right) - \phi\left(P_{\text{start}}\right)$ - independent of the path between the two points.
- $\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = 0 \quad \forall C \subset \Omega$

ϕ is the potential function for **F**, so that $\left[\dfrac{\partial\phi}{\partial x} \quad \dfrac{\partial\phi}{\partial y} \quad \dfrac{\partial\phi}{\partial z} \right]^{\mathrm{T}} = \left[F_1 \;\; F_2 \;\; F_3 \right]^{\mathrm{T}}$.

11. Major Classifications of Common PDEs

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

$D = B^2 - 4AC$

Hyperbolic, wherever (x, y) is such that $D > 0$;

Parabolic, wherever (x, y) is such that $D = 0$;

Elliptic, wherever (x, y) is such that $D < 0$.

d'Alembert Solution

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = r(x,y)$$

A.E.: $A\lambda^2 + B\lambda + C = 0$

C.F.: $u_C(x,y) = f(y+\lambda_1 x) + g(y+\lambda_2 x)$, [except when $D = 0$]

where $\lambda_1 = \dfrac{-B-\sqrt{D}}{2A}$ and $\lambda_2 = \dfrac{-B+\sqrt{D}}{2A}$ and $D = B^2 - 4AC$

When $D = 0$, $u_C(x,y) = f(y+\lambda x) + h(x,y)\,g(y+\lambda x)$,

where $h(x,y)$ is a linear function that is neither zero nor a multiple of $(y+\lambda x)$.

P.S.: if RHS = n^{th} order polynomial in x and y, then try an $(n+2)^{\text{th}}$ order polynomial.

12. The Wave Equation – Vibrating Finite String

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \text{for } 0 < x < L \text{ and } t > 0 \text{ with } y(0,t) = y(L,t) = 0 \text{ for } t \ge 0,$$

$$y(x,0) = f(x) \text{ for } 0 \le x \le L \quad \text{and} \quad \left.\frac{\partial y}{\partial t}\right|_{(x,0)} = g(x) \quad \text{for } 0 \le x \le L$$

Substitute $y(x,t) = X(x)T(t)$ into the PDE. ... leads, via Fourier series, to

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$
$$+ \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

The Heat Equation

If the temperature $u(x,t)$ in a bar satisfies $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ together with the boundary conditions $u(0,t) = T_1$ and $u(L,t) = T_2$ and the initial condition $u(x,0) = f(x)$, then

$u(x,t) = X(x)T(t)$ ... leads to $u(x,t) = v(x,t) + \left(\dfrac{T_2 - T_1}{L}\right)x + T_1$ where

$$v(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L\left(f(z) - \frac{T_2 - T_1}{L}z - T_1\right)\sin\left(\frac{n\pi z}{L}\right)dz\right)\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 k t}{L^2}\right)$$

[Space for Additional Notes]