1. <u>Vector Fields and the Gradient Operator</u>

In this chapter, a review of vectors from previous courses is followed by the introduction of lines of force. The gradient operator is extended to divergence, curl and Laplacian in both Cartesian and general orthonormal curvilinear coordinate systems. Conversion of components of vectors between Cartesian and other coordinate systems is also covered.

Contents of this Chapter:

1.1 Review of Vectors

This first section is a brief review of concepts that were introduced during terms 1 and 2.

The **displacement vector** of a particle can be represented in Cartesian components by

$$\vec{\mathbf{r}}(t) = \langle x(t), y(t), z(t) \rangle$$

where t is a **parameter** (time or angle or distance along a curve, etc.)

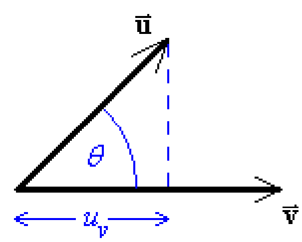
The **distance** of the particle from the origin at any value of t is given by the scalar function

$$r(t) = \left|\vec{\mathbf{r}}(t)\right| = \sqrt{(x(t))^2 + (y(t))^2 + (z(t))^2}$$

Note the various alternative conventions for a vector and its magnitude:

$\mathbf{r} \equiv \vec{\mathbf{r}} \equiv \underline{\mathbf{r}}$ and $r \equiv \|\vec{\mathbf{r}}\| \equiv |\vec{\mathbf{r}}|$

Scalar product (dot product):



$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = uv\cos\theta$$

The component of vector $\vec{\mathbf{u}}$ in the direction of vector $\vec{\mathbf{v}}$ is

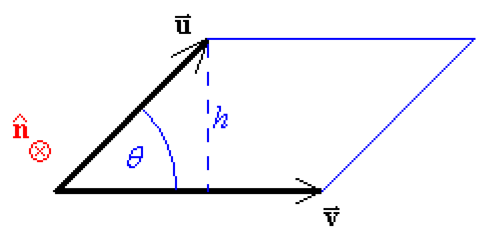
$$u_v = \vec{\mathbf{u}} \bullet \hat{\mathbf{v}} = u\cos\theta$$

$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = 0 \quad \Rightarrow \quad \vec{\mathbf{u}} = \vec{\mathbf{0}}$ or $\vec{\mathbf{v}} = \vec{\mathbf{0}}$ or $\vec{\mathbf{u}} \perp \vec{\mathbf{v}}$ ("$\vec{\mathbf{u}}$ at right angles to $\vec{\mathbf{v}}$")

The scalar product is commutative:

$$\vec{\mathbf{v}} \bullet \vec{\mathbf{u}} = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}} \qquad \forall \vec{\mathbf{u}}, \vec{\mathbf{v}}$$

Vector product (cross product):



The vector product of two vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ is in a direction $\hat{\mathbf{n}}$ at right angles to both $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$.

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = (uv\sin\theta)\hat{\mathbf{n}}$$

The area of the parallelogram is

$$A = \left|\vec{\mathbf{u}} \times \vec{\mathbf{v}}\right| \quad \text{and} \quad h = u\sin\theta = \left|\vec{\mathbf{u}} \times \hat{\mathbf{v}}\right|$$

The orientation of the plane containing vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is defined by the direction of the vector product $\vec{\mathbf{n}} = \vec{\mathbf{u}} \times \vec{\mathbf{v}}$.

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \vec{\mathbf{u}} = \vec{\mathbf{0}} \text{ or } \vec{\mathbf{v}} = \vec{\mathbf{0}} \text{ or } \vec{\mathbf{u}} \parallel \vec{\mathbf{v}} \qquad (\text{"}\vec{\mathbf{u}} \text{ parallel to } \vec{\mathbf{v}}\text{"})$$

The vector product is anti-commutative:

$$\vec{\mathbf{v}} \times \vec{\mathbf{u}} \equiv -\vec{\mathbf{u}} \times \vec{\mathbf{v}} \qquad \Rightarrow \quad \vec{\mathbf{v}} \times \vec{\mathbf{v}} \equiv \vec{\mathbf{0}}$$

The product rule of differentiation is valid for scalar and vector products:

$$\frac{d}{dt}\left(\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}\right) = \frac{d\vec{\mathbf{u}}}{dt} \bullet \vec{\mathbf{v}} + \vec{\mathbf{u}} \bullet \frac{d\vec{\mathbf{v}}}{dt} \quad \text{and}$$

$$\frac{d}{dt}\left(\vec{\mathbf{u}} \times \vec{\mathbf{v}}\right) = \frac{d\vec{\mathbf{u}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \frac{d\vec{\mathbf{v}}}{dt} = -\vec{\mathbf{v}} \times \frac{d\vec{\mathbf{u}}}{dt} + \vec{\mathbf{u}} \times \frac{d\vec{\mathbf{v}}}{dt}$$

Example 1.1.1

For the vectors $\vec{\mathbf{u}} = \langle 4, 3, 2 \rangle$ and $\vec{\mathbf{v}} = \langle 2, -1, -2 \rangle$
find $\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}$, u_v, $\vec{\mathbf{u}} \times \vec{\mathbf{v}}$, θ (= the angle between $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$)
and the equation of the plane parallel to $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ that passes through the origin.

$$u = \left|\vec{\mathbf{u}}\right| = \sqrt{16+9+4} = \sqrt{29}$$

$$v = \left|\vec{\mathbf{v}}\right| = \sqrt{4+1+4} = 3$$

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = 4(2) + 3(-1) + 2(-2) = \underline{\underline{1}}$$

$$u_v = \vec{\mathbf{u}} \bullet \hat{\mathbf{v}} = \frac{\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}}{v} = \underline{\underline{\frac{1}{3}}}$$

$$\vec{\mathbf{u}} \times \vec{\mathbf{v}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & 3 & 2 \\ 2 & -1 & -2 \end{vmatrix} = \underline{\underline{\langle -4, 12, -10 \rangle}} = -2\underbrace{\langle 2, -6, 5 \rangle}_{\vec{\mathbf{n}}}$$

$$\cos\theta = \frac{\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}}{uv} = \frac{1}{3\sqrt{29}} \approx 0.0619 \qquad \Rightarrow \quad \theta \approx \underline{\underline{86.5^\circ}}$$

Example 1.1.1 (continued)

A normal to the plane is $\vec{\mathbf{n}} = \langle 2, -6, 5 \rangle$.

A point on the plane is $\vec{\mathbf{a}} = \langle 0, 0, 0 \rangle \quad \Rightarrow \quad \vec{\mathbf{a}} \bullet \vec{\mathbf{n}} = 0$.

The equation of the plane is $\vec{\mathbf{r}} \bullet \vec{\mathbf{n}} = \vec{\mathbf{a}} \bullet \vec{\mathbf{n}} \quad \Rightarrow \quad \underline{\underline{2x - 6y + 5z = 0}}$

The **arc length** s is the distance travelled along the curve. It is related to the displacement vector $\mathbf{r}$ by

$$\frac{ds}{dt} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right|$$

The distance along a curve between two points whose parameter values are t_0 and t_1 is

$$L = \int_{t_0}^{t_1} ds = \int_{t_0}^{t_1} \frac{ds}{dt}\, dt = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

A **unit vector** $\hat{\mathbf{u}}$ has a magnitude of 1: $|\hat{\mathbf{u}}| = 1$

Any non-zero vector $\mathbf{r}$ can be decomposed into its magnitude r and its direction:

$$\vec{\mathbf{r}} \equiv r\,\hat{\mathbf{r}} \ , \quad \text{where } r \equiv \left|\vec{\mathbf{r}}\right| > 0$$

The **unit tangent** at any point on the curve is

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \left| \frac{d\vec{\mathbf{r}}}{dt} \right| = \frac{d\vec{\mathbf{r}}}{ds}$$

The **unit principal normal** at any point on the curve is

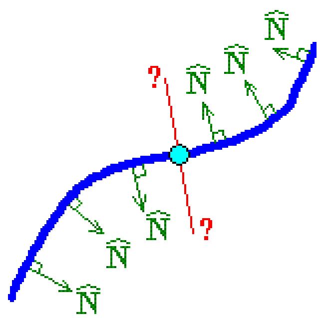
$$\hat{\mathbf{N}} = \rho \frac{d\hat{\mathbf{T}}}{ds} = \frac{d\hat{\mathbf{T}}}{dt} \div \left| \frac{d\hat{\mathbf{T}}}{dt} \right|$$

where ρ = **radius of curvature** (SI units: metre)
κ = **curvature** (SI units: m^{-1})

and $\rho = \dfrac{1}{\kappa}$

Of all circles that touch a curve on the "inside" at a particular point (and which therefore all share the same unit tangent vector [or its negative] there), the one whose radius is ρ is the best fit to that curve at that point. In general, curvature varies along most curves.

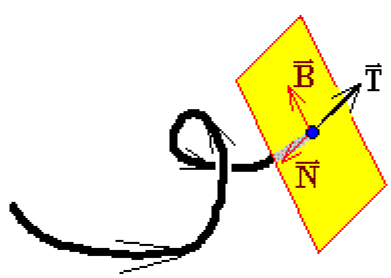
At points of inflexion, $\kappa = 0$



The **unit binormal** is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$$

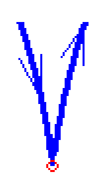
The three vectors form an **orthonormal** set (they are mutually perpendicular and each one is a unit vector; that is, magnitude = 1)



$\hat{\mathbf{B}}$ = constant $\Rightarrow$

the curve lies entirely in one plane (the plane defined by $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$).

Note that $\hat{\mathbf{N}}$ and $\hat{\mathbf{B}}$ are undefined at points of inflexion.
All three unit vectors are undefined where the curve suddenly reverses direction, (such as at a cusp).



Velocity and Acceleration

If the parameter t is the time, then the **velocity** of a particle [a vector function] is

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}}{dt}$$

and its **speed** [a scalar function] is

$$v(t) = \left|\vec{\mathbf{v}}(t)\right| = \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \frac{ds}{dt}$$

and

$$\vec{\mathbf{v}} = v\hat{\mathbf{T}}$$

The **acceleration** [vector] is

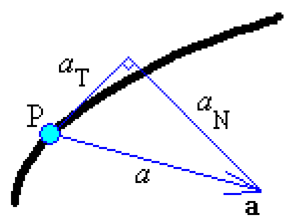
$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right\rangle = a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}}$$

where

$$a_T = \frac{dv}{dt} = \vec{\mathbf{a}}\bullet\hat{\mathbf{T}} = \frac{\vec{\mathbf{v}}\bullet\vec{\mathbf{a}}}{v}$$

and

$$a_N = \kappa v^2 = \sqrt{a^2 - a_T{}^2} = \vec{\mathbf{a}}\bullet\hat{\mathbf{N}} = \frac{\left|\vec{\mathbf{v}}\times\vec{\mathbf{a}}\right|}{v}$$



Example 1.1.2

The equation of a curve in $\mathbb{R}^3$ is given parametrically by

$\vec{\mathbf{r}} = e^t \sin t\,\hat{\mathbf{i}} - \hat{\mathbf{j}} + e^t \cos t\,\hat{\mathbf{k}}$

Find $\vec{\mathbf{v}}, \vec{\mathbf{a}}, v, a_T, a_N, \kappa, \hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}$.

Show that the curve lies entirely in one plane and find the equation of that plane.

Let $c = \cos t$ and $s = \sin t$ then $\vec{\mathbf{r}} = \left\langle e^t s,\ -1,\ e^t c \right\rangle$

$$\Rightarrow \quad \vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \underline{\underline{\left\langle e^t(s+c),\ 0,\ e^t(c-s) \right\rangle}}$$

$$\Rightarrow \quad v = e^t \sqrt{(s+c)^2 + 0^2 + (c-s)^2} = e^t \sqrt{s^2 + 2sc + c^2 \ + \ c^2 - 2sc + s^2}$$

$$\therefore \quad v = \underline{\underline{e^t \sqrt{2}}}$$

$$\hat{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{v} = \underline{\underline{\frac{\sqrt{2}}{2}\langle s+c, 0, c-s \rangle}}$$

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = e^t \left\langle (s+c+c-s), 0, (c-s-s-c) \right\rangle = \underline{\underline{2e^t \langle c, 0, -s \rangle}}$$

$$a_T = \frac{dv}{dt} = \frac{d}{dt}\left(e^t\sqrt{2}\right) = \underline{\underline{e^t\sqrt{2}}}$$

OR

$$a_T = \frac{\vec{\mathbf{v}} \bullet \vec{\mathbf{a}}}{v} = \vec{\mathbf{a}} \bullet \hat{\mathbf{T}} = 2e^t \langle c, 0, -s \rangle \bullet \frac{\sqrt{2}}{2} \langle s+c, 0, c-s \rangle$$

$$= e^t \sqrt{2}\left(\left(sc + c^2\right) + 0 + \left(-cs + s^2\right)\right) = \underline{\underline{e^t\sqrt{2}}}$$

Example 1.1.2 (continued)

$$a_N = \sqrt{a^2 - a_T{}^2} = e^t \sqrt{4\left(c^2 + (-s)^2\right) - 2} = e^t \sqrt{4-2} = \underline{\underline{e^t \sqrt{2}}}$$

OR

$$a_N = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{v} = \frac{1}{v} \left| \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ e^t(s+c) & 0 & e^t(c-s) \\ 2e^t c & 0 & -2e^t s \end{bmatrix} \right|$$

$$= \frac{1}{v} \left| \left\langle 0,\ 2e^{2t}\left(c^2 - cs + s^2 + sc\right),\ 0 \right\rangle \right| = \frac{2e^{2t}}{e^t \sqrt{2}} = \underline{\underline{e^t \sqrt{2}}}$$

$$\kappa = \frac{a_N}{v^2} = \frac{e^t \sqrt{2}}{\left(e^t \sqrt{2}\right)^2} = \underline{\underline{\frac{\sqrt{2}}{2} e^{-t}}} \qquad \Rightarrow \quad \rho = \frac{1}{\kappa} = e^t \sqrt{2}$$

OR evaluate κ from

$$\kappa = \frac{a_N}{v^2} = \frac{|\vec{\mathbf{v}} \times \vec{\mathbf{a}}|}{v^3}$$

$$\frac{d\hat{\mathbf{T}}}{dt} = \frac{\sqrt{2}}{2} \langle c-s,\ 0,\ -s-c \rangle$$

$$\Rightarrow \quad \left| \frac{d\hat{\mathbf{T}}}{dt} \right| = \frac{\sqrt{2}}{2} \sqrt{c^2 - 2cs + s^2 + s^2 + 2sc + c^2} = \frac{\sqrt{2}\sqrt{2}}{2} = 1$$

$$\Rightarrow \quad \hat{\mathbf{N}} = \frac{d\hat{\mathbf{T}}}{dt} \div \left| \frac{d\hat{\mathbf{T}}}{dt} \right| = \underline{\underline{\frac{\sqrt{2}}{2} \langle c-s,\ 0,\ -s-c \rangle}}$$

OR

$$\vec{\mathbf{a}} = a_T \hat{\mathbf{T}} + a_N \hat{\mathbf{N}} \qquad \Rightarrow \quad \hat{\mathbf{N}} = \frac{1}{a_N}\left(\vec{\mathbf{a}} - a_T \hat{\mathbf{T}}\right)$$

$$= \frac{1}{e^t \sqrt{2}} \left(2e^t \langle c,\ 0,\ -s \rangle \ - \ \left(e^t \sqrt{2}\right) \frac{\sqrt{2}}{2} \langle s+c,\ 0,\ c-s \rangle \right) = \frac{1}{\sqrt{2}} \langle 2c-s-c,\ 0,\ -2s-c+s \rangle$$

$$\Rightarrow \quad \hat{\mathbf{N}} = \underline{\underline{\frac{\sqrt{2}}{2} \langle c-s,\ 0,\ -s-c \rangle}}$$

Example 1.1.2 (continued)

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}} = \left(\frac{\sqrt{2}}{2}\right)^2 \det\begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ s+c & 0 & c-s \\ c-s & 0 & -(s+c) \end{bmatrix}$$

$$= \frac{1}{2}\left\langle 0, \left(c^2 - 2cs + s^2 + s^2 + 2sc + c^2\right), 0 \right\rangle = \frac{1}{2}\langle 0, 2, 0 \rangle$$

$$\therefore \quad \hat{\mathbf{B}} = \underline{\underline{\hat{\mathbf{j}}}}$$

$$\Rightarrow \quad \frac{d\hat{\mathbf{B}}}{ds} = \vec{\mathbf{0}} \qquad \Rightarrow \quad \tau = 0$$

where τ is the torsion (the measure of the rate at which the curve is twisting out of the plane defined by $\hat{\mathbf{T}}$ and $\hat{\mathbf{N}}$).

$\tau = 0 \;\Rightarrow\;$ the curve lies in one plane, with plane normal $\vec{\mathbf{n}} = \hat{\mathbf{B}} = \hat{\mathbf{j}}$.

A point on the curve can be found by setting the parameter $t = 0$:

$$\vec{\mathbf{r}}(0) = \langle 0, -1, 1 \rangle = \text{"}\vec{\mathbf{a}}\text{"}$$

The equation of the plane containing the curve is

$$\vec{\mathbf{n}} \bullet \vec{\mathbf{r}} = \vec{\mathbf{n}} \bullet \vec{\mathbf{a}} \qquad \Rightarrow \quad 0x + 1y + 0z = 0(0) + 1(-1) + 0(1)$$

$$\Rightarrow \quad \underline{\underline{y = -1}}$$

Frenet-Serret formulæ:

1. $$\frac{d\hat{\mathbf{T}}}{ds} = \kappa \hat{\mathbf{N}}$$
2. $$\frac{d\hat{\mathbf{N}}}{ds} = \tau \hat{\mathbf{B}} - \kappa \hat{\mathbf{T}}$$
3. $$\frac{d\hat{\mathbf{B}}}{ds} = -\tau \hat{\mathbf{N}}$$

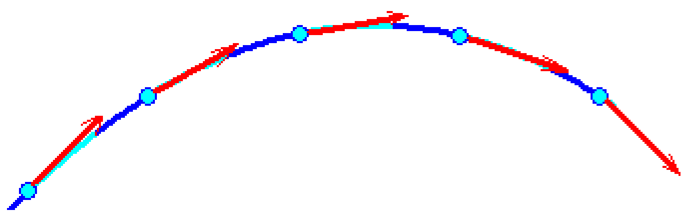
The proofs are in Problem Set 1.

1.2 Lines of Force

A vector function of n variables in $\mathbb{R}^n$ is a **vector field**.

$\mathbf{F}(x, y, z) = \langle f_1(x, y, z),\ f_2(x, y, z),\ f_3(x, y, z) \rangle$
(and the f_i form scalar fields).
The domain must be defined. If not explicitly mentioned, the domain is assumed to be all of that part of $\mathbb{R}^n$ for which all of the scalar fields f_i are defined.

A vector field defines a vector $\mathbf{F}$ at each point in the domain.
If these vectors are tangents to a family of curves, then those curves are **streamlines** or **flow lines** or **lines of force**.



Let $\mathbf{F}(x, y, z)$ be a vector field to a family of lines of force $\mathbf{r}(x, y, z)$. Then

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds} \parallel \vec{\mathbf{F}} \qquad \Rightarrow \quad \frac{d\vec{\mathbf{r}}}{ds} = k\vec{\mathbf{F}}(s)$$

where k is some scalar.

$$\Rightarrow \quad \langle x'(s), y'(s), z'(s) \rangle = k \langle f_1(s), f_2(s), f_3(s) \rangle$$

$$\Rightarrow \quad \frac{dx}{ds} = k\, f_1\,, \quad \frac{dy}{ds} = k\, f_2\,, \quad \frac{dz}{ds} = k\, f_3$$

$$\Rightarrow \quad (k\, ds = \ ) \quad \boxed{\frac{dx}{f_1} = \frac{dy}{f_2} = \frac{dz}{f_3}}$$

(provided f_1, f_2, f_3 are all non-zero).

$$\left[\ f_1 \equiv 0 \qquad \Rightarrow \quad \frac{dx}{ds} \equiv 0 \qquad \Rightarrow \quad x \equiv \text{constant}\,, \ \text{etc.} \right]$$

Example 1.2.1

Find the lines of force associated with the vector field $\mathbf{F} = \left\langle e^z, 0, -x^2 \right\rangle$ and find the line of force that passes through the point (4, 2, 0).

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds} \parallel \vec{\mathbf{F}} \quad \Rightarrow \quad \frac{d\vec{\mathbf{r}}}{ds} = k\vec{\mathbf{F}}(s)$$

$$\frac{dx}{ds} = k\,e^z \,, \quad \frac{dy}{ds} = 0 \,, \quad \frac{dz}{ds} = -k\,x^2 \quad \Rightarrow \quad y = A \quad \text{and}$$

$$k\,ds = \frac{dx}{e^z} = \frac{dz}{-x^2} \quad \Rightarrow \quad -x^2 dx = e^z dz$$

$$\Rightarrow \quad -\int x^2 dx = \int e^z dz \quad \Rightarrow \quad -\frac{x^3}{3} + B = e^z$$

$$\Rightarrow \quad z = \ln\left(B - \tfrac{1}{3}x^3\right)$$

The lines of force are therefore

$$\vec{\mathbf{r}}(x,y,z) = \underline{\underline{\left\langle x, A, \ln\left(B - \tfrac{1}{3}x^3\right)\right\rangle}}$$

We want the line of force through (4, 2, 0)

$$\Rightarrow \quad \left\langle 4, A, \ln\left(B - \tfrac{64}{3}\right)\right\rangle = \langle 4, 2, 0\rangle \quad \Rightarrow \quad A = 2 \;\text{ and }\; B - \tfrac{64}{3} = e^0 = 1$$

$$\Rightarrow \quad B = 1 + \tfrac{64}{3} = \tfrac{67}{3}$$

The required line of force is

$$\vec{\mathbf{r}}(x,y,z) = \underline{\underline{\left\langle x, 2, \ln\left(\frac{67 - x^3}{3}\right)\right\rangle}}$$

$\left(\text{for} \quad x < \sqrt[3]{67} \;\text{ only}\right)$

Example 1.2.2

Find the lines of force associated with the vector field $\mathbf{F} = \langle x^2, 2y, -1 \rangle$
and find the line of force that passes through the point (–1, 6, 2).

$$\frac{d\vec{\mathbf{r}}}{ds} = k\vec{\mathbf{F}} \quad \Rightarrow \quad \left\langle \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right\rangle = k\langle x^2, 2y, -1 \rangle$$

$$\Rightarrow \quad k\,ds = \frac{dx}{x^2} = \frac{dy}{2y} = \frac{dz}{-1} \qquad (x \not\equiv 0 \text{ and } y \not\equiv 0)$$

$$\Rightarrow \quad dz = \frac{-1}{x^2}dx = \frac{-1}{2y}dy \qquad \Rightarrow \quad \int dz = \int \frac{-1}{x^2}dx = \int \frac{-1}{2y}dy$$

$$\Rightarrow \quad z = \frac{1}{x} + c_1 \quad \text{and} \quad z = -\tfrac{1}{2}\ln y + c_2$$

$$\Rightarrow \quad x = \frac{1}{z - c_1} \quad \text{and} \quad \ln y = -2z - 2c_2$$

$$\Rightarrow \quad \vec{\mathbf{r}} = \underline{\underline{\left\langle \frac{1}{z - c_1}, c_3 e^{-2z}, z \right\rangle}}$$

But the required line of force passes through (–1, 6, 2):

$$\Rightarrow \quad \left\langle \frac{1}{2 - c_1}, c_3 e^{-4}, 2 \right\rangle = \langle -1, 6, 2 \rangle \qquad \Rightarrow \quad c_1 = 2 + 1 = 3\,,\ c_3 = 6e^4$$

$$\Rightarrow \quad \vec{\mathbf{r}} = \underline{\underline{\left\langle \frac{1}{z - 3},\ 6e^{4-2z},\ z \right\rangle}} \qquad (z \neq 3)$$

<u>Example 1.2.3</u>

Find the streamlines associated with the velocity field $\vec{\mathbf{v}} = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2}, 0 \right\rangle$

and find the streamline through the point (1, 0, 0).

$$\frac{d\vec{\mathbf{r}}}{ds} = k\,\vec{\mathbf{v}} \quad \Rightarrow \quad \left\langle \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right\rangle = k\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle$$

$$\Rightarrow \quad \frac{dz}{ds} = 0 \quad \text{and} \quad k\,ds = \frac{x^2+y^2}{x}\,dy = \frac{x^2+y^2}{-y}\,dx$$

$$\therefore \quad z = B\,.$$

Provided $(x, y) \neq (0,0)$, $\quad -y\,dy = x\,dx \quad \Rightarrow \quad -\int y\,dy = \int x\,dx$

$$\Rightarrow \quad -\frac{y^2}{2} = \frac{x^2}{2} + C \quad \Rightarrow \quad x^2 + y^2 = A^2$$

The streamlines are therefore a family of circles, parallel to the x-y plane, centre the z-axis, lying on a concentric set of circular cylinders:

$$x^2 + y^2 = A^2\,, \quad z = B \qquad \text{or} \quad \vec{\mathbf{r}} = \underline{\underline{\left\langle x, \pm\sqrt{A^2 - x^2}, B \right\rangle}}$$

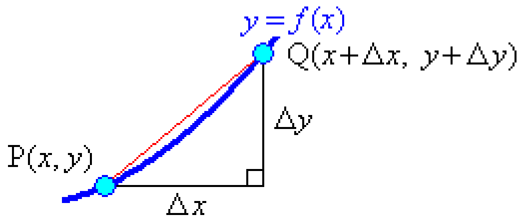
$(x, y, z) = (1,0,0) \quad \Rightarrow \quad 1^2 + 0^2 = A^2\,, \quad 0 = B$

The required streamline is therefore the unit circle

$$\underline{\underline{x^2 + y^2 = 1\,, \quad z = 0}}$$

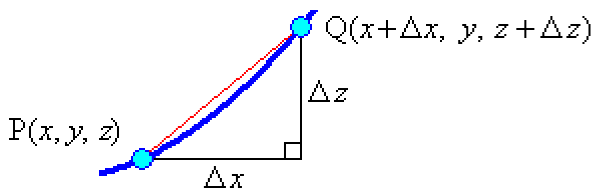
1.3 The Gradient Vector

If a curve in $\mathbb{R}^2$ is represented by $y = f(x)$, then



$$\frac{dy}{dx} = \lim_{Q \to P} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

If a surface in $\mathbb{R}^3$ is represented by $z = f(x, y)$, then in a slice y = constant,

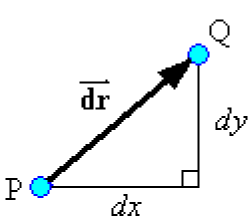


$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y, z + \Delta z) - f(x, y, z)}{\Delta x}$$

Similarly,

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y, z + \Delta z) - f(x, y, z)}{\Delta y}$$

In the plane of the independent variables:



$$f(P) = f$$
$$f(Q) = f + df$$
$$\mathbf{dr} = \langle dx, dy, 0 \rangle$$

Chain rule:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

$$\Rightarrow \quad df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\Rightarrow \quad df = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \bullet \langle dx, dy \rangle = \vec{\nabla} f \bullet d\vec{\mathbf{r}}$$

where ∇f (pronounced as "del f") is the **gradient vector**.

At any point (x, y) in the domain, the value of the function $f(x, y)$ changes at different rates when one moves in different directions on the xy-plane.

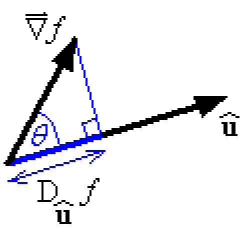
∇f is a vector in the plane of the independent variables (the xy-plane).
The magnitude of ∇f at a point (x, y) is the maximum instantaneous rate of increase of f at that point. The direction of ∇f at that point is the direction in which one would have to start moving on the xy-plane in order to experience that maximum rate of increase, (which is also at right angles to the contour $f(x, y) = \text{constant}$ at that point).

Points where $\vec{\nabla} f = \vec{\mathbf{0}}$ are critical points of f, (maximum, minimum or saddle point).

The **directional derivative** of f in the direction of the unit vector $\hat{\mathbf{u}}$ is

$$\boxed{D_{\hat{\mathbf{u}}} f = \vec{\nabla} f \bullet \hat{\mathbf{u}}}$$

Both vectors are in the plane of the independent variables.
The directional derivative is the component of ∇f in the direction of $\hat{\mathbf{u}}$.



$$D_{\hat{\mathbf{u}}} f = \left|\vec{\nabla} f\right| \left|\hat{\mathbf{u}}\right| \cos\theta$$

$$\Rightarrow \quad \max\left(D_{\hat{\mathbf{u}}} f\right) = \left|\vec{\nabla} f\right| \qquad \text{when} \quad \hat{\mathbf{u}} \parallel \vec{\nabla} f$$

$$\text{and} \quad \min\left|D_{\hat{\mathbf{u}}} f\right| = 0 \qquad \text{when} \quad \hat{\mathbf{u}} \perp \vec{\nabla} f$$

(when $\hat{\mathbf{u}}$ is tangential to the contours of f).

The results above can be extended to functions of more than two variables.
For the hypersurface $z = f(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^{n+1}$, the chain rule becomes

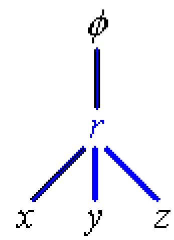
$$\frac{df}{dt} = \vec{\nabla} f \bullet \frac{d\vec{\mathbf{r}}}{dt}, \quad \text{where} \quad \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right\rangle \quad \text{and} \quad \frac{d\vec{\mathbf{r}}}{dt} = \left\langle \frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_n}{dt} \right\rangle$$

<u>Example 1.3.1</u>

The electrostatic potential ϕ at a point $P(x, y, z)$ in $\mathbb{R}^3$ due to a point charge Q at the origin is

$$\phi = \frac{1}{4\pi\varepsilon} \cdot \frac{Q}{r}, \quad \text{where} \quad r = \sqrt{x^2 + y^2 + z^2} \,.$$

Find the rate of change of ϕ at the point (1, 2, 2) in the direction $\mathbf{k} - 2\mathbf{i}$.
Find the maximum value of the directional derivative over all directions at any point.
Find the level surfaces.



$$\phi = \frac{1}{4\pi\varepsilon} \cdot \frac{Q}{r} \quad \Rightarrow \quad \frac{\partial\phi}{\partial x} = \frac{Q}{4\pi\varepsilon} \cdot \frac{d}{dr}\left(\frac{1}{r}\right) \cdot \frac{\partial r}{\partial x}$$

But $\dfrac{\partial r}{\partial x} = \dfrac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)^{1/2}$

$$= \frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2} \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2\right)$$

$$= \frac{1}{2} r^{-1}(2x + 0 + 0) = \frac{x}{r}$$

$$\Rightarrow \quad \frac{\partial\phi}{\partial x} = \frac{-Q}{4\pi\varepsilon r^2} \cdot \frac{x}{r} = \frac{-Qx}{4\pi\varepsilon r^3}$$

This is the x component of the gradient vector.

Obtain $\dfrac{\partial\phi}{\partial y}, \dfrac{\partial\phi}{\partial z}$ by symmetry.

$$\Rightarrow \quad \vec{\nabla}\phi = \frac{-Q}{4\pi\varepsilon r^3}\langle x, y, z\rangle = \frac{-Q}{4\pi\varepsilon r^3}\vec{\mathbf{r}}$$

At (1, 2, 2) $r = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$.

$$\Rightarrow \quad \vec{\nabla}\phi\Big|_P = \frac{-Q}{4\pi\varepsilon \times 27}\langle 1, 2, 2\rangle$$

$$\vec{\mathbf{u}} = \hat{\mathbf{k}} - 2\hat{\mathbf{i}} \quad \Rightarrow \quad \hat{\mathbf{u}} = \frac{1}{\sqrt{5}}\langle -2, 0, 1\rangle$$

$$\Rightarrow \quad \mathrm{D}_{\hat{\mathbf{u}}}\phi\Big|_P = \frac{-Q}{4\pi\varepsilon \times 27}\langle 1, 2, 2\rangle \bullet \frac{1}{\sqrt{5}}\langle -2, 0, 1\rangle = \frac{-Q}{4 \times 27\pi\varepsilon\sqrt{5}}\big(1(-2) + 2(0) + 2(1)\big)$$

$$\Rightarrow \quad \mathrm{D}_{\hat{\mathbf{u}}}\phi\Big|_P = \underline{\underline{0}}$$

Example 1.3.1 (continued)

$$\max\left(\mathrm{D}_{\hat{\mathbf{u}}}\phi\right) = \left|\vec{\nabla}\phi\right| = \frac{+|Q|}{4\pi\varepsilon r^3} r = \underline{\underline{\frac{|Q|}{4\pi\varepsilon r^2}}}$$

This occurs when

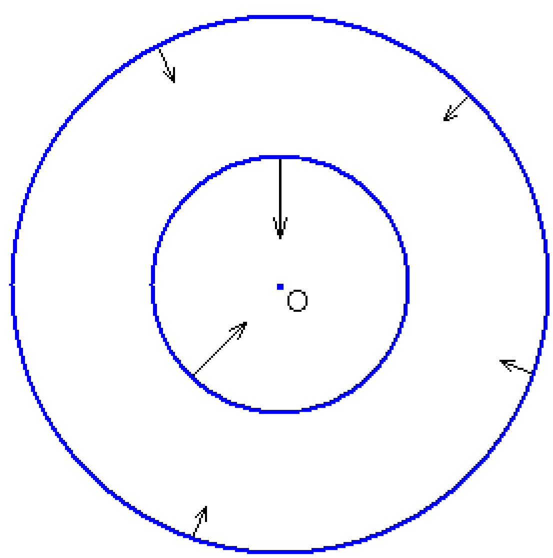
$$\hat{\mathbf{u}} \parallel \begin{cases} -\vec{\mathbf{r}} & (Q > 0) \\ +\vec{\mathbf{r}} & (Q < 0) \end{cases}$$

Level surfaces (contours):

$$\phi = c \quad \Rightarrow \quad \frac{Q}{4\pi\varepsilon r} = c \quad \Rightarrow \quad r = \frac{Q}{4\pi\varepsilon c} = A$$

But $r = A$ is a sphere, centre O.

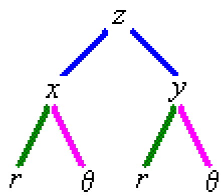
Therefore the level surfaces are a **family of concentric spheres**.



[$Q > 0$ case]

Change of coordinates:

Suppose $z = f(x, y)$ (where (x, y) are Cartesian coordinates) and $\dfrac{\partial z}{\partial r}$ is wanted, (where (r, θ) are plane polar coordinates). Then



$$\Rightarrow \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

But $\quad x = r\cos\theta, \; y = r\sin\theta$

$$\Rightarrow \quad \frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}(r\cos\theta)$ can be found in a similar way.

In matrix form, the chain rule can be expressed concisely as

$$\begin{bmatrix} \dfrac{\partial z}{\partial r} \\ \dfrac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{\partial z}{\partial x} \\ \dfrac{\partial z}{\partial y} \end{bmatrix} .$$

Note that

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \text{abs}\left(\det \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} \right) \text{ is the } \textbf{Jacobian}$$

For the transformation from Cartesian to plane polar coordinates in $\mathbb{R}^2$, the Jacobian is

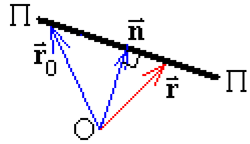
$$\frac{\partial(x, y)}{\partial(r, \theta)} = \left\| \begin{matrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{matrix} \right\| = \left| r\cos^2\theta + r\sin^2\theta \right| = r$$

Integrals over areas can therefore be transformed using the Jacobian:

$$\iint_A f(x, y)\, dx\, dy = \iint_A f(x, y) \frac{\partial(x, y)}{\partial(r, \theta)} dr\, d\theta = \iint_A g(r, \theta)\, r\, dr\, d\theta$$

where $f(x, y) = g(r, \theta)$ at all points in the area A of integration.
We shall return to this topic later.

Planes



Let $\mathbf{n}$ = a normal vector to a plane = $\langle A, B, C \rangle$
and $\mathbf{r}_o$ = the position vector of a point on the plane = $\langle x_o, y_o, z_o \rangle$
where A, B, C, x_o, y_o, z_o are all constants.
Let $\mathbf{r}$ = the position vector of a general point in $\mathbb{R}^3$ = $\langle x, y, z \rangle$
Then the point (x, y, z) is on the plane if and only if

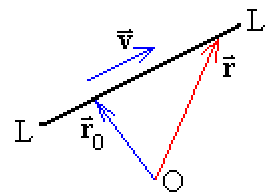
$$\boxed{\vec{\mathbf{n}} \bullet \vec{\mathbf{r}} = \vec{\mathbf{n}} \bullet \vec{\mathbf{r}}_o}$$

which is a vector equation of the plane.
Evaluating the scalar [dot] products generates the Cartesian equation of the plane:

$$\boxed{Ax + By + Cz = D}$$

where $D = Ax_o + By_o + Cz_o$.

Lines



Let $\mathbf{v}$ = a vector parallel to the line = $\langle v_1, v_2, v_3 \rangle$
and $\mathbf{r}_o$ = the position vector of a point on the line = $\langle x_o, y_o, z_o \rangle$
then the vector parametric form of the equation of the line is

$$\boxed{\vec{\mathbf{r}} = \vec{\mathbf{r}}_o + t\vec{\mathbf{v}}}$$

and the Cartesian equivalent, (in the case where none of v_1, v_2, v_3 is zero) is

$$\boxed{\frac{x - x_o}{v_1} = \frac{y - y_o}{v_2} = \frac{z - z_o}{v_3}}$$

Surfaces

The general Cartesian equation of a surface in $\mathbb{R}^3$ is of the form

$$f(x, y, z) = c$$

At every point on the surface where ∇f exists as a non-zero vector, ∇f is orthogonal (perpendicular) to the level surface of the function f that passes through that point.
Therefore, at every point on the surface $f(x, y, z) = c$,
the gradient vector ∇f is normal to the tangent plane.

The tangent plane at the point $P(x_o, y_o, z_o)$ to the surface $f(x, y, z) = c$ has the equation

$$\boxed{\vec{\mathbf{n}} \bullet \vec{\mathbf{r}} = \vec{\mathbf{n}} \bullet \vec{\mathbf{r}}_o, \quad \text{where} \quad \vec{\mathbf{n}} = \vec{\nabla} f \Big|_P}$$

Example 1.3.2

Find the Cartesian equations of the tangent plane and normal line to the surface $z = x^2 + y$ at the point $(-1, 1, 2)$.

The implicit form of the equation of the surface is $f = x^2 + y - z = 0$

$$\vec{\nabla} f = \langle 2x, 1, -1 \rangle$$

$$\Rightarrow \quad \vec{\nabla} f \big|_P = \langle -2, 1, -1 \rangle = \vec{\mathbf{n}}$$

$$\vec{\mathbf{n}} \cdot \vec{\mathbf{r}}_o = -2(-1) + 1(1) - 1(2) = 1$$

Therefore the equation of the tangent plane is $-2x + y - z = 1$ or

$$\underline{\underline{2x - y + z + 1 = 0}}$$

The equation of the normal line follows immediately:

In vector parametric form: $\vec{\mathbf{r}} = \langle -1, 1, 2 \rangle + t\langle -2, 1, -1 \rangle$

or, in Cartesian parametric form, $x = -1 - 2t\,, \quad y = 1 + t\,, \quad z = 2 - t$

or, in Cartesian symmetric form, $\dfrac{x+1}{-2} = \dfrac{y-1}{1} = \dfrac{z-2}{-1}$

or

$$\underline{\underline{\frac{x+1}{2} = \frac{y-1}{-1} = \frac{z-2}{1}}}$$

<u>Example 1.3.3</u>

Find the angle between the surfaces $x^2 + y^2 + z^2 = 4$ and $z^2 + x^2 = 2$ at the point $(1, \sqrt{2}, 1)$.

First note that these surfaces are a sphere, radius 2, centre the origin and a circular cylinder of radius $\sqrt{2}$ aligned along the y axis. Their intersection will therefore be a pair of parallel circles, equally spaced on either side of the x-z plane.

The point $P\ (1, \sqrt{2}, 1)$ is clearly on both surfaces [satisfies the equations of both surfaces].

$$f = x^2 + y^2 + z^2 = 4 \quad \Rightarrow \quad \vec{\nabla} f = \langle 2x, 2y, 2z \rangle = 2\langle x, y, z \rangle = 2\vec{\mathbf{n}}_1$$

At P, $\vec{\mathbf{n}}_1 = \langle 1, \sqrt{2}, 1 \rangle$

$$g = x^2 + z^2 = 2 \quad \Rightarrow \quad \vec{\nabla} g = \langle 2x, 0, 2z \rangle = 2\langle x, 0, z \rangle = 2\vec{\mathbf{n}}_2$$

At P, $\vec{\mathbf{n}}_2 = \langle 1, 0, 1 \rangle$

The angle θ between the surfaces equals the angle between the two normal vectors.

$$\vec{\mathbf{n}}_1 \bullet \vec{\mathbf{n}}_2 = 1(1) + \sqrt{2}(0) + 1(1) = 2$$
$$n_1 = \sqrt{1+2+1} = 2$$
$$n_2 = \sqrt{1+0+1} = \sqrt{2}$$
$$\Rightarrow \quad \cos\theta = \frac{\vec{\mathbf{n}}_1 \bullet \vec{\mathbf{n}}_2}{n_1 n_2} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \theta = \underline{\underline{\frac{\pi}{4}}} \quad (45°)$$

Note: In the event that $\vec{\mathbf{n}}_1 \bullet \vec{\mathbf{n}}_2 < 0$, then the two normal vectors meet at an obtuse angle (they are pointing in approximately opposite directions). In that case use $\left| \vec{\mathbf{n}}_1 \bullet \vec{\mathbf{n}}_2 \right|$.

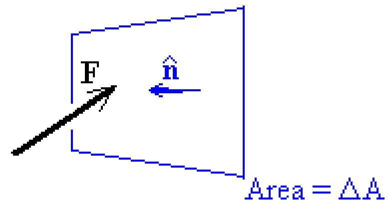
Gradient Operator

For three independent variables (x, y, z), the gradient operator is the "vector"

$$\vec{\nabla} = \hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

It operates on anything immediately to its right; either a scalar function or scalar field, or, via a dot product or cross product, on a vector function or vector field.

1.4 Divergence and Curl



For an elementary area ΔA in a vector field $\mathbf{F}$,

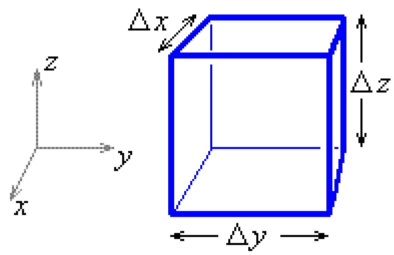
$\hat{\mathbf{n}}$ is an outward unit normal to the surface.

ΔA is sufficiently small that $\mathbf{F} = \langle f_1, f_2, f_3 \rangle$ is approximately constant over ΔA.

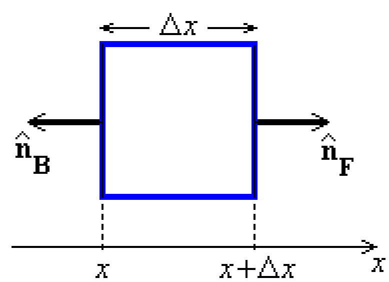
The element of flux $\Delta\phi$ from the vector field through ΔA is approximately

$$\Delta\phi \approx \vec{\mathbf{F}} \bullet \hat{\mathbf{n}}\, \Delta A$$

Now add up the elements of flux passing through the six faces of an elementary cuboid of sides $\Delta x, \Delta y, \Delta z$, volume $\Delta V = \Delta x\, \Delta y\, \Delta z$ and with one corner at (x, y, z).



The front face is at $(x + \Delta x)$ and the back face is at (x).



Back face: $\hat{\mathbf{n}}_{\mathbf{B}} = -\hat{\mathbf{i}}$

$$\Delta A = \Delta y \cdot \Delta z$$

$$\Rightarrow \quad \vec{\mathbf{F}} \bullet \hat{\mathbf{n}}_{\mathbf{B}} = \vec{\mathbf{F}} \bullet \left(-\hat{\mathbf{i}}\right) = -f_1(x, y, z)$$

$$\Rightarrow \quad \Delta\phi_B = -f_1(x, y, z) \cdot \Delta y \cdot \Delta z$$

Front face: $\hat{\mathbf{n}}_{\mathbf{F}} = +\hat{\mathbf{i}}$

$$\Delta A = \Delta y \cdot \Delta z$$

$$\Rightarrow \quad \vec{\mathbf{F}} \bullet \hat{\mathbf{n}}_{\mathbf{F}} = \vec{\mathbf{F}} \bullet \hat{\mathbf{i}} = f_1(x + \Delta x, y, z) \quad \text{[front face is at } x + \Delta x\text{]}$$

$$\Rightarrow \quad \Delta\phi_F = +f_1(x + \Delta x, y, z) \cdot \Delta y \cdot \Delta z$$

Divergence (continued)

$$\Rightarrow \quad \frac{\Delta\phi_F + \Delta\phi_B}{\Delta V} = \frac{f_1(x+\Delta x, y, z) - f_1(x, y, z)}{\Delta x}$$

and $$\lim_{\Delta V \to 0}\left(\frac{\Delta\phi_F + \Delta\phi_B}{\Delta V}\right) = \frac{\partial f_1}{\partial x}$$

Similarly, for the left and right faces,

$$\lim_{\Delta V \to 0}\left(\frac{\Delta\phi_L + \Delta\phi_R}{\Delta V}\right) = \frac{\partial f_2}{\partial y}$$

and for the top and bottom faces,

$$\lim_{\Delta V \to 0}\left(\frac{\Delta\phi_{TOP} + \Delta\phi_{BOT}}{\Delta V}\right) = \frac{\partial f_3}{\partial z}$$

Summing over all six faces of the cuboid, the net rate of flux per unit volume out of a point (x, y, z) is

$$\lim_{\Delta V \to 0}\left(\frac{\Delta\phi}{\Delta V}\right) = \frac{d\phi}{dV} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \vec{\nabla}\bullet\langle f_1, f_2, f_3\rangle = \vec{\nabla}\bullet\vec{\mathbf{F}}$$

which is the **divergence of F**.

No net flux (no sources) $\Rightarrow$ $\operatorname{div}\vec{\mathbf{F}} = 0$

Example 1.4.1

Find the divergence of the vector **F**, given that $\mathbf{F} = -\nabla\phi$, $\phi = \dfrac{Q}{4\pi\varepsilon r}$,
$r = \sqrt{x^2+y^2+z^2}$.

From example 1.3.1,

$$\vec{\nabla}\phi = \frac{-Q}{4\pi\varepsilon\, r^3}\vec{\mathbf{r}} \quad \Rightarrow \quad \vec{\mathbf{F}} = -\vec{\nabla}\phi = \frac{+Q}{4\pi\varepsilon\, r^3}\vec{\mathbf{r}}$$

$$\Rightarrow \quad \operatorname{div}\vec{\mathbf{F}} = \frac{Q}{4\pi\varepsilon}\left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \bullet \left\langle \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right\rangle$$

Also from example 1.3.1, $\dfrac{\partial r}{\partial x} = \dfrac{x}{r}$, $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$, $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1r^3 - x\left(3r^2 \cdot \dfrac{x}{r}\right)}{r^6} = \frac{r\left(r^2 - 3x^2\right)}{r^6} = \frac{r^2 - 3x^2}{r^5}$$

The other two derivatives follow by symmetry.

$$\Rightarrow \quad \operatorname{div}\vec{\mathbf{F}} = \frac{Q}{4\pi\varepsilon}\left(\frac{r^2-3x^2 + r^2-3y^2 + r^2-3z^2}{r^5}\right) = \frac{Q\left(3r^2-3r^2\right)}{4\pi\varepsilon r^5} = 0$$

(unless $r = 0$, where $\operatorname{div}\mathbf{F}$ is undefined).

Therefore the ideal electrostatic field is source free except at the point charge itself.
The flux passing in through any volume not enclosing the charge is balanced by the flux passing out.

Streamlines for Fluid Flow

Let $\mathbf{v}(\mathbf{r})$ be the velocity at any point (x, y, z) in an incompressible fluid. Because the fluid is incompressible, the flow in to any region must be matched by the flow out from that region (except when the region includes a source or a sink). This generates the continuity equation

$$\operatorname{div} \mathbf{v} = 0$$

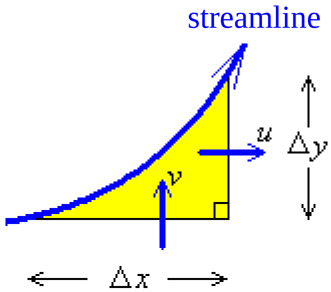
Let us take the case of fluid flow parallel to the x-y plane everywhere, so that we can ignore the third dimension and consider the flow in two dimensions only. Then

$$\vec{\mathbf{v}} = \vec{\mathbf{v}}(x, y) = u(x, y)\hat{\mathbf{i}} + v(x, y)\hat{\mathbf{j}}$$

The continuity equation then becomes

$$\operatorname{div} \vec{\mathbf{v}} = \vec{\nabla} \bullet \vec{\mathbf{v}} = \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0}$$

The fluid flows along streamlines, but never across any streamline.
Consider the flow through a small region bounded partly by a streamline:



$$\Rightarrow \quad v\,\Delta x = u\,\Delta y \qquad \Rightarrow \quad v\,dx - u\,dy = 0$$

This ODE is exact if and only if

$$\frac{\partial}{\partial y}(v) = \frac{\partial}{\partial x}(-u) \qquad \Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

which is automatically true (equation of continuity).

The solution to the ODE is then the **stream function** $\psi(x, y)$, where

$$\boxed{\frac{\partial \psi}{\partial x} = v \quad \text{and} \quad \frac{\partial \psi}{\partial y} = -u}$$

<u>Example 1.4.2</u> (Example 1.2.3 repeated)

Find the streamlines associated with the velocity field $\vec{\mathbf{v}} = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$

and find the streamline through the point (1, 0).

$$u = \frac{-y}{x^2+y^2}, \quad v = \frac{x}{x^2+y^2}$$

Verify that the equation of continuity is satisfied:

$$\vec{\nabla} \bullet \vec{\mathbf{v}} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)$$
$$= \frac{-y(-2x)}{\left(x^2+y^2\right)^2} + \frac{x(-2y)}{\left(x^2+y^2\right)^2} = 0 \qquad \forall (x,y) \neq (0,0)$$

Therefore the stream function $\psi(x,y)$ exists, such that

$$\frac{\partial \psi}{\partial x} = v = \frac{x}{x^2+y^2} \quad \text{and} \quad \frac{\partial \psi}{\partial y} = -u = \frac{+y}{x^2+y^2}$$

$$\Rightarrow \quad \psi(x,y) = \tfrac{1}{2}\ln\left(x^2+y^2\right) = C \qquad \Rightarrow \quad x^2+y^2 = e^{2C} = A^2$$

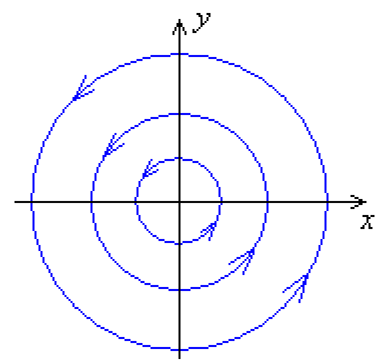
Therefore the streamlines in the *x*-*y* plane are

$$\underline{\underline{x^2 + y^2 = A^2}}$$

$$(x,y) = (1,0) \qquad \Rightarrow \quad 1^2 + 0^2 = A^2$$

The required streamline is therefore the unit circle

$$\underline{\underline{x^2 + y^2 = 1}}$$



The flow is a vortex around the origin.

Divergence (a scalar quantity):

$$\boxed{\operatorname{div} \vec{\mathbf{F}} = \vec{\nabla} \bullet \vec{\mathbf{F}}}$$

Curl (a vector quantity):

$$\boxed{\operatorname{curl} \vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}} = \det \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle}$$

curl **F = 0** everywhere $\Rightarrow$ **F** is an **irrotational vector field**.

Example 1.4.3

Find curl **F** for **F** $= \langle \cos y, -\sin x, 0 \rangle$.
Also find the lines of force for the vector field **F**.

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos y & -\sin x & 0 \end{vmatrix} = \langle 0-0, 0-0, -\cos x + \sin y \rangle$$

$$\Rightarrow \quad \operatorname{curl} \vec{\mathbf{F}} = \underline{\underline{(\sin y - \cos x)\hat{\mathbf{k}}}}$$

Lines of force:

$$\frac{d\vec{\mathbf{r}}}{ds} = k\vec{\mathbf{F}} \quad \Rightarrow \quad \left\langle \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right\rangle = k \langle \cos y, -\sin x, 0 \rangle$$

$$\Rightarrow \quad \frac{dz}{ds} = 0 \quad \text{and} \quad k\,ds = \frac{dx}{\cos y} = \frac{dy}{-\sin x}$$

$$\Rightarrow \quad z = B \quad \text{and} \quad \int -\sin x \, dx = \int \cos y \, dy$$

The lines of force are therefore

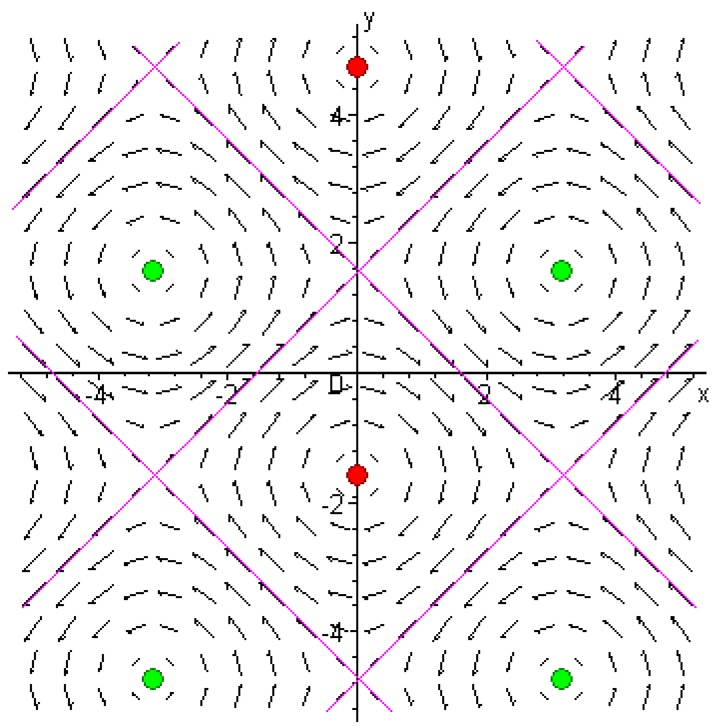
$$\underline{\underline{\sin y = A + \cos x, \quad z = B}}$$

In order to be real, $-2 \le A \le 2$

It then follows that curl **F** $= A\,\mathbf{k}$ and is constant along any one line of force.

Example 1.4.3 (continued)

Direction field plot for the vector field $\mathbf{F} = \langle \cos y, \ -\sin x, 0 \rangle$:



$$\text{curl } \mathbf{F} = (\sin y - \cos x)\, \mathbf{k}$$

Lines of force: $\sin y = A + \cos x \quad (-2 \le A \le 2)$

and $\text{curl } \mathbf{F} = A\, \mathbf{k}$.

Along the highlighted lines (at 45° angles), $A = 0$ (and therefore curl $\mathbf{F} = \mathbf{0}$).
Where those lines cross, $\mathbf{F} = \mathbf{0}$ also, (in addition to $A = 0$ and curl $\mathbf{F} = \mathbf{0}$).
Half way along the lines between those intersections, $|\,\mathbf{F}\,|$ is at a maximum ($\sqrt{2}$).
At the highlighted dots, | curl $\mathbf{F}$ | achieves its maximum value of 2 and $\mathbf{F} = \mathbf{0}$ and
where the lines of force near a dot are in an anticlockwise direction, curl $\mathbf{F} = +2\,\mathbf{k}$;
where the lines of force near a dot are in a clockwise direction, curl $\mathbf{F} = -2\,\mathbf{k}$.

All differentiable gradient-vector fields are irrotational:

$$\boxed{\text{curl grad } \phi \equiv \vec{\nabla} \times \vec{\nabla} \phi \equiv \vec{\mathbf{0}}}$$

Proof:

$$\text{curl } \vec{\mathbf{F}} = \vec{\nabla} \times \vec{\nabla} \phi = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = \left\langle \phi_{zy} - \phi_{yz}, \phi_{xz} - \phi_{zx}, \phi_{yx} - \phi_{xy} \right\rangle = \vec{\mathbf{0}}$$

Also:

$$\boxed{\text{div curl } \vec{\mathbf{F}} \equiv \vec{\nabla} \bullet \vec{\nabla} \times \vec{\mathbf{F}} \equiv 0}$$

Proof:

Let $\vec{\mathbf{F}} = \langle f_1, f_2, f_3 \rangle$ and $f_{1x} = \dfrac{\partial f_1}{\partial x}$ etc., then

$$\vec{\nabla} \bullet \vec{\nabla} \times \vec{\mathbf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \bullet \left\langle f_{3y} - f_{2z}, f_{1z} - f_{3x}, f_{2x} - f_{1y} \right\rangle$$

$$= \left(\cancel{f_{3yx}} - \cancel{f_{2zx}} \right) + \left(\cancel{f_{1zy}} - \cancel{f_{3xy}} \right) + \left(\cancel{f_{2xz}} - \cancel{f_{1yz}} \right) = 0$$

The **Laplacian** of a twice-differentiable scalar field ϕ is:

$$\text{div grad } \phi \equiv \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \bullet \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \equiv \nabla^2 \phi$$

Laplace's equation is

$$\nabla^2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Some Vector Identities

$$\vec{\nabla} \times \vec{\nabla} f \;=\; \text{curl grad } f \;=\; \vec{\mathbf{0}}$$
$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{\mathbf{F}} \;=\; \text{div curl } \vec{\mathbf{F}} \;=\; 0$$

$$\text{Laplacian of } V \;=\; \nabla^2 V \;=\; \vec{\nabla} \cdot \left(\vec{\nabla} V \right) \;=\; \text{div grad } V$$

$$\vec{\nabla}^2 \vec{\mathbf{F}} \;=\; \vec{\nabla}\left(\vec{\nabla} \cdot \vec{\mathbf{F}}\right) - \vec{\nabla} \times \left(\vec{\nabla} \times \vec{\mathbf{F}}\right) \;=\; \text{grad div } \vec{\mathbf{F}} \;-\; \text{curl curl } \vec{\mathbf{F}}$$

$$\vec{\nabla}(f\,g) \;=\; f\,\vec{\nabla} g \;+\; \left(\vec{\nabla} f\right) g$$

$$\vec{\nabla} \cdot \left(g\,\vec{\mathbf{F}}\right) \;=\; \left(\vec{\nabla} g\right) \cdot \vec{\mathbf{F}} \;+\; g\left(\vec{\nabla} \cdot \vec{\mathbf{F}}\right)$$
$$\text{div } (g\,\mathbf{F}) \;=\; (\text{grad } g) \bullet \mathbf{F} \;+\; g \text{ div } \mathbf{F}$$

$$\vec{\nabla} \times \left(g\,\vec{\mathbf{F}}\right) \;=\; \left(\vec{\nabla} g\right) \times \vec{\mathbf{F}} \;+\; g\left(\vec{\nabla} \times \vec{\mathbf{F}}\right)$$
$$\text{curl } (g\,\mathbf{F}) \;=\; (\text{grad } g) \times \mathbf{F} \;+\; g \text{ curl } \mathbf{F}$$

$$\vec{\nabla} \cdot \left(\vec{\mathbf{F}} \times \vec{\mathbf{G}}\right) \;=\; \left(\vec{\nabla} \times \vec{\mathbf{F}}\right) \cdot \vec{\mathbf{G}} \;-\; \vec{\mathbf{F}} \cdot \left(\vec{\nabla} \times \vec{\mathbf{G}}\right)$$
$$\text{div } (\mathbf{F} \times \mathbf{G}) \;=\; (\text{curl } \mathbf{F}) \bullet \mathbf{G} \;-\; \mathbf{F} \bullet (\text{curl } \mathbf{G})$$

$$\vec{\nabla} \times \left(\vec{\mathbf{F}} \times \vec{\mathbf{G}}\right) \;=\; \left(\vec{\mathbf{G}} \cdot \vec{\nabla}\right)\vec{\mathbf{F}} \;-\; \left(\vec{\mathbf{F}} \cdot \vec{\nabla}\right)\vec{\mathbf{G}} \;+\; \left(\vec{\nabla} \cdot \vec{\mathbf{G}}\right)\vec{\mathbf{F}} \;-\; \left(\vec{\nabla} \cdot \vec{\mathbf{F}}\right)\vec{\mathbf{G}},$$

where
$$\left(\vec{\mathbf{F}} \cdot \vec{\nabla}\right)\vec{\mathbf{G}} \;=\; \left(F_1 \frac{\partial G_1}{\partial x} + F_2 \frac{\partial G_1}{\partial y} + F_3 \frac{\partial G_1}{\partial z}\right)\hat{\mathbf{i}} + \left(F_1 \frac{\partial G_2}{\partial x} + F_2 \frac{\partial G_2}{\partial y} + F_3 \frac{\partial G_2}{\partial z}\right)\hat{\mathbf{j}} + \left(F_1 \frac{\partial G_3}{\partial x} + F_2 \frac{\partial G_3}{\partial y} + F_3 \frac{\partial G_3}{\partial z}\right)\hat{\mathbf{k}}$$

so that $\left(\vec{\mathbf{F}} \cdot \vec{\nabla}\right)$ is the operator $\left(F_1 \dfrac{\partial}{\partial x} + F_2 \dfrac{\partial}{\partial y} + F_3 \dfrac{\partial}{\partial z}\right)$

$$\vec{\nabla}\left(\vec{\mathbf{F}} \cdot \vec{\mathbf{G}}\right) \;=\; \left(\vec{\mathbf{G}} \cdot \vec{\nabla}\right)\vec{\mathbf{F}} \;+\; \left(\vec{\mathbf{F}} \cdot \vec{\nabla}\right)\vec{\mathbf{G}} \;+\; \vec{\mathbf{G}} \times \left(\vec{\nabla} \times \vec{\mathbf{F}}\right) \;+\; \vec{\mathbf{F}} \times \left(\vec{\nabla} \times \vec{\mathbf{G}}\right)$$

1.5 Conversions between Coordinate Systems

In general, the conversion of a vector $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}$ from Cartesian coordinates (x, y, z) to another orthonormal coordinate system (u, v, w) in $\mathbb{R}^3$ (where "orthonormal" means that the new basis vectors $\hat{\mathbf{a}}_u, \hat{\mathbf{a}}_v, \hat{\mathbf{a}}_w$ are mutually orthogonal and of unit length) is given by $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}} = F_u\hat{\mathbf{a}}_u + F_v\hat{\mathbf{a}}_v + F_w\hat{\mathbf{a}}_w$.

However, $F_u = \vec{\mathbf{F}}\bullet\hat{\mathbf{a}}_u = \left(F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}}\right)\bullet\hat{\mathbf{a}}_u = \left(\hat{\mathbf{i}}\bullet\hat{\mathbf{a}}_u\right)F_x + \left(\hat{\mathbf{j}}\bullet\hat{\mathbf{a}}_u\right)F_y + \left(\hat{\mathbf{k}}\bullet\hat{\mathbf{a}}_u\right)F_z$.

F_v and F_w are defined similarly in terms of the Cartesian components F_x, F_y, F_z.
In matrix form

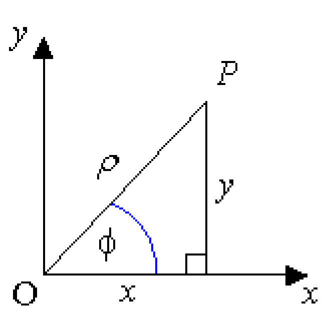
$$\begin{bmatrix} F_u \\ F_v \\ F_w \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}\bullet\hat{\mathbf{a}}_u & \hat{\mathbf{j}}\bullet\hat{\mathbf{a}}_u & \hat{\mathbf{k}}\bullet\hat{\mathbf{a}}_u \\ \hat{\mathbf{i}}\bullet\hat{\mathbf{a}}_v & \hat{\mathbf{j}}\bullet\hat{\mathbf{a}}_v & \hat{\mathbf{k}}\bullet\hat{\mathbf{a}}_v \\ \hat{\mathbf{i}}\bullet\hat{\mathbf{a}}_w & \hat{\mathbf{j}}\bullet\hat{\mathbf{a}}_w & \hat{\mathbf{k}}\bullet\hat{\mathbf{a}}_w \end{bmatrix} \begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}.$$

The matrices on the right hand side of the equation will contain a mixture of expressions in the new (u, v, w) and old (x, y, z) coordinates. This needs to be converted into a set of expressions in (u, v, w) only.

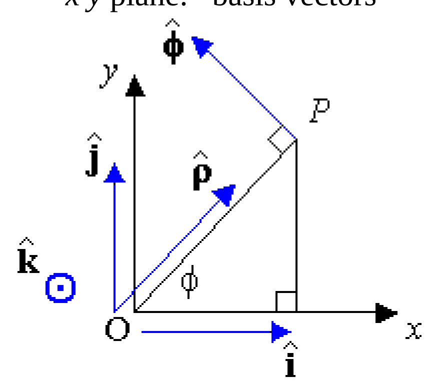
Example 1.5.1

Express the vector $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j} + z\,\mathbf{k}$ in cylindrical polar coordinates.

$x\,y$ plane: coordinates



$x\,y$ plane: basis vectors



$\Rightarrow \quad x = \rho\cos\phi \qquad \hat{\mathbf{i}}\bullet\hat{\rho} = \left|\hat{\mathbf{i}}\right|\bullet\left|\hat{\rho}\right|\cos\phi = \cos\phi$

$y = \rho\sin\phi \qquad \hat{\mathbf{j}}\bullet\hat{\rho} = 1\times1\times\cos\left(\tfrac{\pi}{2}-\phi\right) = \sin\phi$

$z = z \qquad \hat{\mathbf{k}}\bullet\hat{\rho} = 1\times1\times\cos\tfrac{\pi}{2} = 0$

Example 1.5.1 (continued)

$\hat{\mathbf{i}} \bullet \hat{\boldsymbol{\phi}} = 1 \times 1 \times \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi$ ** $\qquad \hat{\mathbf{i}} \bullet \hat{\mathbf{k}} = 0$

$\hat{\mathbf{j}} \bullet \hat{\boldsymbol{\phi}} = 1 \times 1 \times \cos\phi = \cos\phi \qquad \hat{\mathbf{j}} \bullet \hat{\mathbf{k}} = 0$

$\hat{\mathbf{k}} \bullet \hat{\boldsymbol{\phi}} = 1 \times 1 \times \cos\frac{\pi}{2} = 0 \qquad \hat{\mathbf{k}} \bullet \hat{\mathbf{k}} = 1$

The coefficient conversion matrix from Cartesian to cylindrical polar is therefore

$$\begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Letting $c \equiv \cos\phi$, $s \equiv \sin\phi$: $\vec{\mathbf{F}} = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} = \rho s\,\hat{\mathbf{i}} - \rho c\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$

$$\vec{\mathbf{F}}_{\text{polar}} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho s \\ -\rho c \\ z \end{bmatrix} = \begin{bmatrix} (c\rho s - s\rho c + 0) \\ (-s\rho s - c\rho c + 0) \\ (0 + 0 + z) \end{bmatrix} = \begin{bmatrix} 0 \\ -\rho \\ z \end{bmatrix}$$

Therefore
$\vec{\mathbf{F}} = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} = \underline{\underline{-\rho\,\hat{\boldsymbol{\phi}} + z\,\hat{\mathbf{k}}}}$

** This result can be obtained from the trigonometric identity

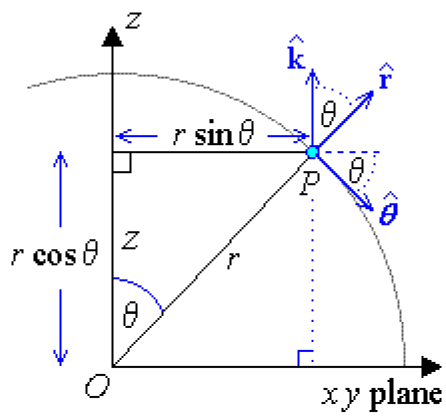
$$\cos(A + B) \equiv \cos A \cos B - \sin A \sin B$$

Setting $A = \frac{\pi}{2}$ and $B = \phi$,

$$\cos\left(\frac{\pi}{2} + \phi\right) \equiv \cos\frac{\pi}{2}\cos\phi - \sin\frac{\pi}{2}\sin\phi \equiv 0 - 1 \times \sin\phi$$

We can also generate the coordinate transformation matrix from Cartesian coordinates (x, y, z) to **spherical polar** coordinates (r, θ, ϕ).
[θ is the declination (angle down from the north pole, $0 \le \theta \le \pi$) and
ϕ is the azimuth (angle around the equator $0 \le \phi < 2\pi$).]

[Vertical] Plane containing *z*-axis and radial vector $\mathbf{r}$:



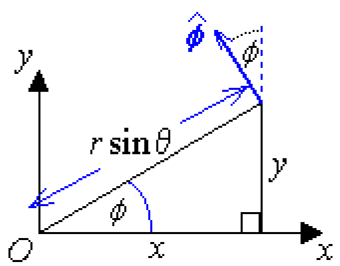
$$z = r\cos\theta$$

The projection of the radial vector $\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$ onto the plane $z = r\cos\theta$ has length $r\sin\theta$

The angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}}$ is θ
$\Rightarrow \quad \hat{\mathbf{k}} \cdot \hat{\mathbf{r}} = \cos\theta$

The angle between $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{k}}$ is $\frac{\pi}{2} + \theta$
$\Rightarrow \quad \hat{\mathbf{k}} \cdot \hat{\boldsymbol{\theta}} = \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

[Horizontal] Plane $z = r\cos\theta$:



The projection of $r\,\hat{\mathbf{r}}$ onto the x axis ($\hat{\mathbf{i}}$)
is $x = (r\sin\theta)\cos\phi = (r\,\hat{\mathbf{r}}) \cdot \hat{\mathbf{i}}$
$\Rightarrow \quad \hat{\mathbf{i}} \cdot \hat{\mathbf{r}} = \sin\theta\,\cos\phi$

$$\mathbf{r} = \langle x, y, z \rangle = \langle r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta \rangle .$$

The projection of $r\,\hat{\mathbf{r}}$ onto the y axis ($\hat{\mathbf{j}}$)
is $y = (r\sin\theta)\sin\phi = (r\,\hat{\mathbf{r}}) \cdot \hat{\mathbf{j}}$
$\Rightarrow \quad \hat{\mathbf{j}} \cdot \hat{\mathbf{r}} = \sin\theta\,\sin\phi$

The angle between $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{j}}$ is ϕ $\Rightarrow \quad \hat{\mathbf{j}} \cdot \hat{\boldsymbol{\phi}} = \cos\phi$
The angle between $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{i}}$ is $\frac{\pi}{2} + \phi$ $\Rightarrow \quad \hat{\mathbf{i}} \cdot \hat{\boldsymbol{\phi}} = \cos\left(\frac{\pi}{2} + \phi\right) = -\sin\phi$
The remaining three entries in the coordinate conversion matrix can be found in a similar way.

The conversion matrix from Cartesian to spherical polar coordinates is then

$$\boxed{\begin{bmatrix} \hat{\mathbf{i}}\bullet\hat{\mathbf{r}} & \hat{\mathbf{j}}\bullet\hat{\mathbf{r}} & \hat{\mathbf{k}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{i}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{j}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{k}}\bullet\hat{\boldsymbol{\theta}} \\ \hat{\mathbf{i}}\bullet\hat{\boldsymbol{\phi}} & \hat{\mathbf{j}}\bullet\hat{\boldsymbol{\phi}} & \hat{\mathbf{k}}\bullet\hat{\boldsymbol{\phi}} \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix}}$$

<u>Example 1.5.2</u>

Convert $\vec{\mathbf{F}} = y\,\hat{\mathbf{i}} - x\,\hat{\mathbf{j}}$ to spherical polar coordinates.

Let $c_1 \equiv \cos\theta$, $s_1 \equiv \sin\theta$, $c_2 \equiv \cos\phi$, $s_2 \equiv \sin\phi$

$\Rightarrow \quad y = r s_1 s_2\,, \quad -x = -r s_1 c_2$

$$\vec{\mathbf{F}} = \begin{bmatrix} F_r \\ F_\theta \\ F_\phi \end{bmatrix} = \begin{bmatrix} s_1c_2 & s_1s_2 & c_1 \\ c_1c_2 & c_1s_2 & -s_1 \\ -s_2 & c_2 & 0 \end{bmatrix} \begin{bmatrix} r s_1 s_2 \\ -r s_1 c_2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} r s_1^{\,2}\left(c_2s_2 - s_2c_2\right) \\ -r s_1 c_1\left(c_2s_2 - s_2c_2\right) \\ r s_1\left(-s_2^{\,2} - c_2^{\,2}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -r s_1 \end{bmatrix}$$

Therefore

$$\vec{\mathbf{F}} = \underline{\underline{-r\sin\theta\,\hat{\boldsymbol{\phi}}}}$$

Expressions for the gradient, divergence, curl and Laplacian operators in any orthonormal coordinate system will follow in section 1.7.

Summary for Coordinate Conversion:

To convert a vector expressed in Cartesian components $v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$ into the equivalent vector expressed in **cylindrical polar** coordinates $v_\rho \hat{\boldsymbol{\rho}} + v_\phi \hat{\boldsymbol{\phi}} + v_z \hat{\mathbf{k}}$, express the Cartesian components v_x, v_y, v_z in terms of (ρ, ϕ, z) using
$x = \rho \cos\phi, \quad y = \rho \sin\phi, \quad z = z$; then evaluate

$$\begin{bmatrix} v_\rho \\ v_\phi \\ v_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Use the inverse matrix to transform back to Cartesian coordinates:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_\rho \\ v_\phi \\ v_z \end{bmatrix}$$

To convert a vector expressed in Cartesian components $v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$ into the equivalent vector expressed in **spherical polar** coordinates $v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} + v_\phi \hat{\boldsymbol{\phi}}$, express the Cartesian components v_x, v_y, v_z in terms of (r, θ, ϕ) using
$x = r \sin\theta \cos\phi, \quad y = r \sin\theta \sin\phi, \quad z = r \cos\theta$; then evaluate

$$\begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Use the inverse matrix to transform back to Cartesian coordinates:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix}$$

Note that, in both cases, the transformation matrix A is orthogonal, so that $A^{-1} = A^T$.
This is generally true for transformations between orthonormal coordinate systems.

1.6 Basis Vectors in Other Coordinate Systems

In the Cartesian coordinate system, all three basis vectors are absolute constants:

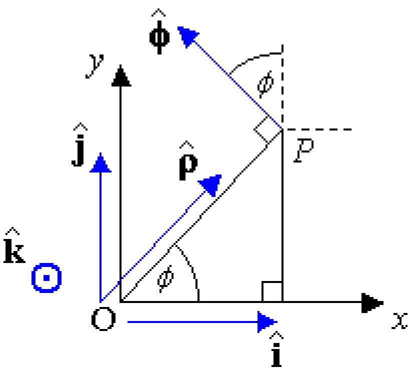
$$\frac{d}{dt}\hat{\mathbf{i}} = \frac{d}{dt}\hat{\mathbf{j}} = \frac{d}{dt}\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

The derivative of a vector is then straightforward to calculate:

$$\frac{d}{dt}\left(f_1\hat{\mathbf{i}} + f_2\hat{\mathbf{j}} + f_3\hat{\mathbf{k}}\right) = \hat{\mathbf{i}}\frac{df_1}{dt} + \hat{\mathbf{j}}\frac{df_2}{dt} + \hat{\mathbf{k}}\frac{df_3}{dt}$$

But many non-Cartesian basis vectors are not constant.

Cylindrical Polar:



$$\hat{\boldsymbol{\rho}} = \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}$$

$$\hat{\mathbf{k}} = \hat{\mathbf{k}}$$

Chain rule diff'n

$\cos\phi$
|
ϕ
|
t

Let $\dot{\mathbf{v}} \equiv \dfrac{d\vec{\mathbf{v}}}{dt}$ then $\dot{\hat{\boldsymbol{\rho}}} = \left(-\sin\phi\,\dot{\phi}\right)\hat{\mathbf{i}} + \left(\cos\phi\,\dot{\phi}\right)\hat{\mathbf{j}} = \dot{\phi}\,\hat{\boldsymbol{\phi}}$

$$\dot{\hat{\boldsymbol{\phi}}} = \left(-\cos\phi\,\dot{\phi}\right)\hat{\mathbf{i}} + \left(-\sin\phi\,\dot{\phi}\right)\hat{\mathbf{j}} = -\dot{\phi}\,\hat{\boldsymbol{\rho}}$$

$$\dot{\hat{\mathbf{k}}} = \vec{\mathbf{0}}$$

Therefore if a vector $\vec{\mathbf{F}}$ is described in cylindrical polar coordinates
$\vec{\mathbf{F}} = F_\rho\hat{\boldsymbol{\rho}} + F_\phi\hat{\boldsymbol{\phi}} + F_z\hat{\mathbf{k}}$, then

$$\dot{\mathbf{F}} = \left(\dot{F}_\rho\hat{\boldsymbol{\rho}} + F_\rho\dot{\phi}\hat{\boldsymbol{\phi}}\right) + \left(\dot{F}_\phi\hat{\boldsymbol{\phi}} - F_\phi\dot{\phi}\hat{\boldsymbol{\rho}}\right) + \left(\dot{F}_z\hat{\mathbf{k}} + \vec{\mathbf{0}}\right)$$

$$= \left(\dot{F}_\rho - F_\phi\dot{\phi}\right)\hat{\boldsymbol{\rho}} + \left(\dot{F}_\phi + F_\rho\dot{\phi}\right)\hat{\boldsymbol{\phi}} + \left(\dot{F}_z\right)\hat{\mathbf{k}}$$

In particular, the displacement vector is $\vec{\mathbf{r}}(t) = \rho(t)\hat{\boldsymbol{\rho}} + 0\hat{\boldsymbol{\phi}} + z(t)\hat{\mathbf{k}}$, so that the velocity vector is

$$\boxed{\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d\rho}{dt}\hat{\boldsymbol{\rho}} + \rho\frac{d\phi}{dt}\hat{\boldsymbol{\phi}} + \frac{dz}{dt}\hat{\mathbf{k}}}$$

Example 1.6.1

Find the velocity and acceleration in cylindrical polar coordinates for a particle travelling along the helix $x = 3\cos 2t$, $y = 3\sin 2t$, $z = t$.

Cylindrical polar coordinates: $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

$$\Rightarrow \quad \rho^2 = x^2 + y^2, \quad \tan\phi = \frac{y}{x}$$

$$\rho^2 = 9\cos^2 2t + 9\sin^2 2t = 9 \quad \Rightarrow \quad \rho = 3 \quad \Rightarrow \quad \dot{\rho} = 0$$

$$\tan\phi = \frac{3\sin 2t}{3\cos 2t} = \tan 2t \quad \Rightarrow \quad \phi = 2t \quad \Rightarrow \quad \dot{\phi} = 2$$

$$z = t \quad \Rightarrow \quad \dot{z} = 1$$

$$\Rightarrow \quad \vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = 0\hat{\boldsymbol{\rho}} + 3\times 2\hat{\boldsymbol{\phi}} + 1\hat{\mathbf{k}} = \underline{\underline{6\hat{\boldsymbol{\phi}} + \hat{\mathbf{k}}}}$$

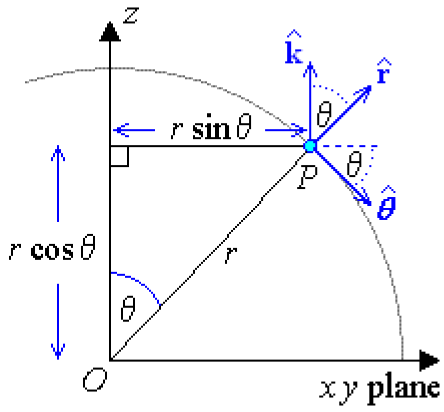
[The velocity has no radial component – the helix remains the same distance from the *z* axis at all times.]

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = 6\dot{\hat{\boldsymbol{\phi}}} + \dot{\hat{\mathbf{k}}} = -6\dot{\phi}\,\hat{\boldsymbol{\rho}} + \vec{\mathbf{0}} = \underline{\underline{-12\,\hat{\boldsymbol{\rho}}}}$$

[The acceleration vector points directly at the *z* axis at all times.]

Spherical Polar Coordinates

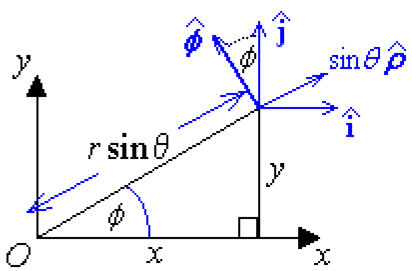
Vertical plane containing z-axis and radial vector $\mathbf{r}$:



$$\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{i}} + \sin\theta\sin\phi\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}$$

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{i}} + \cos\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}}$$

Equatorial plane ($\theta = 0$):



[$\sin\theta\,\hat{\boldsymbol{\rho}}$ is the projection of $\hat{\mathbf{r}}$ onto the equatorial plane.]

$$\hat{\boldsymbol{\phi}} = -\sin\phi\,\hat{\mathbf{i}} + \cos\phi\,\hat{\mathbf{j}}$$

[This reproduces the three rows of the coordinate conversion matrix in section 1.5.]

$$\Rightarrow \quad \frac{d\hat{\mathbf{r}}}{dt} = \left(\cos\theta\frac{d\theta}{dt}\cos\phi - \sin\theta\sin\phi\frac{d\phi}{dt}\right)\hat{\mathbf{i}}$$

$$+ \left(\cos\theta\frac{d\theta}{dt}\sin\phi + \sin\theta\cos\phi\frac{d\phi}{dt}\right)\hat{\mathbf{j}} + \left(-\sin\theta\frac{d\theta}{dt}\right)\hat{\mathbf{k}}$$

$$\Rightarrow \quad \boxed{\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = \left(-\sin\theta\frac{d\theta}{dt}\cos\phi - \cos\theta\sin\phi\frac{d\phi}{dt}\right)\hat{\mathbf{i}}$$

$$+ \left(-\sin\theta\frac{d\theta}{dt}\sin\phi + \cos\theta\cos\phi\frac{d\phi}{dt}\right)\hat{\mathbf{j}} + \left(-\cos\theta\frac{d\theta}{dt}\right)\hat{\mathbf{k}}$$

$$\Rightarrow \quad \boxed{\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}} + \cos\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}}$$

Spherical Polar Coordinates (continued)

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = \left(-\cos\phi \frac{d\phi}{dt}\right)\hat{\mathbf{i}} + \left(-\sin\phi \frac{d\phi}{dt}\right)\hat{\mathbf{j}}$$

But $\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}} = \sin^2\theta\cos\phi\,\hat{\mathbf{i}} + \sin^2\theta\sin\phi\,\hat{\mathbf{j}} + \sin\theta\cos\theta\,\hat{\mathbf{k}}$

$$+ \cos^2\theta\cos\phi\,\hat{\mathbf{i}} + \cos^2\theta\sin\phi\,\hat{\mathbf{j}} - \sin\theta\cos\theta\,\hat{\mathbf{k}}$$

$$= \cos\phi\,\hat{\mathbf{i}} + \sin\phi\,\hat{\mathbf{j}}$$

$$\Rightarrow \boxed{\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)\frac{d\phi}{dt}}$$

In particular, the displacement vector is $\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$, so that the velocity vector is

$$\vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\left(\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}\right)$$

$$\boxed{\Rightarrow \quad \vec{\mathbf{v}} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}}$$

It can be shown that the acceleration vector in the spherical polar coordinate system is

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \left(\frac{d^2r}{dt^2} - r\left(\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\phi}{dt}\right)^2\sin^2\theta\right)\right)\hat{\mathbf{r}}$$

$$+ \left(\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - \frac{r}{2}\left(\frac{d\phi}{dt}\right)^2\sin 2\theta\right)\hat{\boldsymbol{\theta}}$$

$$+ \left(\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\frac{d\phi}{dt}\sin^2\theta\right)\right)\hat{\boldsymbol{\phi}}$$

Compare this to the Cartesian equivalent $\vec{\mathbf{a}} = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} + \frac{d^2z}{dt^2}\hat{\mathbf{k}}$!

Example 1.6.2

Find the velocity vector **v** for a particle whose displacement vector **r**, in spherical polar coordinates, is given by $r = 4,\quad \theta = t\,,\quad \phi = 2t\,,\quad (0 < t < \pi)$.

$$r = 4\,,\quad \theta = t\,,\quad \phi = 2t \quad \Rightarrow \quad \frac{dr}{dt} = 0\,,\quad \frac{d\theta}{dt} = 1\,,\quad \frac{d\phi}{dt} = 2$$

$$\vec{\mathbf{v}} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$= 0\,\hat{\mathbf{r}} + 4\times 1\,\hat{\boldsymbol{\theta}} + 4(\sin t)\times 2\,\hat{\boldsymbol{\phi}}$$

$$\therefore \quad \underline{\underline{\vec{\mathbf{v}}(t) = 4\,\hat{\boldsymbol{\theta}} + 8\sin t\,\hat{\boldsymbol{\phi}}}}$$

[This describes a path spiralling around a sphere of radius 4, from pole to pole.]

Summary:

Cylindrical Polar:

$$\frac{d}{dt}\hat{\boldsymbol{\rho}} = \frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d}{dt}\hat{\boldsymbol{\phi}} = -\frac{d\phi}{dt}\hat{\boldsymbol{\rho}}$$

$$\frac{d}{dt}\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{r}} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{k}}$$

$$\Rightarrow \quad \vec{\mathbf{v}} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\mathbf{e}}_z$$

Spherical Polar:

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \sin\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}} + \cos\theta\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)\frac{d\phi}{dt}$$

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad \Rightarrow \quad \vec{\mathbf{v}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}$$

1.7 Gradient Operator in Other Coordinate Systems

For any orthogonal curvilinear coordinate system (u_1, u_2, u_3) in $\mathbb{R}^3$,
the unit tangent vectors along the curvilinear axes are $\hat{\mathbf{e}}_i = \hat{\mathbf{T}}_i = \dfrac{1}{h_i}\dfrac{\partial \vec{\mathbf{r}}}{\partial u_i}$,

where the scale factors $h_i = \left|\dfrac{\partial \vec{\mathbf{r}}}{\partial u_i}\right|$.

The displacement vector $\vec{\mathbf{r}}$ can then be written as $\vec{\mathbf{r}} = u_1\hat{\mathbf{e}}_1 + u_2\hat{\mathbf{e}}_2 + u_3\hat{\mathbf{e}}_3$, where the unit vectors $\hat{\mathbf{e}}_i$ form an **orthonormal basis** for $\mathbb{R}^3$.

$$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} = \begin{cases} 0 & (i \neq j) \\ 1 & (i = j) \end{cases}$$

[δ_{ij} is the “Kronecker delta”.]

The differential displacement vector **dr** is (by the Chain Rule)

$$\mathbf{d}\vec{\mathbf{r}} = \frac{\partial \vec{\mathbf{r}}}{\partial u_1}du_1 + \frac{\partial \vec{\mathbf{r}}}{\partial u_2}du_2 + \frac{\partial \vec{\mathbf{r}}}{\partial u_3}du_3 = h_1\,du_1\,\hat{\mathbf{e}}_1 + h_2\,du_2\,\hat{\mathbf{e}}_2 + h_3\,du_3\,\hat{\mathbf{e}}_3$$

and the differential arc length ds is

$$ds^2 = \mathbf{d}\vec{\mathbf{r}} \bullet \mathbf{d}\vec{\mathbf{r}} = (h_1\,du_1)^2 + (h_2\,du_2)^2 + (h_3\,du_3)^2$$

The element of volume dV is

$$dV = h_1 h_2 h_3\; du_1 du_2 du_3 = \left|\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}\right| du_1 du_2 du_3$$

$$= \begin{vmatrix} \dfrac{\partial x}{\partial u_1} & \dfrac{\partial y}{\partial u_1} & \dfrac{\partial z}{\partial u_1} \\ \dfrac{\partial x}{\partial u_2} & \dfrac{\partial y}{\partial u_2} & \dfrac{\partial z}{\partial u_2} \\ \dfrac{\partial x}{\partial u_3} & \dfrac{\partial y}{\partial u_3} & \dfrac{\partial z}{\partial u_3} \end{vmatrix} du_1 du_2 du_3$$

Gradient operator $$\vec{\nabla} = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3}$$

Gradient $$\vec{\nabla} V = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial V}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial V}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial V}{\partial u_3}$$

Divergence $$\vec{\nabla} \bullet \vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 f_1)}{\partial u_1} + \frac{\partial(h_3 h_1 f_2)}{\partial u_2} + \frac{\partial(h_1 h_2 f_3)}{\partial u_3}\right)$$

Curl
$$\vec{\nabla} \times \vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$$

Laplacian
$$\nabla^2 V = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial V}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial V}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial V}{\partial u_3} \right) \right)$$

Cartesian: $h_x = h_y = h_z = 1$.

Cylindrical polar: $h_\rho = h_z = 1 , \quad h_\phi = \rho$.

Spherical polar: $h_r = 1 , \quad h_\theta = r , \quad h_\phi = r \sin\theta$.

The familiar expressions then follow for the Cartesian coordinate system.

In **<u>cylindrical polar coordinates</u>**, naming the three basis vectors as $\hat{\boldsymbol{\rho}}, \hat{\boldsymbol{\phi}}, \hat{\mathbf{k}}$, we have:
$$\vec{\mathbf{r}} = \rho \hat{\boldsymbol{\rho}} + 0 \hat{\boldsymbol{\phi}} + z \hat{\mathbf{k}} = \langle \rho, 0, z \rangle$$
The relationship to the Cartesian coordinate system is
$$x = \rho \cos\phi , \quad y = \rho \sin\phi , \quad z = z \quad \Rightarrow \quad \rho^2 = x^2 + y^2 , \quad \tan\phi = \frac{y}{x}$$
One scale factor is
$$h_\rho = \left| \frac{\partial \vec{\mathbf{r}}}{\partial \rho} \right| = \sqrt{\left(\frac{\partial x}{\partial \rho} \right)^2 + \left(\frac{\partial y}{\partial \rho} \right)^2 + \left(\frac{\partial z}{\partial \rho} \right)^2}$$
$$= \sqrt{\left(\frac{\partial}{\partial \rho} (\rho \cos\phi) \right)^2 + \left(\frac{\partial}{\partial \rho} (\rho \sin\phi) \right)^2 + \left(\frac{\partial}{\partial \rho} (z) \right)^2}$$
$$= \sqrt{\cos^2\phi + \sin^2\phi + 0} = 1$$

In a similar way, we can confirm that $h_\phi = \rho$ and $h_z = 1$.

In cylindrical polar coordinates,

$$dV = h_\rho h_\phi h_z \, d\rho \, d\phi \, dz = \rho \, d\rho \, d\phi \, dz$$

$$ds^2 = \left(h_\rho \, d\rho\right)^2 + \left(h_\phi \, d\phi\right)^2 + \left(h_z \, dz\right)^2 = \left(d\rho\right)^2 + \left(\rho \, d\phi\right)^2 + \left(dz\right)^2$$

$$\vec{\nabla} = \hat{\boldsymbol{\rho}} \frac{\partial}{\partial \rho} + \frac{\hat{\boldsymbol{\phi}}}{\rho} \frac{\partial}{\partial \phi} + \hat{\mathbf{k}} \frac{\partial}{\partial z}$$

$$\vec{\nabla} V = \hat{\boldsymbol{\rho}} \frac{\partial V}{\partial \rho} + \frac{\hat{\boldsymbol{\phi}}}{\rho} \frac{\partial V}{\partial \phi} + \hat{\mathbf{k}} \frac{\partial V}{\partial z}$$

$$\vec{\nabla} \bullet \vec{\mathbf{F}} = \frac{1}{1 \times \rho \times 1} \left(\frac{\partial \left(\rho \times 1 f_\rho\right)}{\partial \rho} + \frac{\partial \left(1 \times 1 f_\phi\right)}{\partial \phi} + \frac{\partial \left(1 \times \rho f_z\right)}{\partial z} \right)$$

$$= \frac{\partial f_\rho}{\partial \rho} + \frac{f_\rho}{\rho} + \frac{1}{\rho} \frac{\partial f_\phi}{\partial \phi} + \frac{\partial f_z}{\partial z}$$

$$\vec{\nabla} \times \vec{\mathbf{F}} = \frac{1}{1 \times \rho \times 1} \begin{vmatrix} \hat{\boldsymbol{\rho}} & \rho \hat{\boldsymbol{\phi}} & \hat{\mathbf{k}} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ f_\rho & \rho f_\phi & f_z \end{vmatrix}$$

$$\nabla^2 V = \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} \left(\frac{\rho \times 1}{1} \frac{\partial V}{\partial \rho} \right) + \frac{\partial}{\partial \phi} \left(\frac{1 \times 1}{\rho} \frac{\partial V}{\partial \phi} \right) + \frac{\partial}{\partial z} \left(\frac{1 \times \rho}{1} \frac{\partial V}{\partial z} \right) \right)$$

$$= \frac{\partial^2 V}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial V}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

All of the above are undefined on the *z*-axis ($\rho = 0$), where there is a coordinate singularity. However, by taking the limit as $\rho \to 0$, we may obtain well-defined values for some or all of the above expressions.

Example 1.7.1

Given that the gradient operator in a general curvilinear coordinate system is

$\vec{\nabla} = \left(\frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3} \right)$, why isn't the divergence of

$\vec{\mathbf{F}} = F_1\,\hat{\mathbf{e}}_1 + F_2\,\hat{\mathbf{e}}_2 + F_3\,\hat{\mathbf{e}}_3$ equal, in general, to $\left(\frac{1}{h_1}\frac{\partial F_1}{\partial u_1} + \frac{1}{h_2}\frac{\partial F_2}{\partial u_2} + \frac{1}{h_3}\frac{\partial F_3}{\partial u_3} \right)$?

The quick answer is that the differential operators operate not just on the components F_1, F_2, F_3, but also on the basis vectors $\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3$. In most orthonormal coordinate systems, these basis vectors are not constant. The divergence therefore contains additional terms.

$$\left(\frac{\mathbf{e}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\mathbf{e}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\mathbf{e}_3}{h_3}\frac{\partial}{\partial u_3} \right) \bullet \left(F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 \right) =$$

$$\begin{aligned}
&\left(\frac{\mathbf{e}_1 \bullet \mathbf{e}_1}{h_1}\frac{\partial F_1}{\partial u_1} + \frac{F_1}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_1}{\partial u_1} \right) + \left(\cancel{\frac{\mathbf{e}_2 \bullet \mathbf{e}_1}{h_2}\frac{\partial F_1}{\partial u_2}} + \frac{F_1}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_1}{\partial u_2} \right) + \left(\cancel{\frac{\mathbf{e}_3 \bullet \mathbf{e}_1}{h_3}\frac{\partial F_1}{\partial u_3}} + \frac{F_1}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_1}{\partial u_3} \right) + \\
&\left(\cancel{\frac{\mathbf{e}_1 \bullet \mathbf{e}_2}{h_1}\frac{\partial F_2}{\partial u_1}} + \frac{F_2}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_2}{\partial u_1} \right) + \left(\frac{\mathbf{e}_2 \bullet \mathbf{e}_2}{h_2}\frac{\partial F_2}{\partial u_2} + \frac{F_2}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_2}{\partial u_2} \right) + \left(\cancel{\frac{\mathbf{e}_3 \bullet \mathbf{e}_2}{h_3}\frac{\partial F_2}{\partial u_3}} + \frac{F_2}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_2}{\partial u_3} \right) + \\
&\left(\cancel{\frac{\mathbf{e}_1 \bullet \mathbf{e}_3}{h_1}\frac{\partial F_3}{\partial u_1}} + \frac{F_3}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_3}{\partial u_1} \right) + \left(\cancel{\frac{\mathbf{e}_2 \bullet \mathbf{e}_3}{h_2}\frac{\partial F_3}{\partial u_2}} + \frac{F_3}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_3}{\partial u_2} \right) + \left(\frac{\mathbf{e}_3 \bullet \mathbf{e}_3}{h_3}\frac{\partial F_3}{\partial u_3} + \frac{F_3}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_3}{\partial u_3} \right) =
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{1}{h_1}\frac{\partial F_1}{\partial u_1} + \frac{F_1}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_1}{\partial u_1} \right) + \left(\frac{F_1}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_1}{\partial u_2} \right) + \left(\frac{F_1}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_1}{\partial u_3} \right) + \\
&\left(\frac{F_2}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_2}{\partial u_1} \right) + \left(\frac{1}{h_2}\frac{\partial F_2}{\partial u_2} + \frac{F_2}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_2}{\partial u_2} \right) + \left(\frac{F_2}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_2}{\partial u_3} \right) + \\
&\left(\frac{F_3}{h_1}\mathbf{e}_1 \bullet \frac{\partial \mathbf{e}_3}{\partial u_1} \right) + \left(\frac{F_3}{h_2}\mathbf{e}_2 \bullet \frac{\partial \mathbf{e}_3}{\partial u_2} \right) + \left(\frac{1}{h_3}\frac{\partial F_3}{\partial u_3} + \frac{F_3}{h_3}\mathbf{e}_3 \bullet \frac{\partial \mathbf{e}_3}{\partial u_3} \right)
\end{aligned}$$

For Cartesian coordinates, all derivatives of any basis vector are zero, which leaves the familiar Cartesian expression for the divergence. But for most non-Cartesian coordinate systems, at least some of these partial derivatives are not zero. More complicated expressions for the divergence therefore arise.

Example 1.7.1 (continued)

For **cylindrical polar coordinates**, we have

$$\left(\frac{1}{1}\frac{\partial F_\rho}{\partial \rho} + \cancel{\frac{F_\rho}{1}\hat{\boldsymbol{\rho}} \bullet \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \rho}}\right) + \left(\frac{F_\rho}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi}\right) + \left(\cancel{\frac{F_\rho}{1}\hat{\mathbf{k}} \bullet \frac{\partial \hat{\boldsymbol{\rho}}}{\partial z}}\right) +$$

$$\left(\cancel{\frac{F_\phi}{1}\hat{\boldsymbol{\rho}} \bullet \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \rho}}\right) + \left(\frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{F_\phi}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}\right) + \left(\cancel{\frac{F_\phi}{1}\hat{\mathbf{k}} \bullet \frac{\partial \hat{\boldsymbol{\phi}}}{\partial z}}\right) +$$

$$\left(\cancel{\frac{F_z}{1}\hat{\boldsymbol{\rho}} \bullet \frac{\partial \hat{\mathbf{k}}}{\partial \rho}}\right) + \left(\cancel{\frac{F_z}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\mathbf{k}}}{\partial \phi}}\right) + \left(\frac{1}{1}\frac{\partial F_z}{\partial z} + \cancel{\frac{F_z}{1}\hat{\mathbf{k}} \bullet \frac{\partial \hat{\mathbf{k}}}{\partial z}}\right)$$

But none of the basis vectors varies with ρ or z * and the basis vector $\hat{\mathbf{k}}$ is absolutely constant. Therefore the divergence becomes

$$\left(\frac{1}{1}\frac{\partial F_\rho}{\partial \rho} + 0\right) + \left(\frac{F_\rho}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi}\right) + (0) +$$

$$(0) + \left(\frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \cancel{\frac{F_\phi}{\rho}\hat{\boldsymbol{\phi}} \bullet \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}}\right) + (0) + \; (0) + (0) + \left(\frac{1}{1}\frac{\partial F_z}{\partial z} + 0\right)$$

But $\dfrac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi} = \hat{\boldsymbol{\phi}} \;\Rightarrow\; \left(\dfrac{F_\rho}{\rho}\hat{\boldsymbol{\phi}} \bullet \dfrac{\partial \hat{\boldsymbol{\rho}}}{\partial \phi}\right) = \left(\dfrac{F_\rho}{\rho}\hat{\boldsymbol{\phi}} \bullet \hat{\boldsymbol{\phi}}\right) = \dfrac{F_\rho}{\rho}$

and $\dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = -\hat{\boldsymbol{\rho}} \;\Rightarrow\; \left(\dfrac{F_\phi}{\rho}\hat{\boldsymbol{\phi}} \bullet \dfrac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi}\right) = \left(\dfrac{F_\phi}{\rho}\hat{\boldsymbol{\phi}} \bullet (-\hat{\boldsymbol{\rho}})\right) = 0$

So we recover the cylindrical polar form for the divergence,

$$\operatorname{div} \vec{\mathbf{F}} = \frac{\partial F_\rho}{\partial \rho} + \frac{F_\rho}{\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}$$

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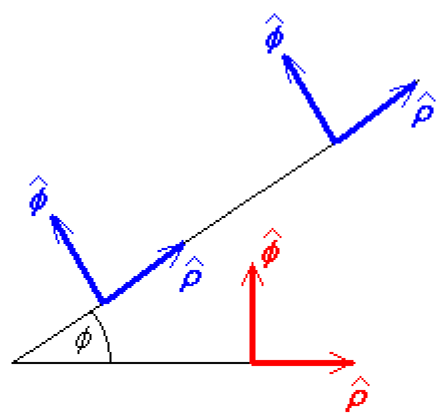
* As shown here, the basis vectors $\hat{\boldsymbol{\rho}}$ and $\hat{\boldsymbol{\phi}}$ clearly vary with ϕ but do not change with ρ.

k is an absolute constant.



In **spherical polar coordinates**, naming the three basis vectors as $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}},$ we have:

$\vec{\mathbf{r}} = r\hat{\mathbf{r}} + 0\hat{\boldsymbol{\theta}} + 0\hat{\boldsymbol{\phi}} = \langle r, 0, 0\rangle$

The relationship to the Cartesian coordinate system is

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$

One of the scale factors is

$$h_\theta = \left|\frac{\partial\vec{\mathbf{r}}}{\partial\theta}\right| = \sqrt{\left(\frac{\partial x}{\partial\theta}\right)^2 + \left(\frac{\partial y}{\partial\theta}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2}$$

$$= \sqrt{\left(\frac{\partial}{\partial\theta}(r\sin\theta\cos\phi)\right)^2 + \left(\frac{\partial}{\partial\theta}(r\sin\theta\sin\phi)\right)^2 + \left(\frac{\partial}{\partial\theta}(r\cos\theta)\right)^2}$$

$$= \sqrt{(r\cos\theta\cos\phi)^2 + (r\cos\theta\sin\phi)^2 + (-r\sin\theta)^2}$$

$$= r\sqrt{\cos^2\theta\,(\cos^2\phi + \sin^2\phi) + \sin^2\theta} = r\sqrt{\cos^2\theta + \sin^2\theta} = r$$

In a similar way, we can confirm that $h_r = 1$ and $h_\phi = r\sin\theta$.

$$dV = (1\times r\times r\sin\theta)\,dr\,d\theta\,d\phi = r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$ds^2 = (1dr)^2 + (r\,d\theta)^2 + (r\sin\theta\,d\phi)^2 = dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

$$\vec{\nabla} = \frac{\hat{\mathbf{r}}}{1}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial\phi}$$

$$\vec{\nabla}V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial V}{\partial\theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial V}{\partial\phi}$$

$$\vec{\nabla}\bullet\vec{\mathbf{F}} = \frac{1}{1\times r\times r\sin\theta}\left(\frac{\partial(r\times r\sin\theta f_r)}{\partial r} + \frac{\partial(r\sin\theta\times 1\,f_\theta)}{\partial\theta} + \frac{\partial(1\times r\,f_\phi)}{\partial\phi}\right)$$

$$= \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial(r^2 f_r)}{\partial r} + r\frac{\partial(\sin\theta\,f_\theta)}{\partial\theta} + r\frac{\partial f_\phi}{\partial\phi}\right)$$

$$= \frac{\partial f_r}{\partial r} + \frac{2}{r}f_r + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{\cot\theta}{r}f_\theta + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$$

$$\vec{\nabla} \times \vec{\mathbf{F}} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\,\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ f_r & r\,f_\theta & r\sin\theta\,f_\phi \end{vmatrix}$$

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{r^2 \sin\theta}{1}\frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{r\sin\theta}{r}\frac{\partial V}{\partial \theta}\right) + \frac{\partial}{\partial \phi}\left(\frac{r}{r\sin\theta}\frac{\partial V}{\partial \phi}\right)\right)$$

$$= \frac{1}{r^2 \sin\theta}\left(\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{\partial V}{\partial \phi}\right)\right)$$

$$= \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$$

All of the above are undefined on the z-axis ($\sin\,\theta = 0$), where there is a coordinate singularity. However, by taking the limit as $\sin\theta \to 0$, we may obtain well-defined values for some or all of the above expressions.

Example 1.7.2

A vector field has the equation, in cylindrical polar coordinates (ρ, ϕ, z),

$$\vec{\mathbf{F}} = \frac{k}{\rho^n}\hat{\mathbf{e}}_\rho = \frac{k}{\rho^n}\hat{\boldsymbol{\rho}}$$

Find the divergence of **F** and the value of n for which the divergence vanishes for all $\rho > 0$.

In cylindrical polar coordinates,

$$\operatorname{div}\vec{\mathbf{F}} = \frac{\partial F_\rho}{\partial \rho} + \frac{F_\rho}{\rho} + \cancel{\frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi}} + \cancel{\frac{\partial F_z}{\partial z}}$$

$$F_\rho = k\rho^{-n}\,, \quad F_\phi = F_z = 0$$

$$\Rightarrow \quad \operatorname{div}\vec{\mathbf{F}} = -kn\rho^{-n-1} + \frac{k\rho^{-n}}{\rho} + 0 + 0 = \underline{\underline{k(1-n)\rho^{-n-1}}}$$

and clearly $\operatorname{div}\vec{\mathbf{F}} = 0$ when $\underline{\underline{n = 1}}$.

$\vec{\mathbf{F}} = \dfrac{k}{\rho}\hat{\boldsymbol{\rho}}$ is therefore a source-free field everywhere except on the z axis.

<u>Example 1.7.3</u>

In spherical polar coordinates,

$$\vec{\mathbf{F}}(r,\theta,\phi) = f(\phi)\cot\theta\,\hat{\mathbf{r}} - 2f(\phi)\,\hat{\boldsymbol{\theta}} + g(r,\theta)\,\hat{\boldsymbol{\phi}}\ ,$$

where $f(\phi)$ is *any* differentiable function of ϕ only
and $g(r, \theta)$ is *any* differentiable function of r and θ only.

Find the divergence of **F**.

$F_r = f(\phi)\cot\theta\,, \quad F_\theta = -2f(\phi)\,, \quad F_\phi = g(r,\theta)$

For spherical polar coordinates,

$$\vec{\nabla}\bullet\vec{\mathbf{F}} = \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial\left(r^2 F_r\right)}{\partial r} + r\frac{\partial\left(\sin\theta\, F_\theta\right)}{\partial\theta} + r\frac{\partial F_\phi}{\partial\phi}\right)$$

$$\vec{\nabla}\bullet\vec{\mathbf{F}} = \frac{1}{r^2\sin\theta}\left(\sin\theta\frac{\partial\left(r^2 f(\phi)\cot\theta\right)}{\partial r} + r\frac{\partial\left(-2\sin\theta\, f(\phi)\right)}{\partial\theta} + r\frac{\partial g(r,\theta)}{\partial\phi}\right)$$

$$= \frac{f(\phi)\cot\theta}{r^2}\frac{\partial\left(r^2\right)}{\partial r} - \frac{2f(\phi)}{r\sin\theta}\frac{\partial(\sin\theta)}{\partial\theta} + 0$$

$$= \frac{2f(\phi)\cot\theta}{r} - \frac{2f(\phi)\cot\theta}{r} = 0$$

everywhere (except possibly on the z axis, where $r\sin\theta = 0$).

Example 1.7.4

Find $\operatorname{curl}\left(\sin\theta\left(\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\right)\right)$, where θ, ϕ are the two angular coordinates in the standard spherical polar coordinate system.

$$\vec{\nabla}\times\vec{\mathbf{F}} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ f_r & r\,f_\theta & r\sin\theta\,f_\phi \end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r\sin\theta\hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ 0 & r\sin\theta & r\sin^2\theta \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left(\left(2r\sin\theta\cos\theta - 0\right)\hat{\mathbf{r}} + r\left(0 - \sin^2\theta\right)\hat{\boldsymbol{\theta}} + r\sin\theta\left(\sin\theta - 0\right)\hat{\boldsymbol{\phi}}\right)$$

$$\therefore\quad \vec{\nabla}\times\vec{\mathbf{F}} = \underline{\underline{\frac{1}{r}\left(2\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\,\hat{\boldsymbol{\phi}}\right)}}$$

Central Force Law

If a potential function $V(x, y, z)$, (due solely to a point source at the origin) depends only on the distance r from the origin, then the functional form of the potential can be deduced. Using spherical polar coordinates:

$$V(r, \theta, \phi) = f(r)$$

$$\Rightarrow \quad \nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{df}{dr}\right) = \frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr}$$

But, in any regions not containing any sources of the vector field, the divergence of the vector field $\vec{\mathbf{F}} = \vec{\nabla} V$ (and therefore the Laplacian of the associated potential function V) must be zero. Therefore, for all $r \neq 0$,

$$\frac{d^2 f}{dr^2} + \frac{2}{r}\frac{df}{dr} = 0$$

Solve this ODE by reduction of order:

Let $y = \frac{df}{dr}$ then $\frac{dy}{dr} + \frac{2}{r} y = 0$

$$\Rightarrow \quad \int \frac{dy}{y} = -2\int \frac{dr}{r} \qquad \Rightarrow \quad \ln y = -2\ln r + C = \ln\left(Br^{-2}\right)$$

$$\Rightarrow \quad \frac{df}{dr} = y = Br^{-2}$$

$$\Rightarrow \quad f = \frac{Br^{-1}}{-1} + A$$

$$\therefore \quad V(r,\theta,\phi) = \underline{\underline{A - \frac{B}{r}}}$$

OR (a much faster solution!)

$$\nabla^2 V = 0 \qquad \Rightarrow \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \qquad \Rightarrow \quad r^2 \frac{dV}{dr} = B \quad (r \neq 0)$$

$$\Rightarrow \quad \frac{dV}{dr} = \frac{B}{r^2} \qquad \Rightarrow \quad V = A - \frac{B}{r}$$

Gravity is an example of a central force law, for which the potential function must be of the form $V(r,\theta,\phi) = A - \dfrac{B}{r}$. The zero point for the potential is usually set at infinity:

$$\lim_{r\to\infty} V = \lim_{r\to\infty}\left(A - \frac{B}{r}\right) = A = 0$$

The force per unit mass due to gravity from a point mass M at the origin is

$$\vec{\mathbf{F}} = -\vec{\nabla} V = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

But, in spherical polar coordinates,

$$\vec{\nabla} V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial V}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial V}{\partial \phi} = \hat{\mathbf{r}}\frac{dV}{dr} = \hat{\mathbf{r}}\frac{B}{r^2}$$

$$\Rightarrow \quad -\frac{GM}{r^2} = -\frac{B}{r^2} \quad \Rightarrow \quad B = GM$$

Therefore the gravitational potential function is

$$V(r) = -\frac{GM}{r}$$

The electrostatic potential function is similar, with a different constant of proportionality.

END OF CHAPTER 1