2. Surface Integrals

This chapter introduces the theorems of Green, Gauss and Stokes. Two different methods of integrating a function of two variables over a curved surface are developed.

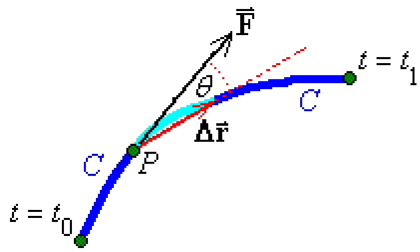
The sections in this chapter are:

2.1 Line Integrals

Two applications of line integrals are treated here: the evaluation of work done on a particle as it travels along a curve in the presence of a [vector field] force; and the evaluation of the location of the centre of mass of a wire.

Work done:

The work done by a force **F** in moving an elementary distance **Δr** along a curve *C* is approximately the product of the component of the force in the direction of **Δr** and the distance | **Δr** | travelled:



$$\Delta W \approx \vec{\mathbf{F}} \bullet \Delta \vec{\mathbf{r}} = F \cos\theta \left| \Delta \vec{\mathbf{r}} \right|$$

Integrating along the curve *C* yields the total work done by the force **F** in moving along the curve *C*:

$$W = \int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$$

$$= \int_C \left(f_1\, dx + f_2\, dy + f_3\, dz \right) = \int_{t_0}^{t_1} \left(f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt} + f_3 \frac{dz}{dt} \right) dt$$

$$\therefore \quad W = \int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_C \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt$$

Example 2.1.1

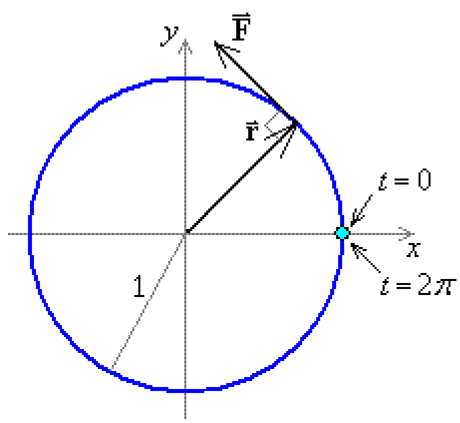
Find the work done by $\vec{\mathbf{F}} = \langle -y, x, z \rangle$ in moving around the curve C (defined in parametric form by $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$).

$$\vec{\mathbf{F}} = \langle -y, x, z \rangle \big|_C = \langle -\sin t, \cos t, 0 \rangle$$

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\langle \cos t, \sin t, 0 \rangle = \langle -\sin t, \cos t, 0 \rangle$$

$$\Rightarrow \quad \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt} = \sin^2 t + \cos^2 t + 0 = 1$$

$$\Rightarrow \quad W = \int_0^{2\pi} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt = \int_0^{2\pi} 1\, dt = \underline{\underline{2\pi}}$$



Note that $F_v = \vec{\mathbf{F}} \bullet \hat{\mathbf{v}} = \dfrac{\vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}}{\left| \frac{d\vec{\mathbf{r}}}{dt} \right|} = 1$ everywhere on the curve C, so that

$W = 1 \times C = 2\pi$ (the length of the path around the circle).

Also note that $\vec{\mathbf{F}} = \langle -y, x, z \rangle \quad \Rightarrow \quad \text{curl}\, \vec{\mathbf{F}} = 2\hat{\mathbf{k}}$ everywhere in $\mathbb{R}^3$.

The lesser curvature of the circular lines of force further away from the z axis is balanced exactly by the increased transverse force, so that curl **F** is the same in all of $\mathbb{R}^3$.

We shall see later (Stokes' theorem, page 2.40) that the work done is also the normal component of the curl integrated over the area enclosed by the closed curve C. In this case

$$W = \left(\vec{\nabla} \times \vec{\mathbf{F}} \bullet \hat{\mathbf{n}} \right) A = \left(2\hat{\mathbf{k}} \bullet \hat{\mathbf{k}} \right) \pi (1)^2 = 2\pi .$$

Example 2.1.1 (continued)

Example 2.1.2

Find the work done by $\vec{\mathbf{F}} = \langle x, y, z \rangle$ in moving around the curve C (defined in parametric form by $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t \le 2\pi$).

$$\vec{\mathbf{F}} = \langle x, y, z \rangle\big|_C = \langle \cos t, \sin t, 0 \rangle = \vec{\mathbf{r}}$$

$$\frac{d\vec{\mathbf{r}}}{dt} = \frac{d}{dt}\langle \cos t, \sin t, 0 \rangle = \langle -\sin t, \cos t, 0 \rangle$$

$$\Rightarrow \quad \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt} = -\cos t \sin t + \sin t \cos t + 0 = 0$$

$$\Rightarrow \quad W = \int_0^{2\pi} 0 \, dt = \underline{\underline{0}}$$

In this case, the force is orthogonal to the direction of motion at all times and no work is done.

If the initial and terminal points of a curve C are identical and the curve meets itself nowhere else, then the curve is said to be a **simple closed curve**.

Notation:

When C is a simple closed curve, write $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ as $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$.

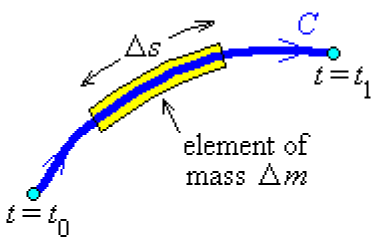
$\mathbf{F}$ is a **conservative vector field** if and only if $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 0$ for all simple closed curves C in the domain.

Be careful of where the endpoints are and of the order in which they appear (the orientation of the curve). The identity $\int_{t_0}^{t_1} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt \equiv -\int_{t_1}^{t_0} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt$ leads to the result

$$\ointclockwise_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = -\ointctrclockwise_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} \quad \forall \text{ simple closed curves } C$$

Another Application of Line Integrals: **The Mass of a Wire**

Let C be a segment $(t_0 \le t \le t_1)$ of wire of line density $\rho(x, y, z)$. Then



$$\Delta m \approx \rho(x, y, z)\, \Delta s$$

$$\Rightarrow \quad m = \int_C \rho\, ds = \int_C \rho \frac{ds}{dt}\, dt = \int_{t_0}^{t_1} \rho \frac{ds}{dt}\, dt$$

First moments about the coordinate planes:

$$\Delta\vec{\mathbf{M}} = \vec{\mathbf{r}}\, \Delta m \approx \rho \vec{\mathbf{r}}\, \Delta s \qquad \Rightarrow \quad \vec{\mathbf{M}} = \int_{t_0}^{t_1} \rho \vec{\mathbf{r}} \frac{ds}{dt}\, dt$$

The location $\langle \vec{\mathbf{r}} \rangle$ of the centre of mass of the wire is $\langle \vec{\mathbf{r}} \rangle = \frac{\vec{\mathbf{M}}}{m}$, where

$$\vec{\mathbf{M}} = \int_{t_0}^{t_1} \rho \vec{\mathbf{r}} \frac{ds}{dt}\, dt, \quad m = \int_{t_0}^{t_1} \rho \frac{ds}{dt}\, dt \quad \text{and} \quad \frac{ds}{dt} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

Example 2.1.3

Find the mass and centre of mass of a wire C (described in parametric form by $x = \cos t\,,\;\; y = \sin t\,,\;\; z = t\,,\;\; -\pi \le t \le \pi$) of line density $\rho = z^2$.

Let $c = \cos t\,,\;\; s = \sin t\,.$

[The shape of the wire is one revolution of a helix, aligned along the z axis, centre the origin.]

$$\vec{\mathbf{r}} = \langle c, s, t \rangle \quad \Rightarrow \quad \frac{d\vec{\mathbf{r}}}{dt} = \langle -s, c, 1 \rangle$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{(-s)^2 + c^2 + 1^2} = \sqrt{2}$$

$$\rho = z^2 = t^2$$

$$\Rightarrow \quad m = \int_C \rho\, ds = \int_{-\pi}^{\pi} \rho \frac{ds}{dt}\, dt = \sqrt{2}\int_{-\pi}^{\pi} t^2\, dt = \sqrt{2}\left[\frac{t^3}{3}\right]_{-\pi}^{\pi}$$

$$\Rightarrow \quad m = \underline{\underline{\frac{2}{3}\sqrt{2}\,\pi^3}}$$

$$\vec{\mathbf{M}} = \int_{t_0}^{t_1} \rho\,\vec{\mathbf{r}}\,\frac{ds}{dt}\, dt = \sqrt{2}\int_{-\pi}^{\pi} \langle t^2 c, t^2 s, t^3 \rangle\, dt$$

x component:

Integration by parts.

D		I
t^2		c
	+	
$2t$		s
	−	
2		$-c$
	+	
0		$-s$

$$\int t^2 c\, dt = \left[\left(t^2 - 2\right)s + 2tc\right]$$

$$\Rightarrow \quad \int_{-\pi}^{\pi} t^2 c\, dt = \left[\left(t^2 - 2\right)s + 2tc\right]_{-\pi}^{\pi}$$

$$= (0 - 2\pi) - (0 + 2\pi) = -4\pi$$

Example 2.1.3 (continued)

y component:

For all integrable functions $f(t)$ and for all constants a note that

$$\int_{-a}^{a} f(t)\,dt = \begin{cases} 0 & \text{if } f(t) \text{ is an ODD function} \\ 2\int_{0}^{a} f(t)\,dt & \text{if } f(t) \text{ is an EVEN function} \end{cases}$$

$t^2 \sin t$ is an odd function

$$\Rightarrow \int_{-\pi}^{\pi} t^2 s\, dt = 0$$

z component:

t^3 is also an odd function

$$\Rightarrow \int_{-\pi}^{\pi} t^3\, dt = 0$$

Therefore $\vec{\mathbf{M}} = -4\pi\sqrt{2}\,\hat{\mathbf{i}}$

$$\langle \vec{\mathbf{r}} \rangle = \frac{\vec{\mathbf{M}}}{m} = \frac{3}{2\pi^3\sqrt{2}} - 4\pi\sqrt{2}\,\hat{\mathbf{i}} = -\frac{6}{\pi^2}\,\hat{\mathbf{i}}$$

The centre of mass is therefore at $\underline{\underline{\left(-\frac{6}{\pi^2}, 0, 0\right)}}$

2.2 <u>Green's Theorem</u>

Some definitions:

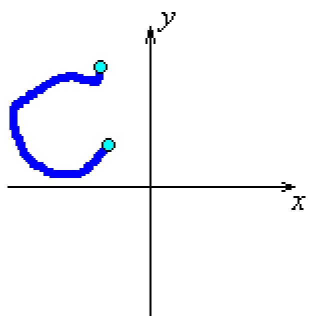
A curve C on $\mathbb{R}^2$ (defined in parametric form by $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}, \; a \le t \le b$) is **closed** iff $(x(a), y(a)) = (x(b), y(b))$.

The curve is **simple** iff $\vec{\mathbf{r}}(t_1) \ne \vec{\mathbf{r}}(t_2)$ for all t_1, t_2 such that $a < t_1 < t_2 < b$;
(that is, the curve neither touches nor intersects itself, except possibly at the end points).

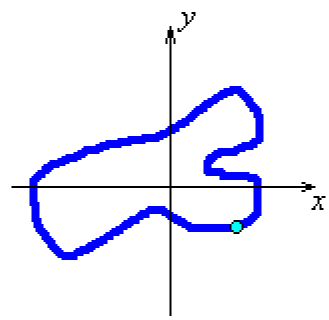
<u>Example 2.2.1</u>

Two simple curves:

open

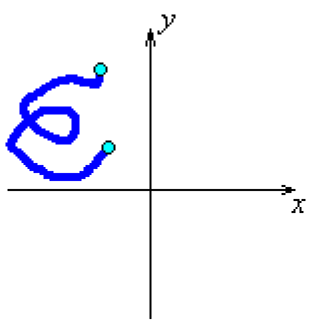


closed

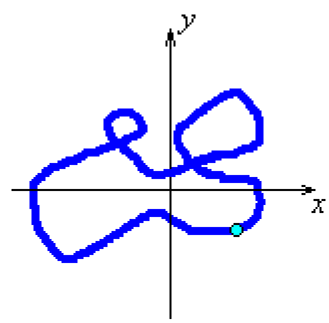


Two non-simple curves:

open



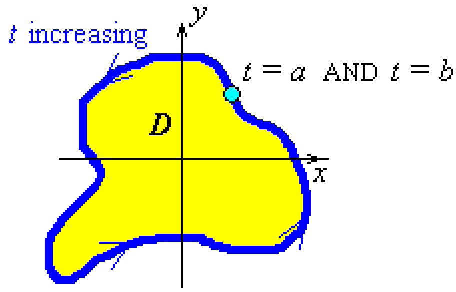
closed



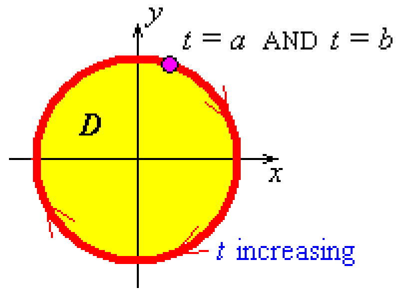
Orientation of closed curves:

A closed curve C has a positive orientation iff a point $\mathbf{r}(t)$ moves around C in an anticlockwise sense as the value of the parameter t increases.

Example 2.2.2



Positive orientation



Negative orientation

Let D be the finite region of $\mathbb{R}^2$ bounded by C. When a particle moves along a curve with positive orientation, D is always to the left of the particle.

For a simple closed curve C enclosing a finite region D of $\mathbb{R}^2$ and for any vector function $\vec{\mathbf{F}} = \langle f_1, f_2 \rangle$ that is differentiable everywhere on C and everywhere in D,
Green's theorem is valid:

$$\oint_C \vec{\mathbf{F}} \cdot \mathbf{d}\vec{\mathbf{r}} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

The region D is entirely in the xy-plane, so that the unit normal vector everywhere on D is $\mathbf{k}$. Let the differential vector $\mathbf{dA} = dA\,\mathbf{k}$, then Green's theorem can also be written as

$$\oint_C \vec{\mathbf{F}} \cdot \mathbf{d}\vec{\mathbf{r}} = \iint_D \left(\vec{\nabla} \times \vec{\mathbf{F}} \right) \cdot \hat{\mathbf{k}}\, dA = \iint_D \left(\text{curl}\, \vec{\mathbf{F}} \right) \cdot \mathbf{d}\vec{\mathbf{A}}$$

$$\vec{\mathbf{F}} \cdot \mathbf{d}\vec{\mathbf{r}} = \langle f_1, f_2 \rangle \cdot \langle dx, dy \rangle \quad \Rightarrow \quad \oint_C \left(f_1\, dx + f_2\, dy \right) = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

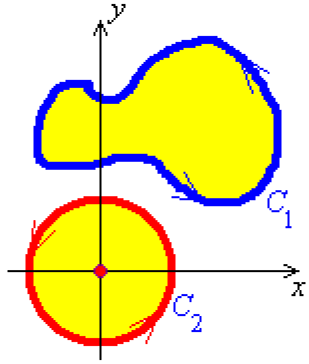
and

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = \det \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \\ f_1 & f_2 \end{bmatrix} = \det \begin{bmatrix} \vec{\nabla}^{\mathrm{T}} \\ \vec{\mathbf{F}}^{\mathrm{T}} \end{bmatrix} = z \text{ component of } \vec{\nabla} \times \vec{\mathbf{F}}$$

Green's theorem is valid if there are no singularities in D.

Example 2.2.3

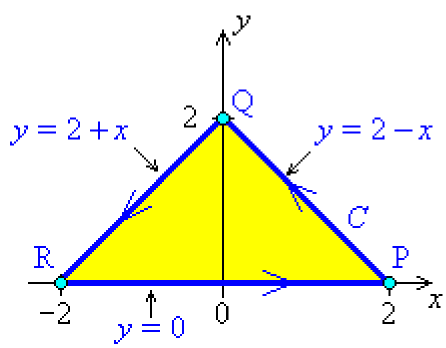
$\vec{\mathbf{F}} = \left\langle \frac{x}{r}, 0 \right\rangle$:



Green's theorem is valid for curve C_1 but not for curve C_2.
There is a singularity at the origin, which curve C_2 encloses.

Example 2.2.4

For $\vec{\mathbf{F}} = \langle x+y, x-y \rangle$ and C as shown, evaluate $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$.



$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{PQ} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{QR} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} + \int_{RP} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Example 2.2.4 (continued)

$$\vec{\mathbf{F}} = \langle x+y,\, x-y \rangle$$

Everywhere on the line segment from P to Q, $y = 2 - x$ (and the parameter t is just x)

$$\Rightarrow \quad \vec{\mathbf{r}} = \langle x,\, 2-x \rangle \quad \Rightarrow \quad \frac{d\vec{\mathbf{r}}}{dx} = \langle 1, -1 \rangle \quad \text{and} \quad \vec{\mathbf{F}} = \langle 2,\, 2x-2 \rangle$$

$$\Rightarrow \quad \int_{PQ} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_2^0 \big(2-(2x-2)\big)\,dx = \int_2^0 (4-2x)\,dx = \left[4x - x^2\right]_2^0$$

$$= (0-0) - (8-4) = -4$$

Everywhere on the line segment from Q to R, $y = 2 + x$

$$\Rightarrow \quad \vec{\mathbf{r}} = \langle x,\, 2+x \rangle \quad \Rightarrow \quad \frac{d\vec{\mathbf{r}}}{dx} = \langle 1, 1 \rangle \quad \text{and} \quad \vec{\mathbf{F}} = \langle 2x+2,\, -2 \rangle$$

$$\Rightarrow \quad \int_{QR} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^{-2} \big((2x+2)-2\big)\,dx = \int_0^{-2} 2x\,dx = \left[x^2\right]_0^{-2}$$

$$= 4 - 0 = 4$$

Everywhere on the line segment from R to P, $y = 0$

$$\Rightarrow \quad \vec{\mathbf{r}} = \langle x,\, 0 \rangle \quad \Rightarrow \quad \frac{d\vec{\mathbf{r}}}{dx} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{\mathbf{F}} = \langle x,\, x \rangle$$

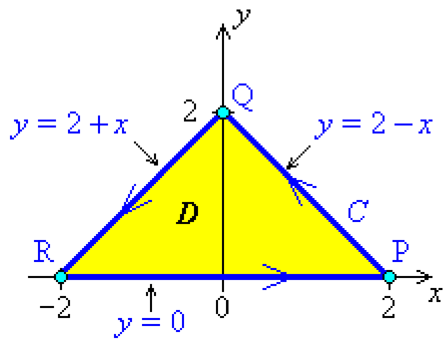
$$\Rightarrow \quad \int_{RP} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{-2}^{2} (x+0)\,dx = \int_{-2}^{2} x\,dx = \left[\frac{x^2}{2}\right]_{-2}^{2}$$

$$= 2 - 2 = 0$$

$$\Rightarrow \quad \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = -4 + 4 + 0 = \underline{\underline{0}}$$

Example 2.2.4 (continued)

OR use Green's theorem!



$$\det\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ f_1 & f_2 \end{bmatrix} = \frac{\partial}{\partial x}(x-y) - \frac{\partial}{\partial y}(x+y) = 1-1 = 0$$

everywhere on D

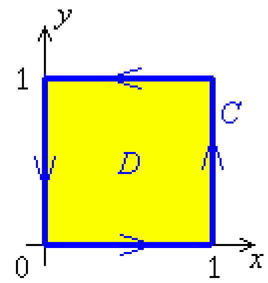
$$\Rightarrow \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA = \iint_D 0 \, dA = 0$$

By Green's theorem it then follows that

$$\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \underline{\underline{0}}$$

Example 2.2.5

Find the work done by the force $\vec{\mathbf{F}} = \langle xy,\, y^2 \rangle$ in one circuit of the unit square.



By Green's theorem,

$$W = \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

$$\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = \frac{\partial}{\partial x}\left(y^2\right) - \frac{\partial}{\partial y}(xy) = 0 - x$$

The region of integration is the square $0 < x < 1,\ 0 < y < 1$

$$\Rightarrow \quad W = \iint_D -x \, dA = \int_0^1 \int_0^1 (-x)\, dy\, dx$$

$$= -\int_0^1 x \left(\int_0^1 1\, dy \right) dx = -\int_0^1 x [y]_0^1 \, dx = -\int_0^1 x(1-0)\, dx$$

$$= -\left[\frac{x^2}{2} \right]_0^1 = -\frac{1}{2} + 0 = -\frac{1}{2}$$

Therefore

$$W = \underline{\underline{-\tfrac{1}{2}}}$$

The alternative method (using line integration instead of Green's theorem) would involve ***four*** line integrals, each with different integrands!

2.3 Path Independence

Gradient Vector Fields:

If $\vec{\mathbf{F}} = \vec{\nabla}\phi$, then $\vec{\mathbf{F}} = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle \Rightarrow \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = \phi_{yx} - \phi_{xy} \equiv 0$

(provided that the second partial derivatives are all continuous).

It therefore follows, for any closed curve C and twice differentiable potential function ϕ that

$$\boxed{\oint_C \vec{\nabla}\phi \bullet d\vec{\mathbf{r}} \equiv 0}$$

Path Independence

If $\vec{\mathbf{F}} = \vec{\nabla}\phi$ $\left(\text{or } \vec{\mathbf{F}} = -\vec{\nabla}\phi\right)$, then ϕ is a **potential function** for **F**.
Let the path C travel from point P_o to point P_1:

$$\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_C \vec{\nabla}\phi \bullet d\vec{\mathbf{r}} = \int_C \left(\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz \right) = \int_C d\phi$$

[chain rule]

$$= [\phi]_{P_0}^{P_1} = \phi(P_1) - \phi(P_0)$$

which is independent of the path C between the two points.

Therefore $\begin{pmatrix} \text{work done} \\ \text{by } \vec{\nabla}\phi \end{pmatrix} = \begin{pmatrix} \text{difference in } \phi \\ \text{between endpoints of } C \end{pmatrix}$

$$\Rightarrow \oint_C \vec{\nabla}\phi \bullet d\vec{\mathbf{r}} = \phi(P) - \phi(P) = 0$$

[work done = potential difference]

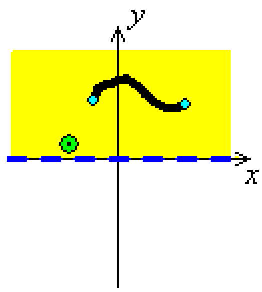
Domain

A region Ω of $\mathbb{R}^2$ is a **domain** if and only if

1) For all points P_o in Ω, there exists a circle, centre P_o, all of whose interior points are inside Ω; and
2) For all points P_o and P_1 in Ω, there exists a piecewise smooth curve C, entirely in Ω, from P_o to P_1.

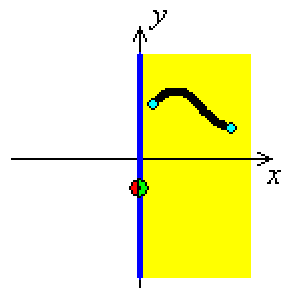
Example 2.3.1 Are these domains?

$\{ (x, y) \mid y > 0 \}$

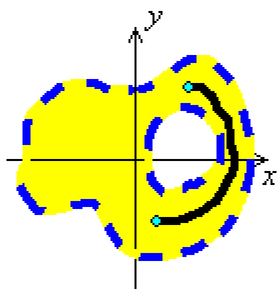


YES (and simply connected)

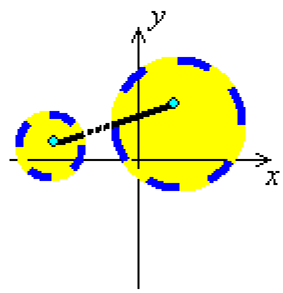
$\{ (x, y) \mid x \geq 0 \}$



NO



YES (but ***not*** simply connected)



NO

If a domain is not specified, then, by default, it is assumed to be all of $\mathbb{R}^2$.

When a vector field **F** is defined on a simply connected domain Ω, these statements are all equivalent (that is, **all** of them are true or all of them are false):

- $\vec{\mathbf{F}} = \vec{\nabla}\phi$ for some scalar field ϕ that is differentiable everywhere in Ω;
- **F** is conservative;
- $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$ is path-independent (has the same value no matter which path within Ω is chosen between the two endpoints, for any two endpoints in Ω);
- $\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \phi_{\text{end}} - \phi_{\text{start}}$ (for any two endpoints in Ω);
- $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 0$ for all closed curves C lying entirely in Ω;
- $\dfrac{\partial f_2}{\partial x} = \dfrac{\partial f_1}{\partial y}$ everywhere in Ω; and
- $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ everywhere in Ω (so that the vector field **F** is irrotational).

There must be no singularities anywhere in the domain Ω in order for the above set of equivalencies to be valid.

Example 2.3.2

Evaluate $\int_C \left((2x+y)\,dx + \left(x+3y^2\right)dy \right)$ where C is any piecewise-smooth curve from (0, 0) to (1, 2).

$\vec{\mathbf{F}} = \left\langle 2x+y,\, x+3y^2 \right\rangle$ is continuous everywhere in $\Omega = \mathbb{R}^2$

$$\frac{\partial f_2}{\partial x} = 1 = \frac{\partial f_1}{\partial y} \quad \Rightarrow \quad \vec{\mathbf{F}} \text{ is conservative and } \vec{\mathbf{F}} = \vec{\nabla}\phi$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial x} = 2x+y \quad \text{and} \quad \frac{\partial \phi}{\partial y} = x+3y^2$$

A potential function that has the correct first partial derivatives is $\phi = x^2 + xy + y^3$

$$\Rightarrow \quad \int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \Big[\ \phi\ \Big]_{(0,0)}^{(1,2)} = (1+2+8) - (0+0+0)$$

Therefore

$$\int_C \left((2x+y)\,dx + \left(x+3y^2\right)dy \right) = \underline{\underline{11}}$$

Example 2.3.3 (A Counterexample)

Evaluate $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}}$, where $\vec{\mathbf{F}} = \left\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2} \right\rangle$ and C is the unit circle, centre at the origin.

$\vec{\mathbf{F}}$ is continuous everywhere except (0, 0)

$\Rightarrow$ Ω is ***not*** simply connected. [Ω is all of $\mathbb{R}^2$ *except* (0, 0).]

$$\frac{\partial f_2}{\partial x} = \frac{x^2 - y^2}{\left(x^2+y^2\right)^2} = \frac{\partial f_1}{\partial y} \quad \text{everywhere in } \Omega$$

We cannot use Green's theorem, because $\vec{\mathbf{F}}$ is ***not*** continuous everywhere inside C (there is a singularity at the origin).

Let $c = \cos t$ and $s = \sin t$ then

$$\vec{\mathbf{r}} = \langle c, s \rangle \quad (0 \le t < 2\pi) \qquad \Rightarrow \quad \vec{\mathbf{r}}' = \langle -s, c \rangle$$

$$\vec{\mathbf{F}} = \left\langle \frac{s}{c^2+s^2}, \frac{-c}{c^2+s^2} \right\rangle = \langle s, -c \rangle$$

$$\Rightarrow \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^{2\pi} \left(-s^2 - c^2\right) dt = -\int_0^{2\pi} 1\, dt = -\left[t\right]_0^{2\pi}$$

Therefore

$$\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \underline{\underline{-2\pi}}$$

Note: $\oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} \neq 0$, but

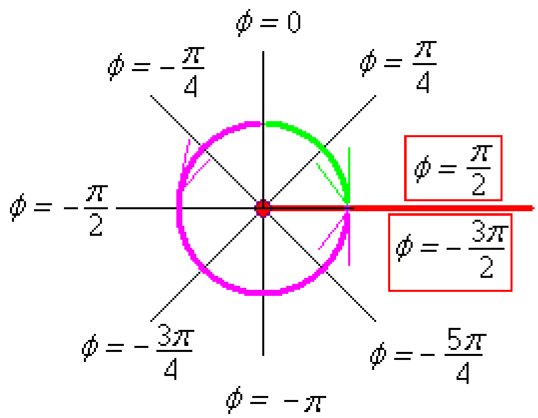
everywhere on Ω, $\vec{\mathbf{F}} = \vec{\nabla}\phi$, where $\phi = \text{Arctan}\left(\frac{x}{y}\right) + k$

The problem is that the arbitrary constant k is ill-defined.

Example 2.3.3 (continued)

Let us explore the case when $k = 0$.

Contour map of $\phi = \text{Arctan}\left(\frac{x}{y}\right) + 0$



We encounter a conflict in the value of the potential function ϕ.

Solution: Change the domain Ω to the simply connected domain

$$\Omega' = \left(\begin{array}{c} \mathbb{R}^2 \text{ except the} \\ \text{non-negative } x \text{ axis} \end{array} \right)$$

then the potential function ϕ can be well-defined, but no curve in Ω' can enclose the origin.

2.4 Surface Integrals - Projection Method

Surfaces in $\mathbb{R}^3$

In $\mathbb{R}^3$ a surface can be represented by a vector parametric equation

$$\vec{\mathbf{r}} = x(u,v)\,\hat{\mathbf{i}} + y(u,v)\,\hat{\mathbf{j}} + z(u,v)\,\hat{\mathbf{k}}$$

where u, v are **parameters**.

Example 2.4.1

The unit sphere, centre O, can be represented by

$$\vec{\mathbf{r}}(\theta,\phi) = \langle \sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta \rangle$$

$$0 \le \theta \le \pi \quad \text{and} \quad 0 \le \phi < 2\pi$$

$\uparrow$ declination $\uparrow$ azimuth

If every vertical line (parallel to the z-axis) in $\mathbb{R}^3$ meets the surface no more than once, then the surface can also be parameterized as

$$\vec{\mathbf{r}}(x, y) = \langle x,\ y,\ f(x, y) \rangle \qquad \text{or as} \quad z = f(x, y)$$

Example 2.4.2

$z = \sqrt{4 - x^2 - y^2}$, $\{(x, y) \mid x^2 + y^2 \le 4\}$ is a **hemisphere, centre O.**

A **simple surface** does not cross itself.
If the following condition is true:

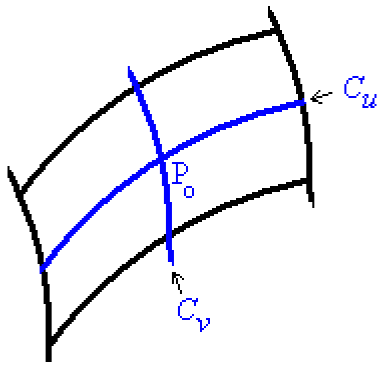
{ $\vec{\mathbf{r}}(u_1,v_1) = \vec{\mathbf{r}}(u_2,v_2) \;\Rightarrow\; (u_1,v_1) = (u_2,v_2)$ for all pairs of points in the domain}

then the surface is simple.

The converse of this statement is not true.
This condition is sufficient, but it is not necessary for a surface to be simple.
The condition may fail on a simple surface at coordinate singularities. For example, one of the angular parameters of the polar coordinate systems is undefined everywhere on the z-axis, so that spherical polar (2, 0, 0) and (2, 0, π) both represent the same Cartesian point (0, 0, 2). Yet a sphere remains simple at its z-intercepts.

Tangent and Normal Vectors to Surfaces

A surface S is represented by $\mathbf{r}(u, v)$. Examine the neighbourhood of a point P_o at $\mathbf{r}(u_o, v_o)$. Hold parameter v constant at v_o (its value at P_o) and allow the other parameter u to vary. This generates a slice through the two-dimensional surface, namely a one-dimensional curve C_u containing P_o and represented by a vector parametric equation $\vec{\mathbf{r}} = \vec{\mathbf{r}}(u, v_o)$ with only one freely-varying parameter (u).

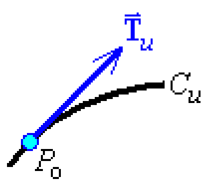


$C_u : \quad \vec{\mathbf{r}}(u, v_o)$

$C_v : \quad \vec{\mathbf{r}}(u_o, v)$

If, instead, u is held constant at u_o and v is allowed to vary, we obtain a different slice containing P_o, the curve $C_v : \quad \vec{\mathbf{r}}(u_o, v)$.

On each curve a unique tangent vector can be defined.



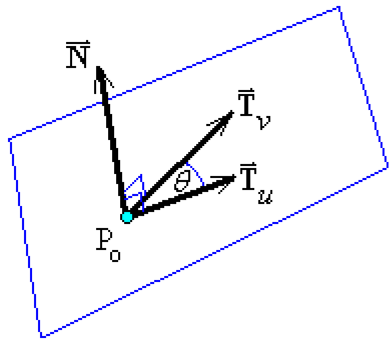
At all points along C_u, a tangent vector is defined by $\vec{\mathbf{T}}_u = \dfrac{\partial}{\partial u}\left(\vec{\mathbf{r}}(u, v_o)\right)$.

[Note that this is not necessarily a *unit* tangent vector.]

At P_o the tangent vector becomes $\vec{\mathbf{T}}_u\big|_{P_o} = \dfrac{\partial}{\partial u}\left(\vec{\mathbf{r}}(u_o, v_o)\right)$.

Similarly, along the other curve C_v, the tangent vector at P_o is $\vec{\mathbf{T}}_v\big|_{P_o} = \dfrac{\partial}{\partial v}\left(\vec{\mathbf{r}}(u_o, v_o)\right)$.

If the two tangent vectors are not parallel and neither of these tangent vectors is the zero vector, then they define the orientation of tangent plane to the surface at P_o.



A normal vector to the tangent plane is $\vec{\mathbf{N}} = \vec{\mathbf{T}}_u \times \vec{\mathbf{T}}_v = \left. \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right|_{(u_o, v_o)}$

$$= \det \left. \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix} \right|_{(u_o, v_o)} = \left. \left\langle \frac{\partial(y,z)}{\partial(u,v)}, \frac{\partial(z,x)}{\partial(u,v)}, \frac{\partial(x,y)}{\partial(u,v)} \right\rangle \right|_{(u_o, v_o)},$$

where $\frac{\partial(x,y)}{\partial(u,v)}$ is the **Jacobian** $\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix}$.

Cartesian parameters

With $u = x, \quad v = y, \quad z = f(x, y)$, the components of the normal vector $\vec{\mathbf{N}} = N_1 \hat{\mathbf{i}} + N_2 \hat{\mathbf{j}} + N_3 \hat{\mathbf{k}}$ are:

$$N_1 = \frac{\partial(y,z)}{\partial(x,y)} = \begin{vmatrix} 0 & \frac{\partial f}{\partial x} \\ 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x} \qquad N_2 = \frac{\partial(z,x)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial f}{\partial x} & 1 \\ \frac{\partial f}{\partial y} & 0 \end{vmatrix} = -\frac{\partial f}{\partial y}$$

$$N_3 = \frac{\partial(x,y)}{\partial(x,y)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$\Rightarrow$ a normal vector to the surface $z = f(x, y)$ at (x_o, y_o, z_o) is

$$\boxed{\vec{\mathbf{N}} = \left. \left\langle -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, +1 \right\rangle \right|_{(x_o, y_o)}}$$

If the normal vector **N** is continuous and non-zero over all of the surface *S*, then the surface is said to be **smooth**.

Example 2.4.3

A sphere is smooth.

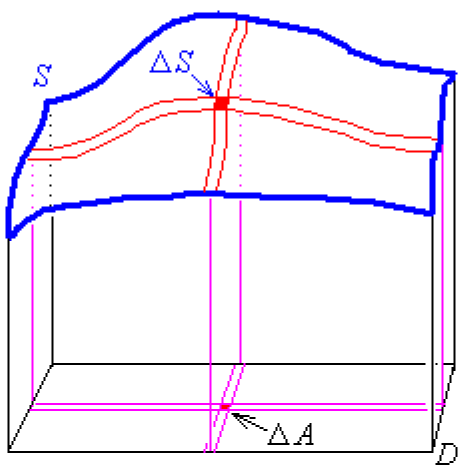
A cube is **piecewise smooth** (six smooth faces)

A cone is **not smooth** ($\vec{\mathbf{N}}$ is undefined at the apex)

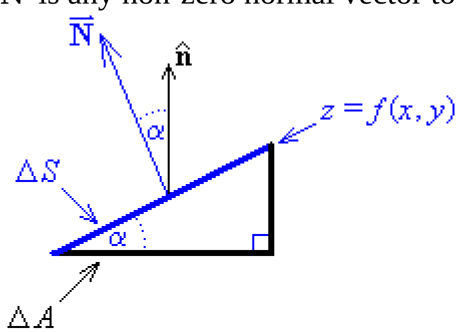
Surface Integrals (Projection Method)

This method is suitable mostly for surfaces which can be expressed easily in the Cartesian form $z = f(x, y)$.

The plane region D is the projection of the surface $S : f(\mathbf{r}) = c$ onto a plane (usually the *xy*-plane) in a 1:1 manner.



The plane containing D has a constant unit normal $\hat{\mathbf{n}}$.
$\vec{\mathbf{N}}$ is any non-zero normal vector to the surface S.



$$\Delta A = \Delta S \cos\alpha$$

but

$$\cos\alpha = \frac{\left|\vec{\mathbf{N}}\bullet\hat{\mathbf{n}}\right|}{\left|\vec{\mathbf{N}}\right|}$$

$$\Rightarrow \quad \iint_S dS = \iint_D \frac{|\vec{\mathbf{N}}|}{|\vec{\mathbf{N}} \bullet \hat{\mathbf{n}}|} \, dA$$

and

$$\iint_S g(\vec{\mathbf{r}}) \, dS = \iint_D g(\vec{\mathbf{r}}) \frac{|\vec{\mathbf{N}}|}{|\vec{\mathbf{N}} \bullet \hat{\mathbf{n}}|} \, dA$$

For $z = f(x, y)$ and D = a region of the xy-plane,

$$\vec{\mathbf{N}} = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle \quad \text{and} \quad \hat{\mathbf{n}} = \hat{\mathbf{k}}$$

$$\Rightarrow \quad |\vec{\mathbf{N}} \bullet \hat{\mathbf{n}}| = 1 \quad \text{and}$$

$$\boxed{\iint_S g(\vec{\mathbf{r}}) \, dS = \iint_D g(\vec{\mathbf{r}}) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \; dA}$$

which is the projection method of integration of $g(x, y, z)$ over the surface $z = f(x, y)$.

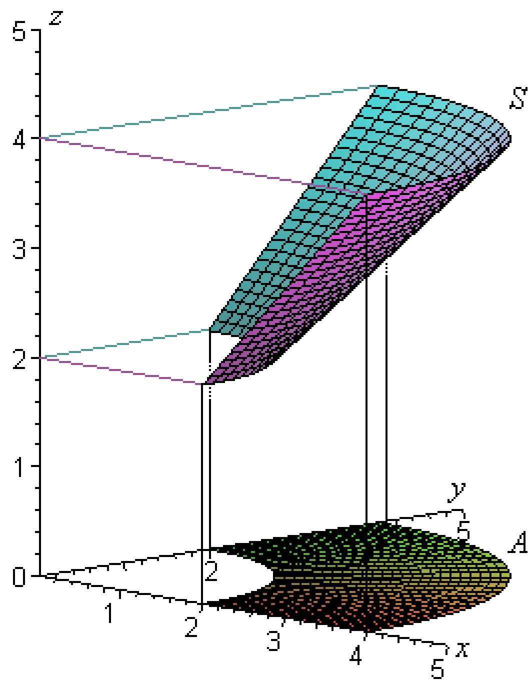
Advantage: Region D can be geometrically simple (often a rectangle in $\mathbb{R}^2$).

Disadvantage: Finding a suitable D (and/or a suitable 1:1 projection) can be difficult.

You may have to split the surface into pieces (such as splitting a sphere into two hemispheres) in order to obtain separate 1:1 projections. The projection fails if part of the surface is vertical (such as a vertical cylinder onto the *xy* plane).

Example 2.4.4

Evaluate $\iint_S z\,dS$, where the surface S is the section of the cone $z^2 = x^2 + y^2$ in the first octant, between z = 2 and z = 4.



$z^2 = x^2 + y^2$

$$\Rightarrow \quad 2z\frac{\partial z}{\partial x} = 2x + 0$$

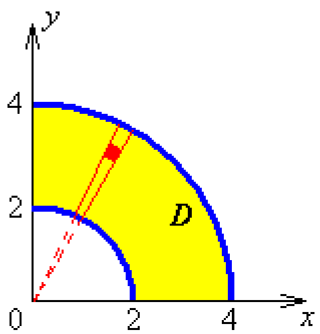
$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{x}{z} = \frac{x}{\sqrt{x^2+y^2}}$$

By symmetry,

$$\frac{\partial z}{\partial y} = \frac{y}{z} = \frac{y}{\sqrt{x^2+y^2}}$$

$$dS = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\;dA = \sqrt{\left(\frac{x^2}{z^2}\right)+\left(\frac{y^2}{z^2}\right)+1}\;dA = \sqrt{\left(\frac{z^2}{z^2}\right)+1}\;dA = \sqrt{2}\,dA$$

Use the polar form for dA :



$dA = r\,dr\,d\theta\,, \qquad 2 \le r \le 4\,, \quad 0 \le \theta \le \frac{\pi}{2}$

$r = \sqrt{x^2+y^2} = z$

$$\Rightarrow \quad \iint_S z\,dS = \int_0^{\pi/2}\int_2^4 r\sqrt{2}\,r\,dr\,d\theta$$

Example 2.4.4 (continued)

$$\Rightarrow \quad \iint_S z\, dS \;=\; \sqrt{2}\int_0^{\pi/2} 1\, d\theta \cdot \int_2^4 r^2\, dr \;=\; \sqrt{2}\Big[\;\theta\;\Big]_0^{\pi/2}\left[\frac{r^3}{3}\right]_2^4$$

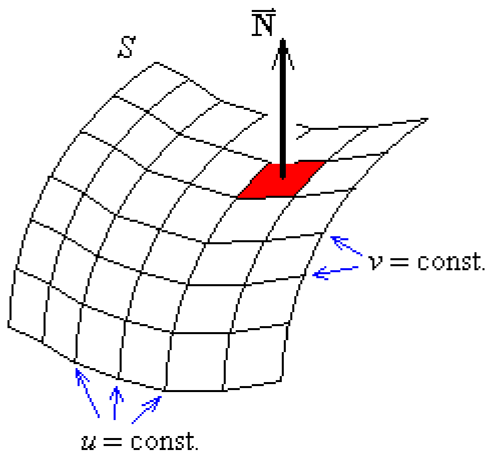
$$= \sqrt{2}\left(\frac{\pi}{2} - 0\right)\left(\frac{64}{3} - \frac{8}{3}\right) \;=\; \frac{\pi\sqrt{2}}{2}\cdot\frac{56}{3}$$

$$\Rightarrow \quad \iint_S z\, dS \;=\; \underline{\underline{\frac{28\pi\sqrt{2}}{3}}}$$

2.5 Surface Integrals - Surface Method

When a surface S is defined in a vector parametric form $\mathbf{r} = \mathbf{r}(u, v)$, one can lay a coordinate grid (u, v) down on the surface S.

A normal vector everywhere on S is $\vec{\mathbf{N}} = \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}$.



$$dS = \left| \mathbf{d}\vec{\mathbf{S}} \right| = \left| \vec{\mathbf{N}} \right| du\, dv$$

$$\iint_S g(\vec{\mathbf{r}})\, dS = \iint_S g(\vec{\mathbf{r}}) \left| \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right| du\, dv$$

Advantage:

- only one integral to evaluate

Disadvantage:

- it is often difficult to find optimal parameters (u, v).

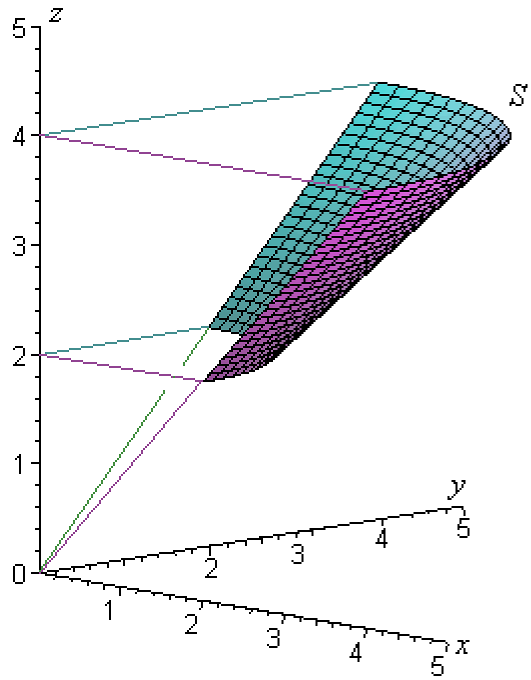
The total flux of a vector field $\vec{\mathbf{F}}$ through a surface S is

$$\Phi = \iint_S \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}} \bullet \hat{\mathbf{N}}\, dS = \iint_S \vec{\mathbf{F}} \bullet \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\, du\, dv$$

(which involves the scalar triple product $\vec{\mathbf{F}} \bullet \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}$).

<u>Example 2.5.1</u>: (same as Example 2.4.4, but using the surface method).

Evaluate $\iint_S z\,dS$, where the surface S is the section of the cone $z^2 = x^2 + y^2$ in the first octant, between $z = 2$ and $z = 4$.



Choose a convenient parametric net:

$$u = r = \sqrt{x^2 + y^2} = z$$

and

$$v = \theta$$

then

$$\vec{\mathbf{r}} = \langle r\cos\theta, r\sin\theta, r \rangle$$
$$\left(2 \le r \le 4, \quad 0 \le \theta \le \tfrac{\pi}{2}\right)$$

$$\Rightarrow \quad \frac{\partial \vec{\mathbf{r}}}{\partial r} = \langle \cos\theta, \sin\theta, 1 \rangle$$

and $\dfrac{\partial \vec{\mathbf{r}}}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$\Rightarrow \quad \vec{\mathbf{N}} = \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \pm\langle -r\cos\theta, -r\sin\theta, r \rangle$$

$$\Rightarrow \quad N = \left|\vec{\mathbf{N}}\right| = r\sqrt{\cos^2\theta + \sin^2\theta + 1} = r\sqrt{2}$$

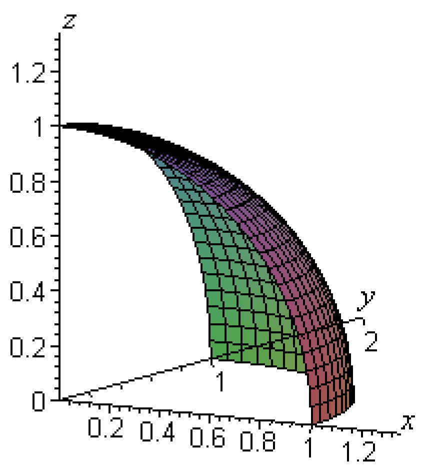
$$\Rightarrow \quad \iint_S z\,dS = \iint_S z\,N\,dr\,d\theta = \int_0^{\pi/2}\int_2^4 r\sqrt{2}\;r\,dr\,d\theta \quad \text{(as before)}$$

$$\Rightarrow \quad \iint_S z\,dS = \underline{\underline{\frac{28\pi\sqrt{2}}{3}}}$$

Just as we used line integrals to find the mass and centre of mass of [one dimensional] wires, so we can use surface integrals to find the mass and centre of mass of [two dimensional] sheets.

Example 2.5.2

Find the centre of mass of the part of the unit sphere (of constant surface density) that lies in the first octant.



Cartesian equation of the sphere:

$$x^2 + y^2 + z^2 = 1; \qquad x > 0,\ y > 0,\ z > 0$$

The radius of the sphere is $r = 1$.

For the parametric net, use the two angular coordinates of the spherical polar coordinate system (r, θ, ϕ).

$$\begin{aligned} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi \\ z &= \cos\theta \end{aligned} \qquad \begin{aligned} &0 < \theta < \tfrac{\pi}{2} \\ &0 < \phi < \tfrac{\pi}{2} \end{aligned}$$

$$\Rightarrow \quad \frac{\partial \vec{\mathbf{r}}}{\partial \theta} = \langle \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta \rangle$$

and $$\frac{\partial \vec{\mathbf{r}}}{\partial \phi} = \langle -\sin\theta\sin\phi, \sin\theta\cos\phi, 0 \rangle$$

$$\Rightarrow \quad \vec{\mathbf{N}} = \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \pm \left\langle \sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta\left(\cos^2\phi + \sin^2\phi\right) \right\rangle$$

$$= \pm\sin\theta \langle \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta \rangle = \pm\sin\theta\,\vec{\mathbf{r}}$$

The outward normal is clearly $\vec{\mathbf{N}} = +\sin\theta\,\vec{\mathbf{r}}$

$$\Rightarrow \quad N = \left|\vec{\mathbf{N}}\right| = \left|\sin\theta\right|\left|\vec{\mathbf{r}}\right| = \sin\theta$$

Example 2.5.2 (continued)

Mass: $m = \iint_S \rho \, dS = \iint_S \rho \left| \vec{\mathbf{N}} \right| d\theta \, d\phi$

$$= \rho \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \, d\theta \, d\phi$$

$$= \rho \int_0^{\pi/2} \sin\theta \, d\theta \cdot \int_0^{\pi/2} d\phi = \rho \left[-\cos\theta \right]_0^{\pi/2} \cdot \left[\phi \right]_0^{\pi/2}$$

$$= \rho \left(0 + 1 \right) \left(\tfrac{\pi}{2} - 0 \right)$$

$$\therefore \quad m = \frac{\rho\pi}{2}$$

OR

Note that the mass of a complete spherical shell of radius r and constant density ρ is $4\pi r^2 \rho$. Therefore the mass of one eighth of a shell of radius 1 is $\dfrac{4\rho\pi}{8} = \dfrac{\rho\pi}{2}$.

By symmetry, the three Cartesian coordinates of the centre of mass are all equal: $\bar{x} = \bar{y} = \bar{z}$.

Taking moments about the *xy* plane:

$$M = \iint_S z \, \rho \, dS = \rho \int_0^{\pi/2} \int_0^{\pi/2} \left(\cos\theta \right) \sin\theta \, d\theta \, d\phi$$

$$= \rho \int_0^{\pi/2} \frac{1}{2} \sin 2\theta \, d\theta \cdot \int_0^{\pi/2} d\phi = \rho \left[-\frac{\cos 2\theta}{4} \right]_0^{\pi/2} \cdot \left[\phi \right]_0^{\pi/2}$$

$$= \rho \left(\frac{1}{4} + \frac{1}{4} \right) \cdot \left(\frac{\pi}{2} - 0 \right) = \frac{1}{2} \cdot \frac{\pi\rho}{2} = \frac{1}{2} m$$

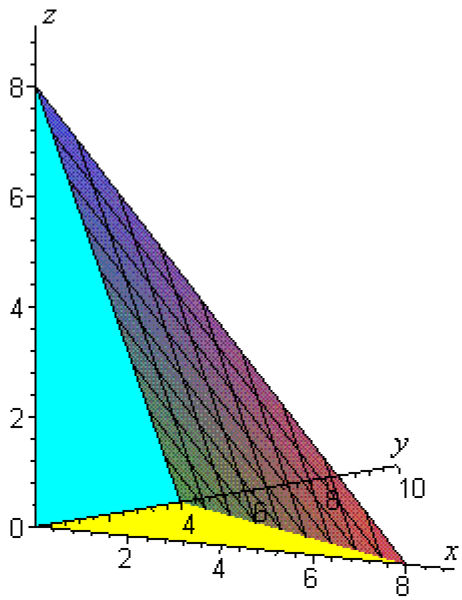
$$\Rightarrow \quad \bar{z} = \frac{M}{m} = \frac{1}{2}$$

Therefore the centre of mass is at

$$\left(\bar{x}, \bar{y}, \bar{z} \right) = \underline{\underline{\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2} \right)}}$$

Example 2.5.3

Find the flux of the field $\vec{\mathbf{F}} = \langle x, y, -z \rangle$ across that part of $x + 2y + z = 8$ that lies in the first octant.



$\vec{\mathbf{N}} = \langle 1, 2, 1 \rangle$

The Cartesian coordinates x, y will serve as parameters for the surface:

$$\vec{\mathbf{r}} = \langle x, y, 8 - x - 2y \rangle$$

$$\Rightarrow \quad \frac{\partial \vec{\mathbf{r}}}{\partial x} = \langle 1, 0, -1 \rangle$$

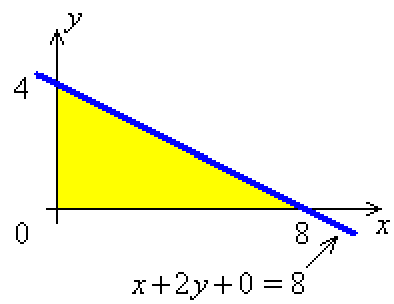
and $\dfrac{\partial \vec{\mathbf{r}}}{\partial y} = \langle 0, 1, -2 \rangle$

$$\Rightarrow \quad \vec{\mathbf{N}} = \pm \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -1 \\ 0 & 1 & -2 \end{vmatrix} = \pm \langle 1, 2, 1 \rangle$$

Choose $\vec{\mathbf{N}}$ to point "outwards".

Range of parameter values:

In the *xy* plane:



$$\Rightarrow \quad 0 \le x \le 8 \quad \text{and} \quad 0 \le y \le 4$$

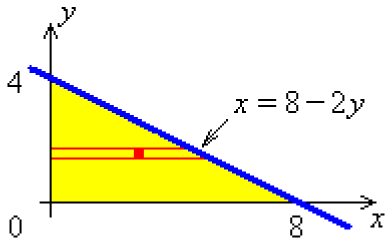
But the area is a triangle, ***not*** a rectangle, so these inequalities do not provide the correct limits for the inner integral.

Example 2.5.3 (continued)

Net flux = $\Phi = \iint_S \vec{\mathbf{F}} \cdot \mathbf{d}\vec{\mathbf{S}} = \iint_S F_N \, dS = \iint_S \left(\vec{\mathbf{F}} \cdot \hat{\mathbf{N}}\right)\left(\left|\vec{\mathbf{N}}\right| dA\right) = \iint_S \vec{\mathbf{F}} \cdot \vec{\mathbf{N}} \, dA$

(where $dA = dx\,dy$)

$$\vec{\mathbf{F}} \cdot \vec{\mathbf{N}} = \langle x, y, -(8-x-2y)\rangle \cdot \langle 1, 2, 1\rangle = x + 2y - 8 + x + 2y = 2(x+2y-4)$$



$$\Rightarrow \quad \Phi = \int_0^4 \int_0^{8-2y} 2(x+2y-4)\,dx\,dy$$

$$= 2\int_0^4 \left[\frac{x^2}{2} + 2xy - 4x\right]_{x=0}^{8-2y} dy$$

$$= 2\int_0^4 \left(\left(\frac{(8-2y)^2}{2} + 2(8-2y)y - 4(8-2y)\right) - (0+0+0)\right) dy$$

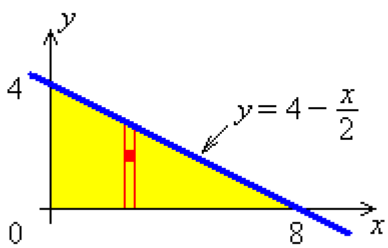
$$= 2\int_0^4 \left(32 - 16y + 2y^2 + 16y - 4y^2 - 32 + 8y\right) dy$$

$$= 4\int_0^4 \left(4y - y^2\right) dy = 4\left[2y^2 - \frac{y^3}{3}\right]_0^4 = 4\left(\left(32 - \frac{64}{3}\right) - (0-0)\right)$$

Therefore the net flux is

$$\Phi = \underline{\underline{\frac{128}{3}}}$$

The iteration could be taken in the other order:



$$\Phi = \int_0^8 \int_0^{4-x/2} 2(x+2y-4)\,dy\,dx$$

$$= 2\int_0^8 \left[xy + xy^2 - 4y\right]_{y=0}^{4-x/2} dx$$

$$= \ldots = \int_0^8 \left(4x - \tfrac{1}{2}x^2\right) dx = \ldots = \frac{128}{3}$$

Example 2.5.4

Find the total flux Φ of the vector field $\vec{\mathbf{F}} = z\hat{\mathbf{k}}$ through the simple closed surface S

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Use the parametric grid (θ, ϕ), such that the displacement vector to any point on the ellipsoid is

$$\vec{\mathbf{r}} = \langle a\sin\theta\cos\phi,\ b\sin\theta\sin\phi,\ c\cos\theta \rangle$$

This grid is a generalisation of the spherical polar coordinate grid and covers the entire surface of the ellipsoid for $0 \le \theta \le \pi,\ 0 \le \phi < 2\pi$.

One can verify that $x = a\sin\theta\cos\phi,\ \ y = b\sin\theta\sin\phi,\ \ z = c\cos\theta$ does lie on the ellipsoid

$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ for all values of (θ, ϕ):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{a^2\sin^2\theta\cos^2\phi}{a^2} + \frac{b^2\sin^2\theta\sin^2\phi}{b^2} + \frac{c^2\cos^2\theta}{c^2}$$

$$= \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = \sin^2\theta\left(\cos^2\phi + \sin^2\phi\right) + \cos^2\theta$$

$$= \sin^2\theta + \cos^2\theta = 1 \qquad \forall\theta \text{ and } \forall\phi$$

The tangent vectors along the coordinate curves ϕ = constant and θ = constant are

$$\frac{d\vec{\mathbf{r}}}{d\theta} = \langle a\cos\theta\cos\phi,\ b\cos\theta\sin\phi,\ -c\sin\theta \rangle \quad \text{and}$$

$$\frac{d\vec{\mathbf{r}}}{d\phi} = \langle -a\sin\theta\sin\phi,\ b\sin\theta\cos\phi,\ 0 \rangle.$$

The normal vector at every point on the ellipsoid follows:

$$\vec{\mathbf{N}} = \frac{d\vec{\mathbf{r}}}{d\theta} \times \frac{d\vec{\mathbf{r}}}{d\phi} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a\cos\theta\cos\phi & b\cos\theta\sin\phi & -c\sin\theta \\ -a\sin\theta\sin\phi & b\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \left\langle bc\sin^2\theta\cos\phi,\ ac\sin^2\theta\sin\phi,\ ab\sin\theta\cos\theta\left(\cos^2\phi + \sin^2\phi\right) \right\rangle$$

(and this vector points away from the origin).

Example 2.5.4 (continued)

On the ellipsoid, $\vec{\mathbf{F}} = z\,\hat{\mathbf{k}} = c\cos\theta\,\hat{\mathbf{k}}$

$$\Rightarrow \quad \vec{\mathbf{F}} \bullet \vec{\mathbf{N}} = c\cos\theta\left(ab\sin\theta\cos\theta\right) = abc\sin\theta\cos^2\theta$$

The total flux of $\vec{\mathbf{F}}$ through the surface S is therefore

$$\Phi = \oiint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \int_0^{2\pi}\int_0^{\pi} \vec{\mathbf{F}} \bullet \vec{\mathbf{N}}\, d\theta\, d\phi = abc\int_0^{2\pi}\int_0^{\pi} \sin\theta\cos^2\theta\, d\theta\, d\phi$$

Let $u = \cos\theta$, then $du = -\sin\theta\, d\theta$ and $\theta = 0 \Rightarrow u = +1$, $\theta = \pi \Rightarrow u = -1$

$$\Rightarrow \quad \oiint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = abc\int_0^{2\pi} 1\, d\phi \cdot \int_{+1}^{-1} -u^2\, du = abc\left[\phi\right]_0^{2\pi}\left[\frac{-u^3}{3}\right]_{+1}^{-1}$$

$$= abc\left(2\pi - 0\right)\left(+\frac{1}{3} + \frac{1}{3}\right) \quad \Rightarrow$$

$$\Phi = \underline{\underline{\frac{4\pi abc}{3}}}$$

For vector fields $\mathbf{F}(\mathbf{r})$,

Line integral: $\displaystyle\int_C \vec{\mathbf{F}}\bullet\mathbf{d}\vec{\mathbf{r}}$

Surface integral:

$$\iint_S \vec{\mathbf{F}}(\vec{\mathbf{r}})\bullet\mathbf{d}\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}}(\vec{\mathbf{r}})\bullet\widehat{\mathbf{N}}\,dS = \iint_S \vec{\mathbf{F}}\bullet\vec{\mathbf{N}}\,du\,dv = \pm\iint_S \vec{\mathbf{F}}\bullet\frac{\partial\vec{\mathbf{r}}}{\partial u}\times\frac{\partial\vec{\mathbf{r}}}{\partial v}\,du\,dv$$

On a closed surface, take the sign such that $\vec{\mathbf{N}}$ points **outward**.

Some Common Parametric Nets

1) The circular plate $(x-x_o)^2 + (y-y_o)^2 \le a^2$ in the plane $z = z_o$.
Let the parameters be r, θ where $0 < r \le a$, $0 \le \theta < 2\pi$
$x = x_o + r\cos\theta$, $y = y_o + r\sin\theta$, $z = z_o$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial r}\times\frac{\partial\vec{\mathbf{r}}}{\partial\theta}\right) = \pm\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \cos\theta & \sin\theta & 0\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix} = \pm r\,\hat{\mathbf{k}}$$

2) The circular cylinder $(x-x_o)^2 + (y-y_o)^2 = a^2$ with $z_o \le z \le z_1$.
Let the parameters be z, θ where $z_o \le z \le z_1$, $0 \le \theta < 2\pi$
$x = a\cos\theta$, $y = a\sin\theta$, $z = z$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial z}\times\frac{\partial\vec{\mathbf{r}}}{\partial\theta}\right) = \pm\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & 0 & 1\\ -a\sin\theta & a\cos\theta & 0\end{vmatrix} = \pm\left(-a\cos\theta\,\hat{\mathbf{i}} - a\sin\theta\,\hat{\mathbf{j}}\right)$$

Outward normal: $\vec{\mathbf{N}} = a\cos\theta\,\hat{\mathbf{i}} + a\sin\theta\,\hat{\mathbf{j}}$

3) The frustrum of the circular cone $w - w_o = a\sqrt{(u-u_o)^2 + (v-v_o)^2}$ where
$w_1 \le w \le w_2$ and $w_o \le w_1$. Let the parameters here be r, θ where

$$\frac{w_1 - w_o}{a} \le r \le \frac{w_2 - w_o}{a}, \qquad 0 \le \theta < 2\pi$$

$x = u = u_o + r\cos\theta$, $y = v = v_o + r\sin\theta$, $z = w = w_o + ar$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial r}\times\frac{\partial\vec{\mathbf{r}}}{\partial\theta}\right) = \pm\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ \cos\theta & \sin\theta & a\\ -r\sin\theta & r\cos\theta & 0\end{vmatrix}$$

$$= \pm\left[(-ar\cos\theta)\,\hat{\mathbf{i}} + (-ar\sin\theta)\,\hat{\mathbf{j}} + r\,\hat{\mathbf{k}}\right]$$

Outward normal: $\vec{\mathbf{N}} = ar\cos\theta\,\hat{\mathbf{i}} + ar\sin\theta\,\hat{\mathbf{j}} - r\,\hat{\mathbf{k}}$

4) The portion of the elliptic paraboloid

$$z - z_o = a^2\left(x - x_o\right)^2 + b^2\left(y - y_o\right)^2 \quad \text{with} \quad z_o \le z_1 \le z \le z_2$$

Let the parameters here be r, θ where

$$\sqrt{\frac{z_1 - z_o}{a^2\cos^2\theta + b^2\sin^2\theta}} \le r \le \sqrt{\frac{z_2 - z_o}{a^2\cos^2\theta + b^2\sin^2\theta}}, \quad 0 \le \theta < 2\pi$$

$x = x_o + r\cos\theta$, $y = y_o + r\sin\theta$, $z = z_o + r^2(a^2\cos^2\theta + b^2\sin^2\theta)$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial r} \times \frac{\partial\vec{\mathbf{r}}}{\partial\theta}\right) = \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 2r\left(a^2\cos^2\theta + b^2\sin^2\theta\right) \\ -r\sin\theta & r\cos\theta & 2r^2\left(b^2 - a^2\right)\sin\theta\cos\theta \end{vmatrix}$$

$$= \pm\left[\left(-2a^2r^2\cos\theta\right)\hat{\mathbf{i}} + \left(-2b^2r^2\sin\theta\right)\hat{\mathbf{j}} + r\,\hat{\mathbf{k}}\right]$$

Outward normal: $\vec{\mathbf{N}} = \left(2a^2r^2\cos\theta\right)\hat{\mathbf{i}} + \left(2b^2r^2\sin\theta\right)\hat{\mathbf{j}} - r\,\hat{\mathbf{k}}$

5) The surface of the sphere $(x - x_\circ)^2 + (y - y_\circ)^2 + (z - z_\circ)^2 = a^2$.

Let the parameters here be θ, ϕ where $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$

$x = x_o + a\sin\theta\cos\phi$, $y = y_o + a\sin\theta\sin\phi$, $z = z_o + a\cos\theta$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial\theta} \times \frac{\partial\vec{\mathbf{r}}}{\partial\phi}\right) = \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \pm a^2\sin\theta\left[\left(\sin\theta\cos\phi\right)\hat{\mathbf{i}} + \left(\sin\theta\sin\phi\right)\hat{\mathbf{j}} + \left(\cos\theta\right)\hat{\mathbf{k}}\right]$$

Outward normal: $\vec{\mathbf{N}} = a^2\sin\theta\left[\left(\sin\theta\cos\phi\right)\hat{\mathbf{i}} + \left(\sin\theta\sin\phi\right)\hat{\mathbf{j}} + \left(\cos\theta\right)\hat{\mathbf{k}}\right]$

6) The part of the plane $A(x - x_\circ) + B(y - y_\circ) + C(z - z_\circ) = 0$ in the first octant with $A, B, C > 0$ and $Ax_o + By_o + Cz_o > 0$.

Let the parameters be x, y where

$$0 \le x \le \frac{Ax_\circ + By_\circ + Cz_\circ - By}{A}; \qquad 0 \le y \le \frac{Ax_\circ + By_\circ + Cz_\circ}{B}$$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial\vec{\mathbf{r}}}{\partial x} \times \frac{\partial\vec{\mathbf{r}}}{\partial y}\right) = \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -A/C \\ 0 & 1 & -B/C \end{vmatrix} = \pm\left[\frac{A}{C}\hat{\mathbf{i}} + \frac{B}{C}\hat{\mathbf{j}} + \hat{\mathbf{k}}\right]$$

2.6 Theorems of Gauss and Stokes; Potential Functions

Gauss' Divergence Theorem

Let S be a piecewise-smooth closed surface enclosing a volume V in $\mathbb{R}^3$ and let $\mathbf{F}$ be a vector field. Then
the net flux of $\mathbf{F}$ out of V is $\oiint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \oiint_S F_N \, dS$.

But the divergence of $\mathbf{F}$ is a flux density, or an "outflow per unit volume" at a point. Integrating $\operatorname{div} \mathbf{F}$ over the entire enclosed volume must match the net flux out through the boundary S of the volume V. **Gauss' divergence theorem** then follows:

$$\oiint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \iiint_V \vec{\nabla} \bullet \vec{\mathbf{F}} \, dV$$

Example 2.6.1 (Example 2.5.4 repeated)

Find the total flux Φ of the vector field $\vec{\mathbf{F}} = z\hat{\mathbf{k}}$ through the simple closed surface S

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Use Gauss' Divergence Theorem: $\oiint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \iiint_V \operatorname{div} \vec{\mathbf{F}} \, dV$

$\vec{\mathbf{F}}$ is differentiable everywhere in $\mathbb{R}^3$, so Gauss' divergence theorem is valid.

$$\operatorname{div} \vec{\mathbf{F}} = \vec{\nabla} \bullet \left(z\hat{\mathbf{k}} \right) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \bullet \langle 0, 0, z \rangle = 0 + 0 + 1 = 1$$

$$\Rightarrow \iiint_V \operatorname{div} \vec{\mathbf{F}} \, dV = \iiint_V 1 \, dV = V = \frac{4\pi abc}{3}$$ – the volume of the ellipsoid !

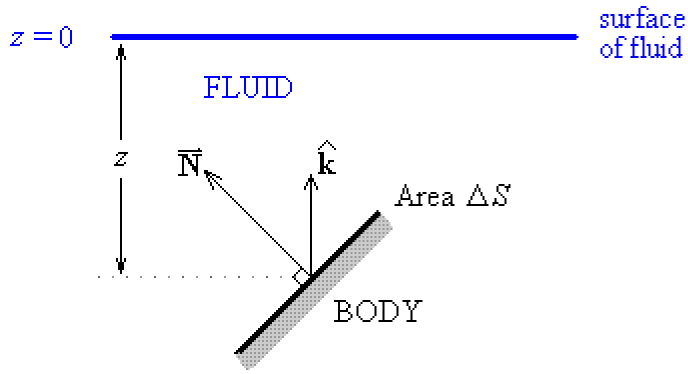
Therefore

$$\Phi = \oiint_S \vec{\mathbf{F}} \bullet d\vec{\mathbf{S}} = \frac{4\pi abc}{3}$$

In fact, the flux of $\vec{\mathbf{F}} = z\hat{\mathbf{k}}$ through *any* simple closed surface is just the volume enclosed by that surface.

<u>Example 2.6.2</u> **Archimedes' Principle**

Gauss' divergence theorem may be used to derive Archimedes' principle for the buoyant force on a body totally immersed in a fluid of constant density ρ (independent of depth). Examine an elementary section of the surface S of the immersed body, at a depth $z < 0$ below the surface of the fluid:



The pressure at any depth z is the weight of fluid per unit area from the column of fluid above that area. Therefore

pressure $= p = -\rho g z$ ρg is the weight of the column
$-z$ is the height of the column (note $z < 0$).

The normal vector $\vec{\mathbf{N}}$ to S is directed outward, but the hydrostatic force on the surface (due to the pressure p) acts inward. The element of hydrostatic force on ΔS is

$$(\text{pressure})\times(\text{area})\times(\text{direction}) = (-\rho g z)(\Delta S)(-\hat{\mathbf{N}}) = (+\rho g z \,\Delta S)\,\hat{\mathbf{N}}$$

The element of buoyant force on ΔS is the component of the hydrostatic force in the direction of **k** (vertically upwards):

$$(+\rho g z \,\Delta S\,\hat{\mathbf{N}})\bullet\hat{\mathbf{k}}$$

Define $\vec{\mathbf{F}} = \rho g z\,\hat{\mathbf{k}}$ and $\mathbf{d}\vec{\mathbf{S}} = \hat{\mathbf{N}}\,dS$.

Summing over all such elements ΔS, the total buoyant force on the immersed object is

$$\iint_S\!\!\!\!\!\!\!\!\!\bigcirc\;\; \rho g z\,\hat{\mathbf{k}}\bullet\hat{\mathbf{N}}\,dS = \iint_S\!\!\!\!\!\!\!\!\!\bigcirc\;\; \vec{\mathbf{F}}\bullet\mathbf{d}\vec{\mathbf{S}} = \iiint_V \vec{\nabla}\bullet\vec{\mathbf{F}}\,dV \quad \text{(by the Gauss Divergence Theorem)}$$

Example 2.6.2 Archimedes' Principle (continued)

$$= \iiint_V \vec{\nabla} \bullet \left(\rho g \, z \, \hat{\mathbf{k}} \right) dV \;=\; \iiint_V \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \bullet \left\langle 0,\, 0,\, \rho g \, z \right\rangle dV$$

$$= \iiint_V \rho g \; dV \quad \left(\text{provided } \frac{\partial}{\partial z}\left(\rho g \right) \equiv 0 \right)$$

= weight of fluid displaced

Therefore the total buoyant force on an object fully immersed in a fluid equals the weight of the fluid displaced by the immersed object (Archimedes' principle).

Gauss' Law

A point charge q at the origin O generates an electric field

$$\vec{\mathbf{E}} = \frac{q}{4\pi\varepsilon\, r^3}\,\vec{\mathbf{r}} = \frac{q}{4\pi\varepsilon\, r^2}\,\hat{\mathbf{r}}$$

If S is a smooth simple closed surface **not** enclosing the charge, then the total flux through S is

$$\oiint_S \vec{\mathbf{E}}\bullet d\vec{\mathbf{S}} = \iiint_V \vec{\nabla}\bullet\vec{\mathbf{E}}\, dV \qquad \text{(Gauss' divergence theorem)}$$

But Example 1.4.1 showed that $\vec{\nabla}\bullet\left(\frac{1}{r^3}\vec{\mathbf{r}}\right) = 0 \quad \forall r \neq 0$.

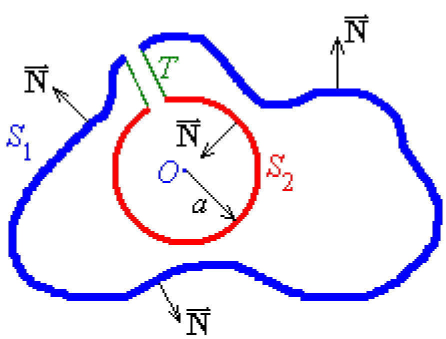
Therefore $\oiint_S \vec{\mathbf{E}}\bullet d\vec{\mathbf{S}} = 0$.

There is no net outflow of electric flux through any closed surface ***not*** enclosing the source of the electrostatic field.

If S does enclose the charge, then one cannot use Gauss' divergence theorem, because

$\vec{\nabla}\bullet\vec{\mathbf{E}}$ is undefined at the origin.

Remedy:
Construct a surface S_1 identical to S except for a small hole cut where a narrow tube T connects it to another surface S_2, a sphere of radius a centre O and entirely inside S. Let $S^* = S_1 \cup T \cup S_2$ (which is a simple closed surface), then O is **outside** S^* !



Applying Gauss' divergence theorem to S^*,

$$\oiint_{S^*} \vec{\mathbf{E}}\bullet d\vec{\mathbf{S}} = \iiint_{V^*} \vec{\nabla}\bullet\vec{\mathbf{E}}\, dV = 0$$

$$\Rightarrow \iint_{S_1} \vec{\mathbf{E}}\bullet d\vec{\mathbf{S}} + \iint_{T} \vec{\mathbf{E}}\bullet d\vec{\mathbf{S}} + \iint_{S_2} \vec{\mathbf{E}}\bullet d\vec{\mathbf{S}} = 0$$

Gauss' Law (continued)

As the tube T approaches zero thickness,

$$\iint_T \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} \to 0 \quad \text{and therefore} \quad \iint_{S_1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} \to -\iint_{S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}}$$

But S_2 is a sphere, centre O, radius a.

Using parameters (θ, ϕ) on the sphere,

$$\vec{\mathbf{r}} = a\langle \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta \rangle$$

Finding $\dfrac{\partial\vec{\mathbf{r}}}{\partial\theta}, \dfrac{\partial\vec{\mathbf{r}}}{\partial\phi}$ as before leads to $\vec{\mathbf{N}} = \pm a \sin\theta\, \vec{\mathbf{r}}$.

But the "outward normal" to S_2 actually points ***towards*** O.

$\Rightarrow \quad \vec{\mathbf{N}} = -a\sin\theta\,\vec{\mathbf{r}}$ on the sphere S_2

and $\vec{\mathbf{E}} = \dfrac{q}{4\pi\varepsilon a^3}\vec{\mathbf{r}}$ everywhere on S_2.

Also $\vec{\mathbf{r}} = a\hat{\mathbf{r}} \quad \Rightarrow \quad \vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = a^2$

$$\Rightarrow \quad \vec{\mathbf{E}} \cdot \vec{\mathbf{N}} = \frac{q}{4\pi\varepsilon a^3}\vec{\mathbf{r}} \cdot \left(-a\sin\theta\,\vec{\mathbf{r}}\right) = \frac{-q\sin\theta}{4\pi\varepsilon a^2}\vec{\mathbf{r}} \cdot \vec{\mathbf{r}} = \frac{-q\sin\theta}{4\pi\varepsilon}$$

Recall that $d\vec{\mathbf{S}} = \widehat{\mathbf{N}}\,dS = \widehat{\mathbf{N}}\,N\,d\theta\,d\phi = \vec{\mathbf{N}}\,d\theta\,d\phi$

$$\iint_{S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = \iint_{S_2} \vec{\mathbf{E}} \cdot \vec{\mathbf{N}}\,d\theta\,d\phi = \frac{-q}{4\pi\varepsilon}\int_0^{2\pi}\int_0^{\pi}\sin\theta\,d\theta\,d\phi$$

$$= \frac{-q}{4\pi\varepsilon}\left[-\cos\theta\right]_0^{\pi} \cdot \left[\phi\right]_0^{2\pi} = \frac{-q}{4\pi\varepsilon}(+1+1)(2\pi - 0) = -\frac{q}{\varepsilon}$$

$$\Rightarrow \quad \iint_{S_1} \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = -\iint_{S_2} \vec{\mathbf{E}} \cdot d\vec{\mathbf{S}} = +\frac{q}{\varepsilon}$$

Gauss' Law (continued)

But, as $a \to 0 \quad (\Rightarrow \quad S_2 \to O), \quad S_1 \to S$

The surface S_1 looks more and more like the surface S as the tube T collapses to a line and the sphere S_2 collapses into a point at the origin. Gauss' law then follows.

Gauss' law for the net flux through any smooth simple closed surface *S*, in the presence of a point charge *q* at the origin, then follows:

$$\oiint_S \vec{\mathbf{E}} \bullet \mathbf{d}\vec{\mathbf{S}} = \begin{cases} \dfrac{q}{\varepsilon} & \text{if } S \text{ encloses } O \\ 0 & \text{otherwise} \end{cases}$$

Example 2.6.3 **Poisson's Equation**

The exact location of the enclosed charge is immaterial, provided it is somewhere inside the volume V enclosed by the surface S. The charge therefore does not need to be a concentrated point charge, but can be spread out within the enclosed volume V. Let the charge density be $\rho(x, y, z)$, then the total charge enclosed by S is

$$q = \iiint_V \rho \, dV$$

Gauss' law $\Rightarrow$ $\oiint_S \vec{\mathbf{E}} \bullet d\vec{\mathbf{S}} = \frac{q}{\varepsilon}$

Apply Gauss' divergence theorem to the left hand side, substitute for q on the right hand side and assume that the permittivity ε is constant throughout the volume:

$$\Rightarrow \iiint_V \vec{\nabla} \bullet \vec{\mathbf{E}} \, dV = \iiint_V \frac{\rho}{\varepsilon} \, dV$$

$$\Rightarrow \iiint_V \left(\vec{\nabla} \bullet \vec{\mathbf{E}} - \frac{\rho}{\varepsilon} \right) dV = 0 \qquad \forall V$$

This identity will hold for all volumes V only if the integrand is zero everywhere.

Poisson's equation then follows:

$$\boxed{\vec{\nabla} \bullet \vec{\mathbf{E}} = \frac{\rho}{\varepsilon}}$$

$\vec{\mathbf{E}} = -\vec{\nabla} V$ and $\vec{\nabla} \bullet \vec{\nabla} V = \nabla^2 V$ $\Rightarrow$ $\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon}}$

This reduces to Laplace's equation $\nabla^2 V = 0$ when $\rho \equiv 0$.

Stokes' Theorem

Let **F** be a vector field acting parallel to the *xy*-plane. Represent its Cartesian components by $\vec{\mathbf{F}} = f_1 \hat{\mathbf{i}} + f_2 \hat{\mathbf{j}} = \langle f_1, f_2, 0 \rangle$. Then

$$\vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & 0 \end{vmatrix} \quad \Rightarrow \quad \left(\vec{\nabla} \times \vec{\mathbf{F}}\right) \bullet \hat{\mathbf{k}} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}$$

Green's theorem can then be expressed in the form

$$\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \iint_D \vec{\nabla} \times \vec{\mathbf{F}} \bullet \hat{\mathbf{k}} \, dA$$

Now let us twist the simple closed curve *C* and its enclosed surface out of the *xy*-plane, so that the normal vector **k** is replaced by a more general normal vector **N**.
If the surface *S* (that is bounded in $\mathbb{R}^3$ by the simple closed curve *C*) can be represented by $z = f(x, y)$, then a normal vector at any point on *S* is

$$\vec{\mathbf{N}} = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle$$

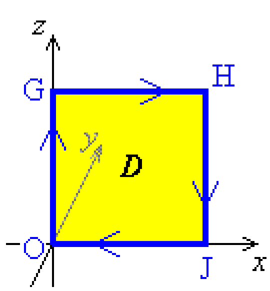
C is oriented coherently with respect to *S* if, as one travels along *C* with **N** pointing from one's feet to one's head, *S* is always on one's left side. The resulting generalization of Green's theorem is **Stokes' theorem**:

$$\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \iint_S \vec{\nabla} \times \vec{\mathbf{F}} \bullet \widehat{\mathbf{N}} \, dS = \iint_S \left(\text{curl } \vec{\mathbf{F}}\right) \bullet \mathbf{d}\vec{\mathbf{S}}$$

This can be extended further, to a non-flat surface *S* with a non-constant normal vector **N**.

Example 2.6.4

Find the circulation of $\vec{\mathbf{F}} = \langle xyz, xz, e^{xy} \rangle$ around C : the unit square in the *xz*-plane.



Because of the right-hand rule, the positive orientation around the square is OGHJ (the *y* axis is directed into the page).

In the *xz* plane $y = 0 \Rightarrow \vec{\mathbf{F}} = \langle 0, xz, 1 \rangle$

Example 2.6.4 (continued)

Computing the line integral around the four sides of the square:

$$OG: \quad \vec{\mathbf{r}} = \langle 0, 0, t \rangle \quad (0 \le t \le 1) \qquad \Rightarrow \quad \frac{d\vec{\mathbf{r}}}{dt} = \langle 0, 0, 1 \rangle$$

$$\text{and} \quad \vec{\mathbf{F}} = \langle 0, 0, 1 \rangle \qquad \Rightarrow \quad \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt} = \langle 0, 0, 1 \rangle \bullet \langle 0, 0, 1 \rangle = 1$$

$$\Rightarrow \quad \int_{OG} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_0^1 1\, dt = \left[t \right]_0^1 = 1 - 0 = 1$$

In a similar way (Problem Set 6 Question 6), it can be shown that

$$\int_{GH} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 0, \quad \int_{HJ} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = -1 \quad \text{and} \quad \int_{JO} \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 0$$

$$\Rightarrow \quad \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = 1 + 0 - 1 + 0 = \underline{\underline{0}}$$

OR use Stokes' theorem:

$$\text{On } D \quad \vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & xz & 1 \end{vmatrix} = \langle -x, 0, z \rangle$$

$$d\vec{\mathbf{A}} = \hat{\mathbf{j}}\, dA$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{\mathbf{F}} \bullet d\vec{\mathbf{A}} = \langle -x, 0, z \rangle \bullet \langle 0, 1, 0 \rangle\, dA = 0\, dA$$

$$\Rightarrow \quad \iint_D \vec{\nabla} \times \vec{\mathbf{F}} \bullet d\vec{\mathbf{A}} = 0 \qquad \Rightarrow \quad \oint_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \underline{\underline{0}}$$

Note that this vector field $\vec{\mathbf{F}}$ is ***not*** conservative, because $\vec{\nabla} \times \vec{\mathbf{F}} \not\equiv 0$.

Domain

A region Ω of $\mathbb{R}^3$ is a **domain** if and only if

1) For all points P_o in Ω, there exists a sphere, centre P_o, all of whose interior points are inside Ω; and
2) For all points P_o and P_1 in Ω, there exists a piecewise smooth curve C, entirely in Ω, from P_o to P_1.

A domain is **simply connected** if it "has no holes".

Example 2.6.5 Are these regions simply-connected domains?

The interior of a sphere. YES

The interior of a torus. NO

The first octant. YES

On a simply-connected domain the following statements are either all true or all false:

- $\mathbf{F}$ is conservative.
- $\mathbf{F} \equiv \nabla \phi$
- $\nabla \times \mathbf{F} \equiv \mathbf{0}$
- $\int_C \vec{\mathbf{F}} \cdot \mathbf{d}\vec{\mathbf{r}} = \phi(P_{\text{end}}) - \phi(P_{\text{start}})$ - independent of the path between the two points.
- $\oint_C \vec{\mathbf{F}} \cdot \mathbf{d}\vec{\mathbf{r}} = 0 \quad \forall C \subset \Omega$

Example 2.6.6

Find a potential function $\phi(x, y, z)$ for the vector field $\vec{\mathbf{F}} = \langle 2x, 2y, 2z \rangle$.

First, check that a potential function exists at all:

$$\text{curl}\,\vec{\mathbf{F}} = \vec{\nabla} \times \vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & 2y & 2z \end{vmatrix} = \langle 0, 0, 0 \rangle = \vec{\mathbf{0}}$$

Therefore $\vec{\mathbf{F}}$ is conservative on $\mathbb{R}^3$.

Example 2.6.6 (continued)

$$\Rightarrow \quad \vec{\mathbf{F}} = \vec{\nabla}\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right\rangle$$

$$\frac{\partial \phi}{\partial x} = 2x \qquad \Rightarrow \quad \phi = x^2 + g(y,z)$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial y} = 0 + \frac{\partial g}{\partial y} = 2y \qquad \Rightarrow \quad g(y,z) = y^2 + h(z)$$

$$\Rightarrow \quad \phi = x^2 + y^2 + h(z)$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial z} = 0 + 0 + \frac{dh}{dz} = 2z \qquad \Rightarrow \quad h(z) = z^2 + c$$

$$\Rightarrow \quad \phi = x^2 + y^2 + z^2 + c$$

We have a free choice for the value of the arbitrary constant c. Choose $c = 0$, then

$$\underline{\underline{\phi(x,y,z) = x^2 + y^2 + z^2}} = r^2$$

Maxwell's Equations (***not*** examinable in this course)

We have seen how Gauss' and Stokes' theorems have led to Poisson's equation, relating the electric intensity vector **E** to the electric charge density ρ:

$$\vec{\nabla}\bullet\vec{\mathbf{E}} = \frac{\rho}{\varepsilon}$$

Where the permittivity is constant, the corresponding equation for the electrical flux density **D** is one of Maxwell's equations: $\boxed{\vec{\nabla}\bullet\vec{\mathbf{D}} = \rho}$.

Another of Maxwell's equations follows from the absence of isolated magnetic charges (no magnetic monopoles): $\vec{\nabla}\bullet\vec{\mathbf{H}} = 0 \Rightarrow \boxed{\vec{\nabla}\bullet\vec{\mathbf{B}} = 0}$, where **H** is the magnetic intensity and **B** is the magnetic flux density.

Faraday's law, connecting electric intensity with the rate of change of magnetic flux density, is $\oint_C \vec{\mathbf{E}}\bullet\mathbf{d}\vec{\mathbf{r}} = -\frac{\partial}{\partial t}\iint_S \vec{\mathbf{B}}\bullet\mathbf{d}\vec{\mathbf{S}}$. Applying Stokes' theorem to the left side produces

$$\boxed{\vec{\nabla}\times\vec{\mathbf{E}} = -\frac{\partial\vec{\mathbf{B}}}{\partial t}}$$

Ampère's circuital law, $I = \oint_C \vec{\mathbf{H}}\bullet\mathbf{d}\vec{\mathbf{l}}$, leads to $\vec{\nabla}\times\vec{\mathbf{H}} = \vec{\mathbf{J}} + \vec{\mathbf{J}}_d$, where the current density is $\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}} = \rho_V\,\vec{\mathbf{v}}$, σ is the conductivity, ρ_V is the volume charge density; and the displacement charge density is $\vec{\mathbf{J}}_d = \frac{\partial\vec{\mathbf{D}}}{\partial t}$

The fourth Maxwell equation is

$$\boxed{\nabla\times\vec{\mathbf{H}} = \vec{\mathbf{J}} + \frac{\partial\vec{\mathbf{D}}}{\partial t}}$$

The four Maxwell's equations together allow the derivation of the equations of propagating electromagnetic waves.

END OF CHAPTER 2