3. Fourier Series

This short chapter offers a very brief review of [discrete] Fourier series.

The **Fourier series** of $f(x)$ on the interval $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\,, \quad (n = 0, 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\,, \quad (n = 1, 2, 3, \ldots)$$

The $\{a_n, b_n\}$ are the Fourier coefficients of $f(x)$.

Note that the cosine functions (and the function 1) are even, while the sine functions are odd.

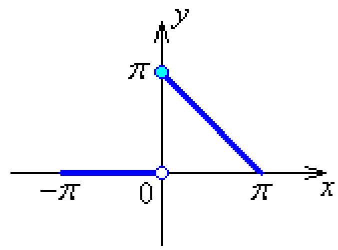
If $f(x)$ is even ($f(-x) = +f(x)$ for all x), then $b_n = 0$ for all n, leaving a Fourier cosine series (and perhaps a constant term) only for $f(x)$.

If $f(x)$ is odd ($f(-x) = -f(x)$ for all x), then $a_n = 0$ for all n, leaving a Fourier sine series only for $f(x)$.

Example 3.1

Expand $f(x) = \begin{cases} 0 & (-\pi < x < 0) \\ \pi - x & (0 \le x < +\pi) \end{cases}$ in a Fourier series.

$L = \pi.$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\,dx$$

$$= 0 + \frac{1}{\pi}\left[\frac{(\pi - x)^2}{-2}\right]_0^{\pi} = \frac{\pi}{2}$$

D		I
$\pi - x$		$\cos nx$
	+	
-1		$\frac{1}{n}\sin nx$
	−	
0		$-\frac{1}{n^2}\cos nx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\sin nx - \cos nx}{n^2}\right]_0^{\pi} = \frac{1 - (-1)^n}{n^2\pi}$$

D		I
$\pi - x$		nx
	+	
-1		$-\frac{1}{n}\cos nx$
	−	
0		$-\frac{1}{n^2}\sin nx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 + \frac{1}{\pi}\int_{0}^{\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\cos nx + \sin nx}{-n^2}\right]_0^{\pi} = \frac{1}{n}$$

Therefore the Fourier series for $f(x)$ is

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \qquad (-\pi < x < +\pi)}$$

Example 3.1 (Additional Notes – also see "`www.engr.mun.ca/~ggeorge/5432/demos/`")

The first few partial sums in the Fourier series

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right) \qquad (-\pi < x < +\pi)$$

are

$$S_0 = \frac{\pi}{4}$$

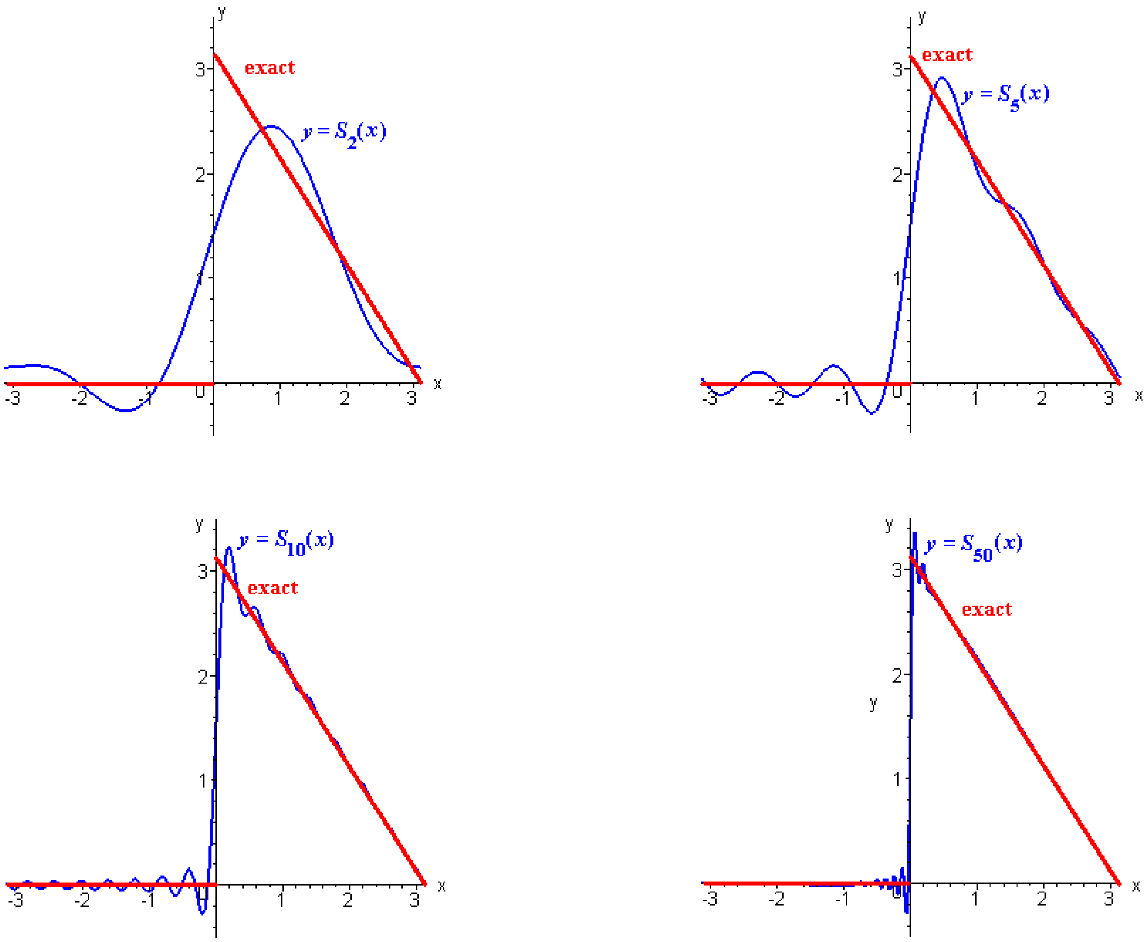
$$S_1 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x$$

$$S_2 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x$$

$$S_3 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x + \frac{2}{9\pi}\cos 3x + \frac{1}{3}\sin 3x$$

and so on.

The graphs of successive partial sums approach $f(x)$ more closely, except in the vicinity of any discontinuities, (where a systematic overshoot occurs, the **Gibbs phenomenon**).



Example 3.2

Find the Fourier series expansion for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

$L = 1$.

The function is odd ($f(-x) = -f(x)$ for all x).
Therefore $a_n = 0$ for all n. We will have a Fourier sine series only.

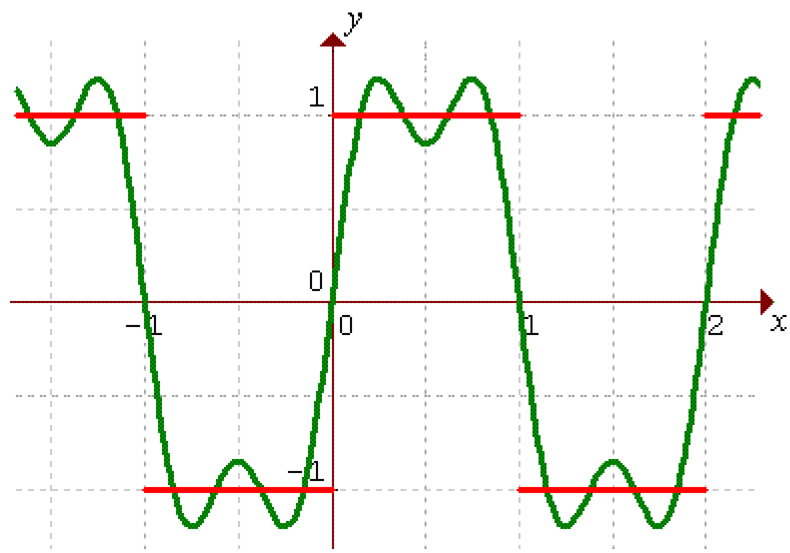
$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0} -\sin n\pi x\,dx + \int_{0}^{1} \sin n\pi x\,dx$$

(can use symmetry)

$$= \left[\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} + \left[\frac{-\cos n\pi x}{n\pi}\right]_{0}^{1} = \frac{2\left(1-(-1)^n\right)}{n\pi}$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\sin n\pi x\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\sin(2k-1)\pi x\right)$$

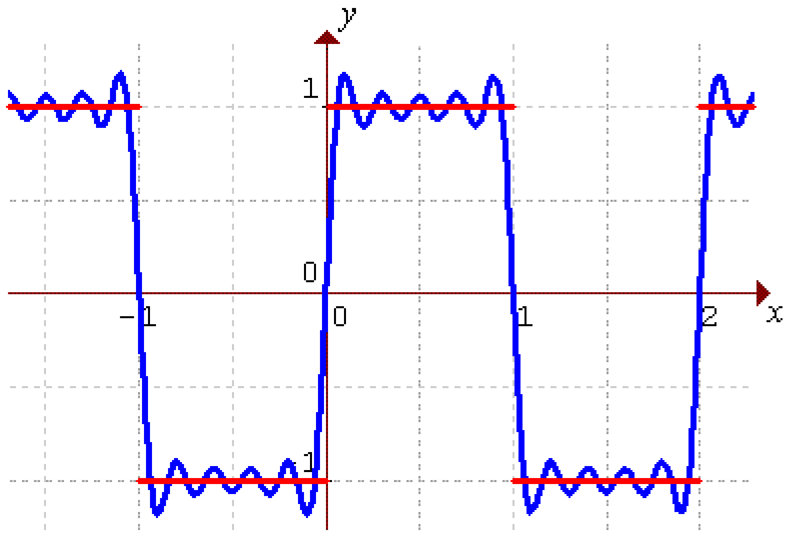
The graphs of the third and ninth partial sums (containing two and five non-zero terms respectively) are displayed here, together with the exact form for $f(x)$, with a **periodic extension** beyond the interval (–1, +1) that is appropriate for the square wave.



$y = S_3(x)$

<u>Example 3.2</u> (continued)

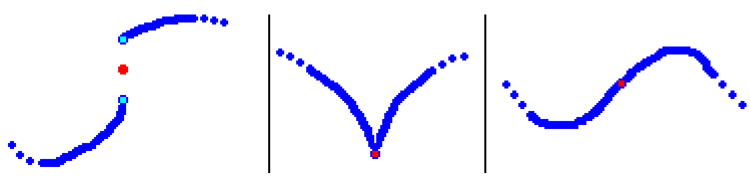
$$y = S_9(x)$$



Convergence

At all points $x = x_o$ in $(-L, L)$ where $f(x)$ is continuous and is either differentiable or the limits $\lim\limits_{x \to x_o^-} f'(x)$ and $\lim\limits_{x \to x_o^+} f'(x)$ both exist, the Fourier series converges to $f(x)$.

At finite discontinuities, (where the limits $\lim\limits_{x \to x_o^-} f'(x)$ and $\lim\limits_{x \to x_o^+} f'(x)$ both exist), the Fourier series converges to $\dfrac{f(x_o -) + f(x_o +)}{2}$,
(using the abbreviations $f(x_o -) = \lim\limits_{x \to x_o^-} f(x)$ and $f(x_o +) = \lim\limits_{x \to x_o^+} f(x)$).



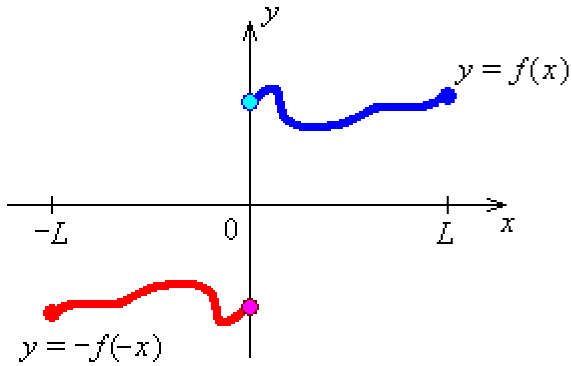
$f(x)$ not continuous at $x = x_o$ | continuous but not differentiable | continuous and differentiable

In all cases, the Fourier series at $x = x_o$ converges to $\dfrac{f(x_o -) + f(x_o +)}{2}$ (the red dot).

Half-Range Fourier Series

A Fourier series for $f(x)$, valid on $[0, L]$, may be constructed by extension of the domain to $[-L, L]$.

An odd extension leads to a **Fourier sine series**:

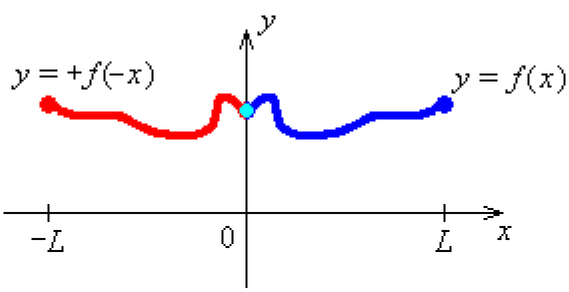


$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

An even extension leads to a **Fourier cosine series**:



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

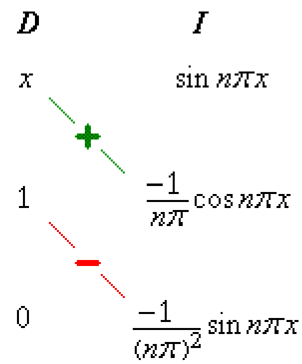
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$$

and there is automatic continuity of the Fourier cosine series at $x = 0$ and at $x = \pm L$.

Example 3.3

Find the Fourier sine series and the Fourier cosine series for $f(x) = x$ on [0, 1].

$f(x) = x$ happens to be an odd function of x for any domain centred on $x = 0$. The odd extension of $f(x)$ to the interval [–1, 1] is $f(x)$ itself.



Evaluating the Fourier sine coefficients,

$$b_n = \frac{2}{1}\int_0^1 x \sin\left(\frac{n\pi x}{1}\right) dx\,, \quad (n = 1, 2, 3, \ldots)$$

$$\Rightarrow \quad b_n = 2\left[-\frac{x}{n\pi}\cos\left(\frac{n\pi x}{1}\right) + \frac{1}{(n\pi)^2}\sin\left(\frac{n\pi x}{1}\right)\right]_0^1$$

$$= \frac{2}{n\pi} \times (-1)^{n+1}$$

Therefore the Fourier sine series for $f(x) = x$ on [0, 1] (which is also the Fourier series for $f(x) = x$ on [–1, 1]) is

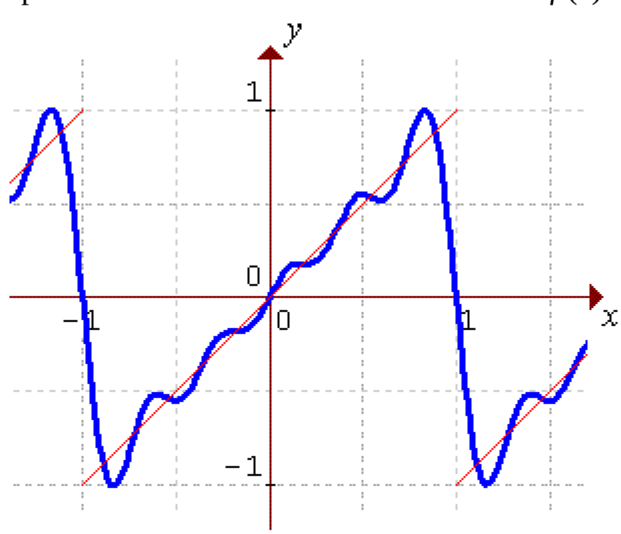
$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi x)}{n\pi}$$

or

$$f(x) = \frac{2}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \frac{\sin 4\pi x}{4} + \ldots\right)$$

This function happens to be continuous and differentiable at $x = 0$, but is clearly discontinuous at the endpoints of the interval ($x = \pm 1$).

Fifth order partial sum of the Fourier sine series for $f(x) = x$ on [0, 1]



<u>Example 3.3</u> (continued)

The even extension of $f(x)$ to the interval $[-1, 1]$ is $f(x) = |x|$.

Evaluating the Fourier cosine coefficients,

D		*I*
x	$+$	$\cos n\pi x$
1	$-$	$\frac{1}{n\pi}\sin n\pi x$
0		$\frac{-1}{(n\pi)^2}\cos n\pi x$

$$a_n = \frac{2}{1}\int_0^1 x\cos\left(\frac{n\pi x}{1}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

$$\Rightarrow \quad a_n = 2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1$$

$$= \frac{2\left((-1)^n - 1\right)}{(n\pi)^2}$$

and $\quad a_0 = \frac{2}{1}\int_0^1 x\,dx = \left[x^2\right]_0^1 = 1$

Evaluating the first few terms,

$$a_0 = 1, \quad a_1 = \frac{-4}{\pi^2}, \quad a_2 = 0, \quad a_3 = \frac{-4}{9\pi^2}, \quad a_4 = 0, \quad a_5 = \frac{-4}{25\pi^2}, \quad a_6 = 0, \ldots$$

or $\quad a_n = \begin{cases} 1 & (n = 0) \\ \frac{-4}{(n\pi)^2} & (n = 1, 3, 5, \ldots) \\ 0 & (n = 2, 4, 6, \ldots) \end{cases}$

Therefore the Fourier cosine series for $f(x) = x$ on [0, 1] (which is also the Fourier series for $f(x) = |x|$ on [–1, 1]) is

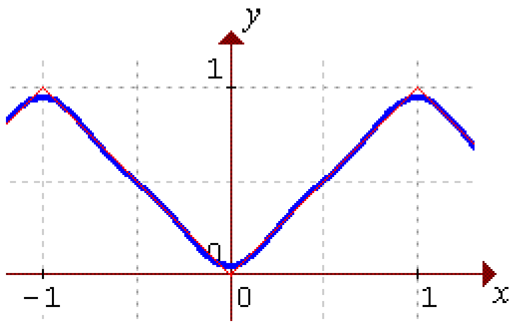
$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\left((2k-1)\pi x\right)}{(2k-1)^2}$$

or

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos\pi x + \frac{\cos 3\pi x}{9} + \frac{\cos 5\pi x}{25} + \frac{\cos 7\pi x}{49} + \ldots\right)$$

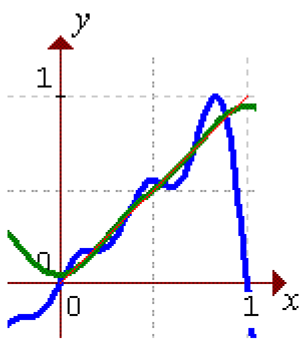
Example 3.3 (continued)

Third order partial sum of the Fourier cosine series for $f(x) = x$ on [0, 1]



Note how rapid the convergence is for the cosine series compared to the sine series.

$S_3(x)$ for cosine series and $S_5(x)$ for sine series for $f(x) = x$ on [0, 1]



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