4. Partial Differential Equations

Partial differential equations (PDEs) are equations involving functions of more than one variable and their partial derivatives with respect to those variables.

Most (but not all) physical models in engineering that result in partial differential equations are of at most second order and are often linear. (Some problems such as elastic stresses and bending moments of a beam can be of fourth order). In this course we shall have time to look at only a very small subset of second order linear partial differential equations.

Sections in this Chapter:

4.1 Major Classifications of Common PDEs

A general second order linear partial differential equation in two Cartesian variables can be written as

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

Three main types arise, based on the value of $D = B^2 - 4AC$ (a discriminant):
Hyperbolic, wherever (x, y) is such that $D > 0$;
Parabolic, wherever (x, y) is such that $D = 0$;
Elliptic, wherever (x, y) is such that $D < 0$.

Among the most important partial differential equations in engineering are:

The **wave equation**: $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\nabla^2 u$

or its one-dimensional special case $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\dfrac{\partial^2 u}{\partial x^2}$ [which is hyperbolic everywhere]

(where u is the displacement and c is the speed of the wave);

The **heat** (or **diffusion**) **equation**: $\mu\rho\dfrac{\partial u}{\partial t} = K\,\nabla^2 u + \vec{\nabla} K \bullet \vec{\nabla} u$

a one-dimensional special case of which is

$$\frac{\partial u}{\partial t} = \frac{K}{\mu\rho}\frac{\partial^2 u}{\partial x^2} \qquad \text{[which is parabolic everywhere]}$$

(where u is the temperature, μ is the specific heat of the medium, ρ is the density and K is the thermal conductivity);

The potential (or **Laplace's**) **equation**: $\nabla^2 u = 0$

a special case of which is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ [which is elliptic everywhere]

The complete solution of a PDE requires additional information, in the form of initial conditions (values of the dependent variable and its first partial derivatives at $t = 0$), boundary conditions (values of the dependent variable on the boundary of the domain) or some combination of these conditions.

4.2 d'Alembert Solution

Example 4.2.1

Show that

$$y(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$

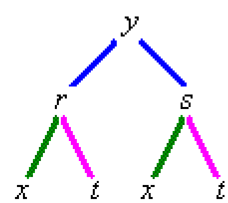
is a solution to the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0$$

with initial conditions $y(x, 0) = f(x)$ and $\left.\frac{\partial}{\partial t} y(x,t)\right|_{t=0} = 0$

for ***any*** twice differentiable function $f(x)$.

Let $r = x + ct$ and $s = x - ct$, then $y(r,s) = \dfrac{f(r) + f(s)}{2}$



and

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial x} = \frac{1}{2}\Big(\big(f'(r)+0\big)\times 1 + \big(0+f'(s)\big)\times 1\Big),$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial y}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial s}\left(\frac{\partial y}{\partial x}\right)\frac{\partial s}{\partial x} = \frac{1}{2}\big(f''(r)\times 1 + f''(s)\times 1\big),$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial t} = \frac{1}{2}\Big(\big(f'(r)+0\big)\times c + \big(0+f'(s)\big)\times(-c)\Big),$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial r}\left(\frac{\partial y}{\partial t}\right)\frac{\partial r}{\partial t} + \frac{\partial}{\partial s}\left(\frac{\partial y}{\partial t}\right)\frac{\partial s}{\partial t} = \frac{1}{2}\big(c\,f''(r)\times c - c\,f''(s)\times(-c)\big),$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{2}\big(f''(r) + f''(s)\big) - \frac{1}{2c^2}\big(c^2 f''(r) + c^2 f''(s)\big) = 0,$$

Therefore $y(x,t) = \dfrac{f(x+ct) + f(x-ct)}{2}$ is a solution to the wave equation for all twice differentiable functions $f(x)$. This is part of the d'Alembert solution.

Example 4.2.1 (continued)

This d'Alembert solution satisfies the initial displacement condition:

$$y(x,0) = \frac{f(x+0) + f(x-0)}{2} = f(x)$$

Also $$\left.\frac{\partial}{\partial t} y(x,t)\right|_{t=0} = \left.\frac{c\,f'(x+ct) - c\,f'(x-ct)}{2}\right|_{t=0} = \frac{c\,f'(x) - c\,f'(x)}{2} = 0$$

The d'Alembert solution therefore satisfies both initial conditions.

A more general d'Alembert solution to the wave equation for an infinitely long string is

$$\boxed{y(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du}$$

This satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } -\infty < x < \infty \quad \text{and} \quad t > 0$$

and

Initial configuration of string: $y(x, 0) = f(x)$ for $x \in \mathbb{R}$

and

Initial speed of string: $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$ for $x \in \mathbb{R}$

for *any* twice differentiable functions $f(x)$ and $g(x)$.

Physically, this represents two identical waves, moving with speed c in opposite directions along the string.

<u>Proof</u> that $y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$ satisfies both initial conditions:

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du \quad \Rightarrow \quad y(x,0) = \frac{1}{2c}\int_{x}^{x} g(u)\,du = 0$$

Using a Leibnitz differentiation of the integral:

$$\frac{\partial y}{\partial t} = \frac{1}{2c}\left(g(x+ct)\cdot\frac{\partial}{\partial t}(x+ct) - g(x-ct)\cdot\frac{\partial}{\partial t}(x-ct) + \int_{x-ct}^{x+ct} \frac{\partial}{\partial t} g(u)\,du \right)$$

$$= \frac{1}{2c}\left(c\,g(x+ct) + c\,g(x-ct) + 0 \right) = \frac{g(x+ct) + g(x-ct)}{2}$$

$$\Rightarrow \left.\frac{\partial y}{\partial t}\right|_{t=0} = \frac{g(x+0) + g(x-0)}{2} = g(x)$$

Example 4.2.2

An elastic string of infinite length is displaced into the form $y = \cos \pi x/2$ on $[-1, 1]$ only (and $y = 0$ elsewhere) and is released from rest. Find the displacement $y(x, t)$ at all locations on the string $x \in \mathbb{R}$ and at all subsequent times ($t > 0$).

For this solution to the wave equation we have initial conditions

$$y(x,0) = f(x) = \begin{cases} \cos\left(\dfrac{\pi x}{2}\right) & (-1 \le x \le 1) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$\frac{\partial y}{\partial t}(x,0) = g(x) = 0$$

The d'Alembert solution is

$$y(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du = \frac{f(x+ct) + f(x-ct)}{2} + 0$$

where $f(x+ct) = \begin{cases} \cos\left(\dfrac{\pi(x+ct)}{2}\right) & (-1-ct \le x \le 1-ct) \\ 0 & (\text{otherwise}) \end{cases}$

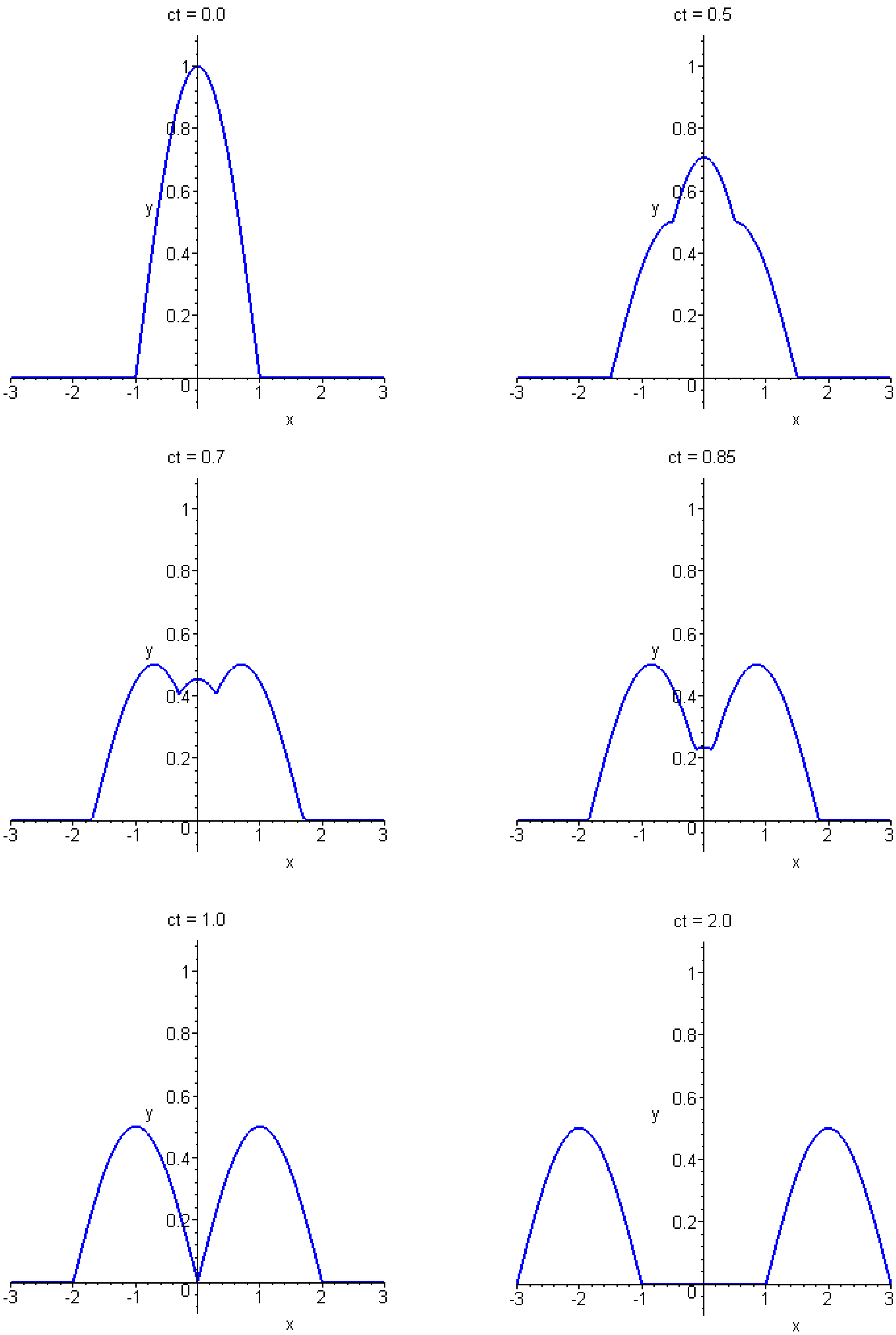
and $f(x-ct) = \begin{cases} \cos\left(\dfrac{\pi(x-ct)}{2}\right) & (-1+ct \le x \le 1+ct) \\ 0 & (\text{otherwise}) \end{cases}$

We therefore obtain two waves, each of the form of a single half-period of a cosine function, moving apart from a superposed state at $x = 0$ at speed c in opposite directions.

See the web page "www.engr.mun.ca/~ggeorge/5432/demos/ex422.html" for an animation of this solution.

Example 4.2.2 (continued)

Some snapshots of the solution are shown here:



A more general case of a d'Alembert solution arises for the homogeneous PDE with constant coefficients

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \, \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

The characteristic (or auxiliary) equation for this PDE is

$$A\lambda^2 + B\lambda + C = 0$$

This leads to the complementary function (which is also the general solution for this homogeneous PDE)

$$\boxed{u(x, y) = f_1(y + \lambda_1 x) + f_2(y + \lambda_2 x),}$$

where

$$\boxed{\lambda_1 = \frac{-B - \sqrt{D}}{2A} \quad \text{and} \quad \lambda_2 = \frac{-B + \sqrt{D}}{2A}}$$

and $D = B^2 - 4AC$

and f_1, f_2 are **arbitrary** twice-differentiable functions of their arguments.

λ_1 and λ_2 are the roots (or eigenvalues) of the characteristic equation.

In the event of equal roots, the solution changes to

$$u(x, y) = f_1(y + \lambda x) + h(x, y) f_2(y + \lambda x)$$

where $h(x, y)$ is any non-trivial linear function of x and/or y (except $y + \lambda x$).

The wave equation is a special case with $y = t$, $A = 1$, $B = 0$, $C = -1/c^2$ and $\lambda = \pm 1/c$.

Example 4.2.3

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$
$$u(x, 0) = -x^2$$
$$u_y(x, 0) = 0$$

(a) Classify the partial differential equation.
(b) Find the value of u at $(x, y) = (0, 1)$.

(a) Compare this PDE to the standard form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

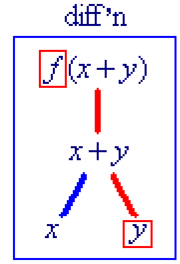
$A = 1\,,\quad B = -3\,,\quad C = 2 \quad\Rightarrow\quad D = 9 - 4\times 2 = 1 > 0$

Therefore the PDE is **hyperbolic** everywhere.

(b) $\lambda = \dfrac{+3 \pm \sqrt{1}}{2} = 1 \text{ or } 2$

The complementary function (and general solution) is
$u(x, y) = f(y + x) + g(y + 2x)$

Chain rule diff'n



$\Rightarrow \quad u_y(x, y) = f'(y + x) + g'(y + 2x)$

Initial conditions:
$u(x, 0) = f(x) + g(2x) = -x^2$ **(1)**
and
$u_y(x, 0) = f'(x) + g'(2x) = 0$ **(2)**

$\dfrac{d}{dx}(\mathbf{1}) = f'(x) + 2g'(2x) = -2x$ **(3)**

(3) $-$ **(2)** $\Rightarrow\quad g'(2x) = -2x \quad\Rightarrow\quad g'(x) = -x$

$\Rightarrow\quad g(x) = -\tfrac{1}{2}x^2 + k \qquad\Rightarrow\quad g(y+2x) = -\tfrac{1}{2}(y+2x)^2 + k$

Example 4.2.3 (continued)

Also **(1)** $\Rightarrow$ $f(x) = -x^2 - g(2x) = -x^2 + ½(2x)^2 - k = x^2 - k$

$\Rightarrow$ $f(y+x) = (y+x)^2 - k$

Therefore $u(x, y) = f(y+x) + g(y+2x)$
$= (y+x)^2 - k - (y+2x)^2 / 2 + k$

$$= \frac{1}{2}\left(2y^2 + 4xy + 2x^2 - y^2 - 4xy - 4x^2\right) = \frac{1}{2}\left(y^2 - 2x^2\right)$$

The complete solution is therefore $u(x, y) = \frac{1}{2}\left(y^2 - 2x^2\right)$

$$\Rightarrow \quad u(0,1) = \frac{1}{2}\left(1^2 - 0^2\right) = \underline{\underline{\mathbf{\frac{1}{2}}}}$$

[It is easy (though tedious) to confirm that $u(x, y) = \frac{1}{2}\left(y^2 - 2x^2\right)$ satisfies the partial differential equation $\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$ together with both initial conditions $u(x, 0) = -x^2$ and $u_y(x, 0) = 0$.]

[Also note that the arbitrary constants of integration for f and g cancelled each other out. This cancellation happens generally for this method of d'Alembert solution.]

Example 4.2.4

Find the complete solution to

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 14,$$
$$u(x, 0) = 2x + 1,$$
$$u_y(x, 0) = 4 - 6x.$$

This PDE is non-homogeneous.

For the particular solution, we require a function such that the combination of second partial derivatives resolves to the constant 14. It is reasonable to try a quadratic function of x and y as our particular solution.

Try $\quad u_P = ax^2 + bxy + cy^2$

$$\Rightarrow \frac{\partial u_P}{\partial x} = 2ax + by \quad \text{and} \quad \frac{\partial u_P}{\partial y} = bx + 2cy$$

$$\Rightarrow \frac{\partial^2 u_P}{\partial x^2} = 2a, \quad \frac{\partial^2 u_P}{\partial x\,\partial y} = b \quad \text{and} \quad \frac{\partial^2 u_P}{\partial y^2} = 2c$$

$$\Rightarrow 6\frac{\partial^2 u_P}{\partial x^2} - 5\frac{\partial^2 u_P}{\partial x\,\partial y} + \frac{\partial^2 u_P}{\partial y^2} = 12a - 5b + 2c = 14$$

We have one condition on three constants, two of which are therefore a free choice.
Choose $b = 0$ and $c = a$, then $14a = 14 \Rightarrow c = a = 1$
Therefore a particular solution is $\quad u = x^2 + y^2$

Complementary function:

$A = 6, \quad B = -5, \quad C = 1 \quad \Rightarrow \quad D = 25 - 4\times 6 = 1 > 0$

Therefore the PDE is **hyperbolic** everywhere.

$$\lambda = \frac{+5 \pm \sqrt{1}}{12} = \frac{1}{3} \text{ or } \frac{1}{2}$$

The complementary function is

$$u_C(x, y) = f\left(y + \tfrac{1}{3}x\right) + g\left(y + \tfrac{1}{2}x\right)$$

and the general solution is

$$u(x, y) = f\left(y + \tfrac{1}{3}x\right) + g\left(y + \tfrac{1}{2}x\right) + x^2 + y^2$$

Example 4.2.4 (continued)

$$u(x,y) = f\left(y+\tfrac{1}{3}x\right) + g\left(y+\tfrac{1}{2}x\right) + x^2+y^2$$
$$\Rightarrow \quad \frac{\partial u}{\partial y} = f'\left(y+\tfrac{1}{3}x\right) + g'\left(y+\tfrac{1}{2}x\right) + 2y$$

Imposing the two boundary conditions:

$$u(x,0) = f\left(\tfrac{1}{3}x\right) + g\left(\tfrac{1}{2}x\right) + x^2 = 2x+1 \qquad \textbf{(1)}$$

and

$$u_y(x,0) = f'\left(\tfrac{1}{3}x\right) + g'\left(\tfrac{1}{2}x\right) + 0 = 4-6x \qquad \textbf{(2)}$$

$$\frac{d}{dx}(\mathbf{1}) = \frac{1}{3}f'\left(\tfrac{1}{3}x\right) + \frac{1}{2}g'\left(\tfrac{1}{2}x\right) + 2x = 2 \qquad \textbf{(3)}$$

$$\textbf{(2)} - \mathbf{2}\times\textbf{(3)} \quad \Rightarrow \quad \frac{1}{3}f'\left(\tfrac{1}{3}x\right) - 4x = 4 - 6x - 4$$
$$\Rightarrow \quad f'\left(\tfrac{1}{3}x\right) = -6x = -18\left(\tfrac{1}{3}x\right) \qquad \Rightarrow \quad f'(x) = -18x$$
$$\Rightarrow \quad f(x) = -9x^2+k$$

$$\textbf{(1)} \quad \Rightarrow \quad g\left(\tfrac{1}{2}x\right) = 2x+1-x^2 - f\left(\tfrac{1}{3}x\right) = 2x+1-x^2 + 9\left(\frac{x^2}{9}\right) - k$$
$$\Rightarrow \quad g(x) = 4x + 1 - k$$

But

$$u(x,y) = f\left(y+\tfrac{1}{3}x\right) + g\left(y+\tfrac{1}{2}x\right) + x^2+y^2$$
$$\Rightarrow \quad u(x,y) = -9\left(y+\tfrac{1}{3}x\right)^2 + k + 4\left(y+\tfrac{1}{2}x\right) + 1 - k + x^2+y^2$$

[again the arbitrary constants cancel - they can be omitted safely.]

$$= -9y^2-6xy-x^2 + 4y + 2x + 1 + x^2+y^2$$

Therefore the complete solution is

$$\boxed{u(x,y) = 1 + 2x + 4y - 6xy - 8y^2}$$

Example 4.2.5

Find the complete solution to

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

$$u = 0 \quad \text{on} \quad x = 0,$$

$$u = x^2 \quad \text{on} \quad y = 1.$$

$A = 1, \quad B = 2, \quad C = 1 \quad \Rightarrow \quad D = 4 - 4\times 1 = 0$

Therefore the PDE is **parabolic** everywhere.

$$\lambda = \frac{-2 \pm \sqrt{0}}{2} = -1 \text{ or } -1$$

The complementary function (and general solution) is

$$u(x,y) = f(y-x) + h(x,y)\,g(y-x)$$

where $h(x, y)$ is any convenient non-trivial linear function of (x, y) except a multiple of $(y-x)$. Choosing, arbitrarily, $h(x, y) = x$,

$$u(x,y) = f(y-x) + x\,g(y-x)$$

Imposing the boundary conditions:

$u(0, y) = 0 \quad \Rightarrow \quad f(y) + 0 = 0$

Therefore the function f is identically zero, for any argument including $(y-x)$.

We now have $u(x, y) = x\,g(y-x)$.

$u(x, 1) = x^2 \quad \Rightarrow \quad x\,g(1-x) = x^2 \quad \Rightarrow \quad g(1-x) = x$

Let $z = 1-x$, then $x = 1-z$ and $g(z) = 1-z \quad \Rightarrow \quad g(x) = 1-x$

Therefore

$u(x, y) = x\,g(y-x) = x\,(1-(y-x))$

The complete solution is

$$\boxed{u(x,y) = x(x-y+1)}$$

Two-dimensional Laplace Equation

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

$A = C = 1, \quad B = 0 \quad \Rightarrow \quad D = 0 - 4 < 0$

This PDE is **elliptic** everywhere.

$$\lambda = \frac{0 \pm \sqrt{-4}}{2} = \pm j$$

The general solution is

$$\boxed{u(x,y) = f(y - jx) + g(y + jx)}$$

where f and g are *any* twice-differentiable functions.

A function $f(x, y)$ is **harmonic** if and only if $\nabla^2 f = 0$ everywhere inside a domain Ω.

Example 4.2.6

Is $u = e^x \sin y$ harmonic on $\mathbb{R}^2$?

$$\frac{\partial u}{\partial x} = e^x \sin y \quad \text{and} \quad \frac{\partial u}{\partial y} = e^x \cos y$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = e^x \sin y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -e^x \sin y$$

$$\Rightarrow \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^x \sin y - e^x \sin y = 0 \qquad \forall (x,y)$$

Therefore **yes**, $u = e^x \sin y$ is harmonic on $\mathbb{R}^2$.

4.3 The Wave Equation – Vibrating Finite String

The wave equation is

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u}$$

If $u(x, t)$ is the vertical displacement of a point at location x on a vibrating string at time t, then the governing PDE is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

If $u(x, y, t)$ is the vertical displacement of a point at location (x, y) on a vibrating membrane at time t, then the governing PDE is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

or, in plane polar coordinates (r, θ), (appropriate for a circular drum),

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

Example 4.3.1

An elastic string of length L is fixed at both ends ($x = 0$ and $x = L$). The string is displaced into the form $y = f(x)$ and is released from rest. Find the displacement $y(x, t)$ at all locations on the string ($0 < x < L$) and at all subsequent times ($t > 0$).

The boundary value problem for the displacement function $y(x, t)$ is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 < x < L \quad \text{and} \quad t > 0$$

Both ends fixed for all time: $y(0, t) = y(L, t) = 0 \quad \text{for} \quad t \geq 0$

Initial configuration of string: $y(x, 0) = f(x) \quad \text{for} \quad 0 \leq x \leq L$

String released from rest: $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = 0 \qquad \text{for } 0 \leq x \leq L$

Example 4.3.1 (continued)

Separation of Variables (or Fourier Method)

Attempt a solution of the form $y(x, t) = X(x)\, T(t)$

Substitute $y(x, t) = X(x)\, T(t)$ into the PDE:

$$\frac{\partial^2}{\partial t^2}\big(X(x)T(t)\big) = c^2 \frac{\partial^2}{\partial x^2}\big(X(x)T(t)\big) \quad \Rightarrow \quad X\frac{d^2T}{dt^2} = c^2 \frac{d^2X}{dx^2}T$$

$$\Rightarrow \quad \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$

The left hand side of this equation is a function of t only. At any instant t it must have the same value at all values of x. Therefore the right hand side, which is a function of x only, must at any one instant have that same value at all values of x.

By a similar argument, the right hand side of this equation is a function of x only. At any location x it must have the same value at all times t. Therefore the left hand side, which is a function of t only, must at any one location have that same value at all times t.

Thus both sides of this differential equation must be the same absolute constant, which we shall represent for now by $-k$.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k \quad \Rightarrow \quad \frac{d^2X}{dx^2} + kX = 0$$

The general solution of this simple second order ODE is a combination of two exponential functions of x if $k < 0$, is a linear function of x if $k = 0$ and is a combination of sine and cosine functions of x if $k > 0$.

It is not possible for both ends of the string to be fixed for all time in the first two cases (unless we admit the trivial solution $y(x, t) \equiv 0$, a string that never moves from its equilibrium position). Therefore $k > 0$. Replace k by λ^2 (guaranteed positive for all real λ except $\lambda = 0$).

We now have the pair of ODEs

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + \lambda^2 c^2 T = 0$$

The general solutions are

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad \text{and} \quad T(t) = C\cos(\lambda ct) + D\sin(\lambda ct)$$

respectively, where A, B, C and D are arbitrary constants.

Example 4.3.1 (continued)

Consider the boundary conditions:

$y(0,t) = X(0)T(t) = 0 \quad \forall t \geq 0$

For a non-trivial solution, this requires $X(0)=0 \quad \Rightarrow \quad A=0$.

$y(L,t) = X(L)T(t) = 0 \quad \forall t \geq 0 \qquad \Rightarrow \quad X(L)=0$

$\Rightarrow \quad B\sin(\lambda L)=0 \quad \Rightarrow \quad \lambda_n = \dfrac{n\pi}{L}, \quad (n \in \mathbb{Z})$

We now have a solution only for a discrete set of eigenvalues λ_n, with corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

Consider the initial condition:

$\left.\dfrac{\partial y}{\partial t}\right|_{(x,0)} = X(x)T'(0) = 0 \quad \forall x \quad \Rightarrow \quad T'(0) = 0$

$T'(t) = -C\lambda c\sin(\lambda ct) + D\lambda c\cos(\lambda ct) \quad \Rightarrow \quad T'(0) = D\lambda c = 0 \quad \Rightarrow \quad D = 0$

Therefore our complete solution for $y(x, t)$ is now some linear combination of

$$y_n(x,t) = X_n(x)T_n(t) = C_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

There is one condition remaining to be satisfied.
The initial configuration of the string is: $y(x, 0) = f(x)$ for $0 \leq x \leq L$.

$$\Rightarrow \quad y(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

This is precisely the Fourier sine series expansion of $f(x)$ on $[0, L]$!
From Fourier series theory (Chapter 3), the coefficients C_n are

$$C_n = \frac{2}{L}\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du$$

Therefore our complete solution is

$$\boxed{y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)}$$

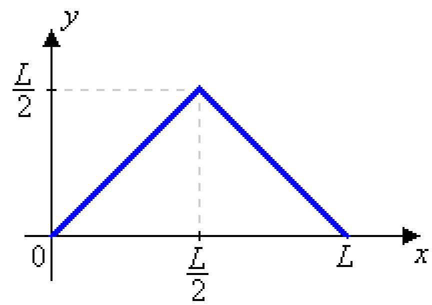
Example 4.3.1 (continued)

This solution is valid for any initial displacement function $f(x)$ that is continuous with a piece-wise continuous derivative on $[0, L]$ with $f(0) = f(L) = 0$.

If the initial displacement is itself sinusoidal $\left(f(x) = a \sin\left(\frac{n\pi x}{L}\right) \text{ for some } n \in \mathbb{N} \right)$, then the complete solution is a single term from the infinite series,

$$y(x,t) = a \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right).$$

Suppose that the initial configuration is triangular:

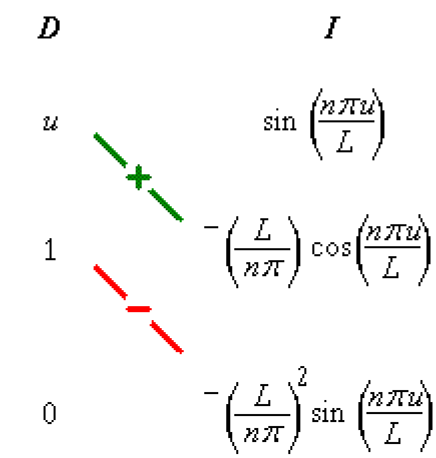


$$y(x,0) = f(x) = \begin{cases} x & \left(0 \le x \le \frac{1}{2}L\right) \\ L - x & \left(\frac{1}{2}L < x \le L\right) \end{cases}$$

Then the Fourier sine coefficients are

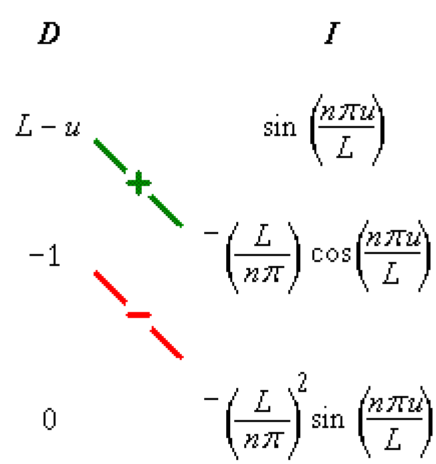
$$C_n = \frac{2}{L}\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du$$

$$= \frac{2}{L}\int_0^{L/2} u \sin\left(\frac{n\pi u}{L}\right)du + \frac{2}{L}\int_{L/2}^{L} (L-u)\sin\left(\frac{n\pi u}{L}\right)du$$



$$= \frac{2}{L}\cdot\left(\frac{L}{n\pi}\right)^2 \left\{ \left[-\left(\frac{n\pi u}{L}\right)\cos\left(\frac{n\pi u}{L}\right) + \sin\left(\frac{n\pi u}{L}\right) \right]_0^{L/2} \right.$$

$$\left. + \left[-\left(\frac{n\pi(L-u)}{L}\right)\cos\left(\frac{n\pi u}{L}\right) - \sin\left(\frac{n\pi u}{L}\right) \right]_{L/2}^{L} \right\}$$



Example 4.3.1 (continued)

$$= \frac{2L}{(n\pi)^2}\left\{\left(-\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right) - (0-0)\right.$$

$$\left. + (-0+0) - \left(-\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right)\right\}$$

$$= \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \qquad \text{But} \quad \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & (n \text{ even}) \\ \pm 1 & (n \text{ odd}) \end{cases}$$

$$\sin\left(\frac{(2k-1)\pi}{2}\right) = (-1)^{k+1}, \quad (k \in \mathbb{N})$$

Therefore sum over the odd integer values of n only ($n = 2k - 1$).

$$C_k = \frac{4L}{((2k-1)\pi)^2}(-1)^{k+1}$$

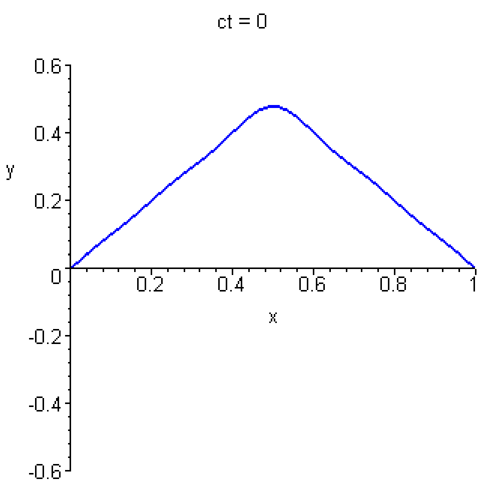
and

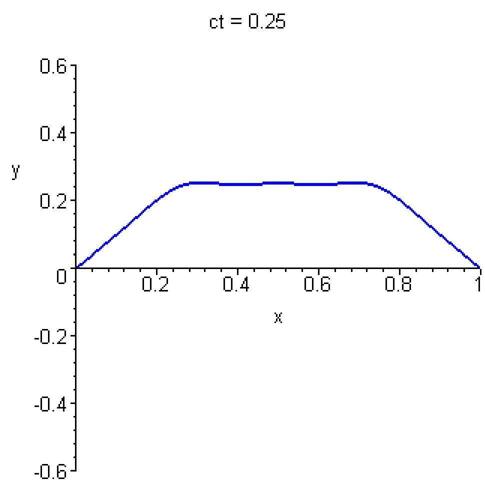
$$\boxed{y(x,t) = \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin\left(\frac{(2k-1)\pi x}{L}\right)\cos\left(\frac{(2k-1)\pi c t}{L}\right)}$$

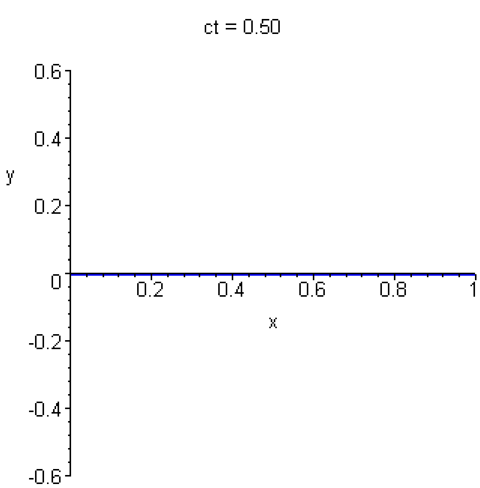
See the web page "www.engr.mun.ca/~ggeorge/5432/demos/ex431.html" for an animation of this solution.

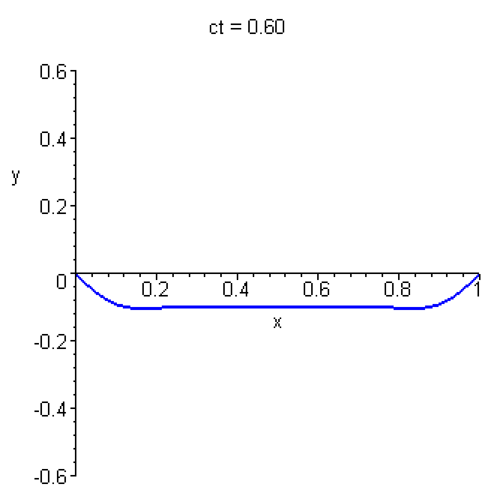
Example 4.3.1 (continued)

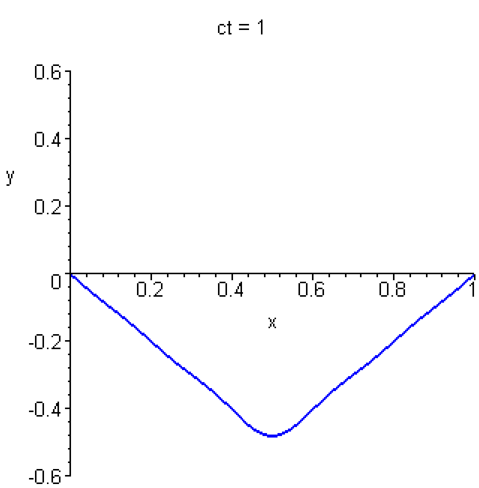
Some snapshots of the solution are shown here:











These graphs were generated from the Fourier series, truncated after the fifth non-zero term.

Example 4.3.2

An elastic string of length L is fixed at both ends ($x = 0$ and $x = L$). The string is initially in its equilibrium state [$y(x, 0) = 0$ for all x] and is released with the initial velocity $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$. Find the displacement $y(x, t)$ at all locations on the string ($0 < x < L$) and at all subsequent times ($t > 0$).

The boundary value problem for the displacement function $y(x, t)$ is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 < x < L \quad \text{and} \quad t > 0$$

Both ends fixed for all time: $y(0, t) = y(L, t) = 0$ for $t \geq 0$

Initial configuration of string: $y(x, 0) = 0$ for $0 \leq x \leq L$

String released with initial velocity: $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$ for $0 \leq x \leq L$

As before, attempt a solution by the method of the **separation of variables**.

Substitute $y(x, t) = X(x)\, T(t)$ into the PDE:

$$\frac{\partial^2}{\partial t^2}\big(X(x)T(t)\big) = c^2 \frac{\partial^2}{\partial x^2}\big(X(x)T(t)\big) \qquad \Rightarrow \quad X\frac{d^2T}{dt^2} = c^2 \frac{d^2X}{dx^2}T$$

Again, each side must be a negative constant.

$$\Rightarrow \quad \frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

We now have the pair of ODEs

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + \lambda^2 c^2 T = 0$$

The general solutions are

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad \text{and} \quad T(t) = C\cos(\lambda ct) + D\sin(\lambda ct)$$

respectively, where A, B, C and D are arbitrary constants.

Example 4.3.2 (continued)

Consider the boundary conditions:

$y(0,t) = X(0)T(t) = 0 \quad \forall t \geq 0$

For a non-trivial solution, this requires $X(0) = 0 \quad \Rightarrow \quad A = 0$.

$y(L,t) = X(L)T(t) = 0 \quad \forall t \geq 0 \qquad \Rightarrow \quad X(L) = 0$

$\Rightarrow \quad B \sin(\lambda L) = 0 \quad \Rightarrow \quad \lambda_n = \dfrac{n\pi}{L}, \quad (n \in \mathbb{Z})$

We now have a solution only for a discrete set of eigenvalues λ_n, with corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

and

$$y_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{L}\right)T_n(t), \quad (n = 1, 2, 3, \ldots)$$

So far, the solution has been identical to Example 4.3.1.

Consider the initial condition $y(x, 0) = 0$:

$$y(x,0) = 0 \quad \Rightarrow \quad X(x)T(0) = 0 \quad \forall x \quad \Rightarrow \quad T(0) = 0$$

The initial value problem for $T(t)$ is now

$$T'' + \lambda^2 c^2 T = 0, \quad T(0) = 0, \qquad \text{where} \quad \lambda = \frac{n\pi}{L}$$

the solution to which is

$$T_n(t) = C_n \sin\left(\frac{n\pi c t}{L}\right), \qquad (n \in \mathbb{N})$$

Our eigenfunctions for y are now

$$y_n(x, t) = X_n(x)T_n(t) = C_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right), \qquad (n \in \mathbb{N})$$

Example 4.3.2 (continued)

Differentiate term by term and impose the initial velocity condition:

$$\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

which is just the Fourier sine series expansion for the function $g(x)$.
The coefficients of the expansion are

$$C_n \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(u) \sin\left(\frac{n\pi u}{L}\right) du$$

which leads to the complete solution

$$\boxed{y(x,t) = \frac{2}{\pi c}\sum_{n=1}^{\infty} \frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)}$$

This solution is valid for any initial velocity function $g(x)$ that is continuous with a piece-wise continuous derivative on $[0, L]$ with $g(0) = g(L) = 0$.

The solutions for Examples 4.3.1 and 4.3.2 may be superposed.

Let $y_1(x, t)$ be the solution for initial displacement $f(x)$ and zero initial velocity.
Let $y_2(x, t)$ be the solution for zero initial displacement and initial velocity $g(x)$.

Then $y(x, t) = y_1(x, t) + y_2(x, t)$ satisfies the wave equation
(the sum of any two solutions of a linear homogeneous PDE is also a solution),
and satisfies the boundary conditions $y(0, t) = y(L, t) = 0$.

$y(x, 0) = y_1(x, 0) + y_2(x, 0) = f(x) + 0$,
which satisfies the condition for initial displacement $f(x)$.

$y_t(x, 0) = y_{1t}(x, 0) + y_{2t}(x, 0) = 0 + g(x)$,
which satisfies the condition for initial velocity $g(x)$.

Therefore the sum of the two solutions is the complete solution for initial displacement $f(x)$ **and** initial velocity $g(x)$:

$$\boxed{\begin{aligned} y(x,t) &= \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right) \\ &\quad + \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right) \end{aligned}}$$

4.4 The Maximum-Minimum Principle

Let Ω be some finite domain on which a function $u(x, y)$ and its second derivatives are defined. Let $\overline{\Omega}$ be the union of the domain with its boundary.
Let m and M be the minimum and maximum values respectively of u on the boundary of the domain.

If $\nabla^2 u \geq 0$ in Ω, then u is **subharmonic** and

$$u(\vec{\mathbf{r}}) < M \quad \text{or} \quad u(\vec{\mathbf{r}}) \equiv M \qquad \forall \vec{\mathbf{r}} \text{ in } \Omega$$

If $\nabla^2 u \leq 0$ in Ω, then u is **superharmonic** and

$$u(\vec{\mathbf{r}}) > m \quad \text{or} \quad u(\vec{\mathbf{r}}) \equiv m \qquad \forall \vec{\mathbf{r}} \text{ in } \Omega$$

If $\nabla^2 u = 0$ in Ω, then u is **harmonic** (both subharmonic and superharmonic) and u is either constant on $\overline{\Omega}$ or $m < u < M$ everywhere on Ω.

Example 4.4.1

$\nabla^2 u = 0$ in $\Omega : x^2 + y^2 < 1$ and $u(x, y) = 1$ on $C :\ x^2 + y^2 = 1$.
Find $u(x, y)$ on Ω.

---

u is harmonic on $\Omega \quad \Rightarrow \quad \min_C(u) \leq \begin{pmatrix} u(x,y) \\ \text{on } \Omega \end{pmatrix} \leq \max_C(u)$

But $\min_C(u) = \max_C(u) = 1$

Therefore $\boldsymbol{u(x, y) = 1}$ **everywhere in $\boldsymbol{\Omega}$**.

Example 4.4.2

$\nabla^2 u = 0$ in the square domain $\Omega : -2 < x < +2,\ -2 < y < +2$.
On the boundary C, on the left and right edges ($x = \pm 2$), $u(x, y) = 4 - y^2$,
while on the top and bottom edges ($y = \pm 2$), $u(x, y) = x^2 - 4$.

Find bounds on the value of $u(x, y)$ inside the domain Ω.

For $-2 \le y \le +2$, $0 \le 4 - y^2 \le 4$.
For $-2 \le x \le +2$, $-4 \le x^2 - 4 \le 0$.
Therefore, on the boundary C of the domain Ω, $-4 \le u(x, y) \le +4$ so that
$m = -4$ and $M = +4$.

$u(x, y)$ is harmonic (because $\nabla^2 u = 0$).

Therefore, everywhere in Ω,

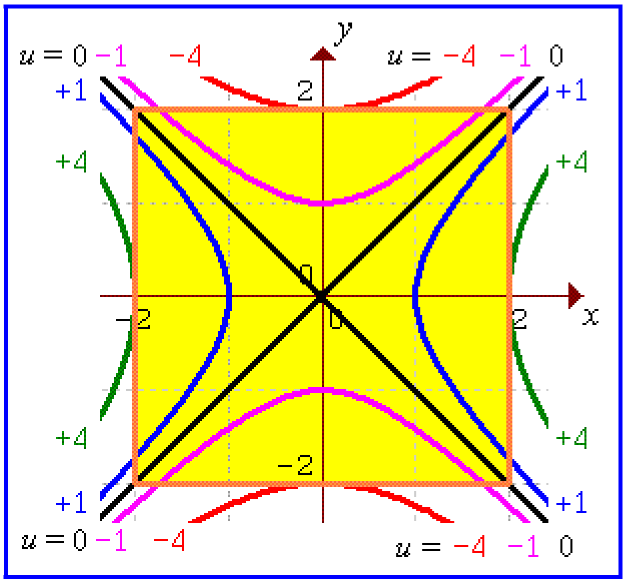
$$-4 < u(x, y) < +4$$

Note:
$u(x, y) = x^2 - y^2$ is consistent with the boundary condition and

$$\frac{\partial u}{\partial x} = 2x - 0\,, \quad \frac{\partial u}{\partial y} = 0 - 2y \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 2\,, \quad \frac{\partial^2 u}{\partial y^2} = -2 = -\frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Contours of constant values of u are hyperbolas.
A contour map illustrates that $-4 < u(x, y) < +4$ within the domain is indeed true.



4.5 The Heat Equation

For a material of constant density ρ, constant specific heat μ and constant thermal conductivity K, the partial differential equation governing the temperature u at any location (x, y, z) and any time t is

$$\frac{\partial u}{\partial t} = k \nabla^2 u \ , \quad \text{where} \quad k = \frac{K}{\mu\rho}$$

Example 4.5.1

Heat is conducted along a thin homogeneous bar extending from $x = 0$ to $x = L$. There is no heat loss from the sides of the bar. The two ends of the bar are maintained at temperatures T_1 (at $x = 0$) and T_2 (at $x = L$). The initial temperature throughout the bar at the cross-section x is $f(x)$.

Find the temperature at any point in the bar at any subsequent time.

The partial differential equation governing the temperature $u(x, t)$ in the bar is

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \qquad \text{[Parabolic]}$$

together with the boundary conditions

$u(0, t) = T_1$ and $u(L, t) = T_2$

and the initial condition

$u(x, 0) = f(x)$

[Note that if an end of the bar is insulated, instead of being maintained at a constant temperature, then the boundary condition changes to $\frac{\partial u}{\partial t}(0,t) = 0$ or $\frac{\partial u}{\partial t}(L,t) = 0$.]

Attempt a solution by the method of separation of variables.

$u(x, t) = X(x)\, T(t)$

$$\Rightarrow \quad X T' = k X'' T \qquad \Rightarrow \quad \frac{T'}{T} = k \frac{X''}{X} = c$$

Again, when a function of t only equals a function of x only, both functions must equal the same absolute constant. Unfortunately, the two boundary conditions cannot both be satisfied unless $T_1 = T_2 = 0$. Therefore we need to treat this more general case as a perturbation of the simpler ($T_1 = T_2 = 0$) case.

Example 4.5.1 (continued)

Let $u(x, t) = v(x, t) + g(x)$
Substitute this into the PDE:

$$\frac{\partial}{\partial t}\big(v(x,t)+g(x)\big) = k\,\frac{\partial^2}{\partial x^2}\big(v(x,t)+g(x)\big) \quad \Rightarrow \quad \frac{\partial v}{\partial t} = k\left(\frac{\partial^2 v}{\partial x^2} + g''(x)\right)$$

This is the standard heat PDE for v if we choose g such that $g''(x) = 0$.
$g(x)$ must therefore be a linear function of x.

We want the perturbation function $g(x)$ to be such that

$u(0, t) = T_1$, $u(L, t) = T_2$

and

$v(0, t) = v(L, t) = 0$

Therefore $g(x)$ must be the linear function for which $g(0) = T_1$ and $g(L) = T_2$.
It follows that

$$g(x) = \left(\frac{T_2 - T_1}{L}\right)x + T_1$$

and we now have the simpler problem

$$\frac{\partial v}{\partial t} = k\,\frac{\partial^2 v}{\partial x^2}$$

together with the boundary conditions

$v(0, t) = v(L, t) = 0$

and the initial condition

$v(x, 0) = f(x) - g(x)$

Now try separation of variables on $v(x, t)$:
$v(x, t) = X(x)\,T(t)$

$$\Rightarrow \quad X\,T' = k\,X''\,T \qquad \Rightarrow \quad \frac{1}{k}\frac{T'}{T} = \frac{X''}{X} = c$$

But $v(0, t) = v(L, t) = 0 \Rightarrow X(0) = X(L) = 0$

This requires c to be a negative constant, say $-\lambda^2$.
The solution is very similar to that for the wave equation on a finite string with fixed ends (section 4.3). The eigenvalues are $\lambda = \dfrac{n\pi}{L}$ and the corresponding eigenfunctions are any non-zero constant multiples of

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

Example 4.5.1 (continued)

The ODE for $T(t)$ becomes

$$T' + \left(\frac{n\pi}{L}\right)^2 kT = 0$$

whose general solution is

$$T_n(t) = c_n e^{-n^2\pi^2 kt/L^2}$$

Therefore

$$v_n(x,t) = X_n(x)T_n(t) = c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$$

If the initial temperature distribution $f(x) - g(x)$ is a simple multiple of $\sin\left(\frac{n\pi x}{L}\right)$ for some integer n, then the solution for v is just $v(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$.

Otherwise, we must attempt a superposition of solutions.

$$v(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)$$

such that $v(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x) - g(x)$.

The Fourier sine series coefficients are $c_n = \frac{2}{L}\int_0^L \left(f(z) - g(z)\right)\sin\left(\frac{n\pi z}{L}\right) dz$

so that the complete solution for $v(x, t)$ is

$$\boxed{v(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L \left(f(z) - \frac{T_2 - T_1}{L}z - T_1\right)\sin\left(\frac{n\pi z}{L}\right)dz\right)\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 kt}{L^2}\right)}$$

and the complete solution for $u(x, t)$ is

$$\boxed{u(x,t) = v(x,t) + \left(\frac{T_2 - T_1}{L}\right)x + T_1}$$

Note how this solution can be partitioned into a transient part $v(x, t)$ (which decays to zero as t increases) and a steady-state part $g(x)$ which is the limiting value that the temperature distribution approaches.

Example 4.5.1 (continued)

As a specific example, let $k = 9$, $T_1 = 100$, $T_2 = 200$, $L = 2$ and
$f(x) = 145x^2 - 240x + 100$, (for which $f(0) = 100$, $f(2) = 200$ and $f(x) > 0 \;\; \forall x$).

Then $g(x) = \dfrac{200-100}{2}x + 100 = 50x + 100$

The Fourier sine series coefficients are

$$c_n = \int_0^2 \left(\left(145z^2 - 240z + 100\right) - \left(50z + 100\right)\right) \sin\left(\frac{n\pi z}{2}\right) dz$$

$$\Rightarrow \quad c_n = 145 \int_0^2 \left(z^2 - 2z\right) \sin\left(\frac{n\pi z}{2}\right) dz$$

	D	**I**
	$z^2 - 2z$	$\sin\left(\frac{n\pi z}{2}\right)$
$+$	$2z - 2$	$-\left(\frac{2}{n\pi}\right)\cos\left(\frac{n\pi z}{2}\right)$
$-$	2	$-\left(\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi z}{2}\right)$
$+$	0	$\left(\frac{2}{n\pi}\right)^3\cos\left(\frac{n\pi z}{2}\right)$

$$\Rightarrow \quad c_n = 145 \left[\left(\left(-z^2 + 2z\right)\frac{2}{n\pi} + \frac{16}{(n\pi)^3} \right) \cos\left(\frac{n\pi z}{2}\right) + \frac{8(z-1)}{(n\pi)^2} \sin\left(\frac{n\pi z}{2}\right) \right]_{z=0}^{z=2}$$

$$\Rightarrow \quad c_n = 145 \left[\left(z(2-z)\frac{2}{n\pi} + \frac{16}{(n\pi)^3} \right) \cos\left(\frac{n\pi z}{2}\right) + \frac{8(z-1)}{(n\pi)^2} \sin\left(\frac{n\pi z}{2}\right) \right]_{z=0}^{z=2}$$

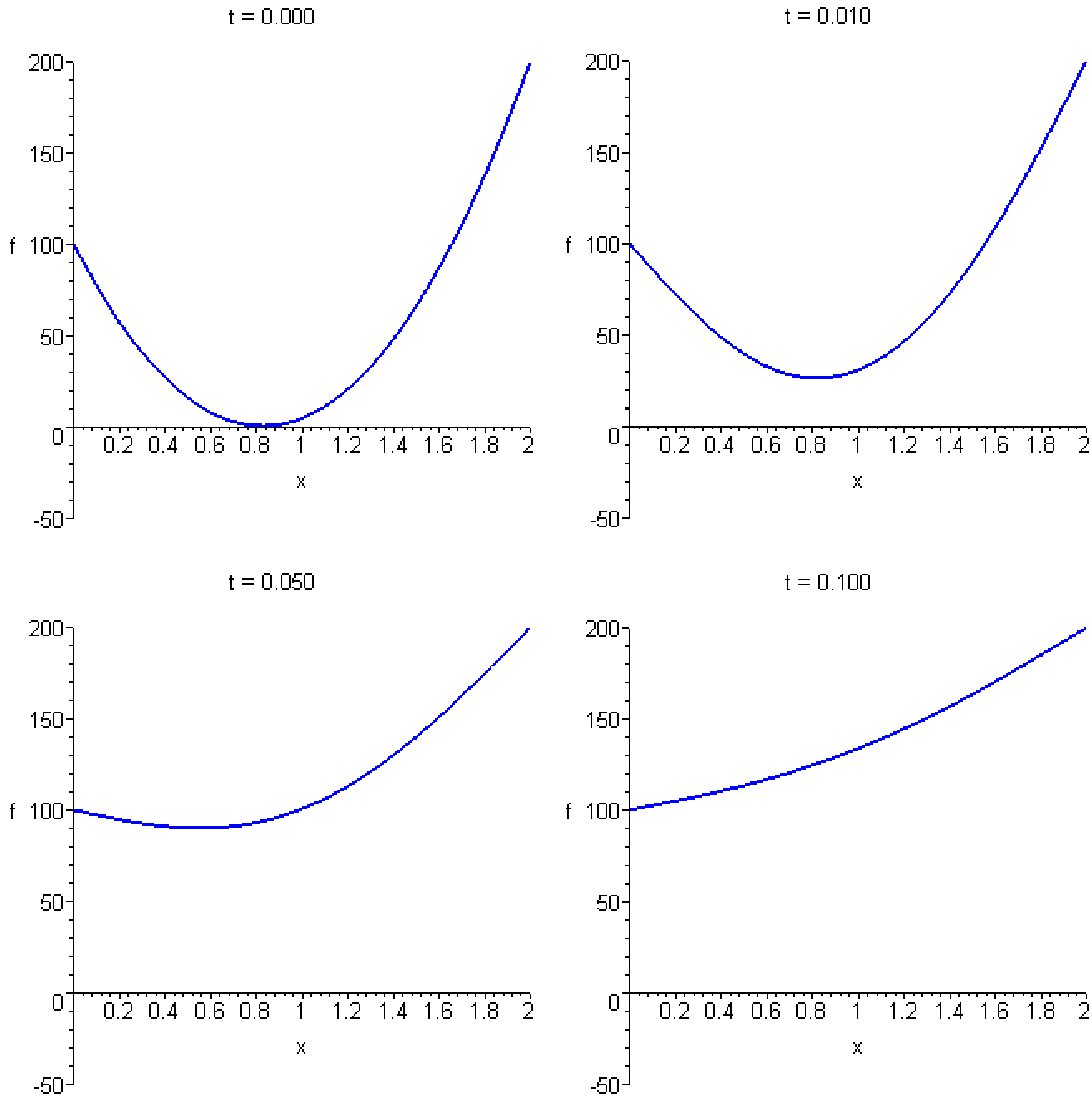
$$\Rightarrow \quad c_n = \frac{2320}{(n\pi)^3}\left((-1)^n - 1\right)$$

The complete solution is

$$\boxed{u(x,t) = 50x + 100 - \frac{2320}{\pi^3} \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n^3} \right) \sin\left(\frac{n\pi x}{2}\right) \exp\left(-\frac{9n^2\pi^2 t}{4}\right)}$$

Some snapshots of the temperature distribution (from the tenth partial sum) from the Maple file at "www.engr.mun.ca/~ggeorge/5432/demos/ex451.mws" are shown on the next page.

Example 4.5.1 (continued)



The steady state distribution is nearly attained in much less than a second!

END OF CHAPTER 4
END OF ENGI. 5432!