5. Suggestions for the Formula Sheets

Below are some suggestions for many more formulae than can be placed easily on both sides of the two standard 8½"×11" sheets of paper for the final examination. It is strongly recommended that students compose their own formula sheets, to suit each individual's needs.

1.1 Vectors

The **component** of vector $\vec{\mathbf{u}}$ in the direction of vector $\vec{\mathbf{v}}$ is $u_v = \vec{\mathbf{u}} \bullet \hat{\mathbf{v}} = u \cos\theta$

$$\frac{d}{dt}\left(\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}\right) = \frac{d\vec{\mathbf{u}}}{dt} \bullet \vec{\mathbf{v}} + \vec{\mathbf{u}} \bullet \frac{d\vec{\mathbf{v}}}{dt} \quad \text{and}$$

$$\frac{d}{dt}\left(\vec{\mathbf{u}} \times \vec{\mathbf{v}}\right) = \frac{d\vec{\mathbf{u}}}{dt} \times \vec{\mathbf{v}} + \vec{\mathbf{u}} \times \frac{d\vec{\mathbf{v}}}{dt} = -\vec{\mathbf{v}} \times \frac{d\vec{\mathbf{u}}}{dt} + \vec{\mathbf{u}} \times \frac{d\vec{\mathbf{v}}}{dt}$$

The **distance along a curve** between two points whose parameter values are t_0 and t_1 is

$$L = \int_{t=t_0}^{t=t_1} ds = \int_{t_0}^{t_1} \frac{ds}{dt}\, dt = \int_{t_0}^{t_1} \left|\frac{d\vec{\mathbf{r}}}{dt}\right| dt = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

The **unit tangent** at any point on a curve is

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{dt} \div \left|\frac{d\vec{\mathbf{r}}}{dt}\right| = \frac{d\vec{\mathbf{r}}}{ds}$$

The **unit principal normal** at any point on a curve is

$$\hat{\mathbf{N}} = \rho \frac{d\hat{\mathbf{T}}}{ds} = \frac{d\hat{\mathbf{T}}}{dt} \div \left|\frac{d\hat{\mathbf{T}}}{dt}\right|$$

The **unit binormal** is

$$\hat{\mathbf{B}} = \hat{\mathbf{T}} \times \hat{\mathbf{N}}$$

Velocity is $\vec{\mathbf{v}}(t) = \dfrac{d\vec{\mathbf{r}}}{dt}$, **speed** is $v(t) = \left|\vec{\mathbf{v}}(t)\right| = \left|\dfrac{d\vec{\mathbf{r}}}{dt}\right| = \dfrac{ds}{dt}$ and $\vec{\mathbf{v}} = v\hat{\mathbf{T}}$

The **acceleration** [vector] is

$$\vec{\mathbf{a}}(t) = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d^2\vec{\mathbf{r}}}{dt^2} = \left\langle \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2} \right\rangle = a_T\hat{\mathbf{T}} + a_N\hat{\mathbf{N}}$$

where $a_T = \dfrac{dv}{dt} = \vec{\mathbf{a}} \bullet \hat{\mathbf{T}} = \dfrac{\vec{\mathbf{v}} \bullet \vec{\mathbf{a}}}{v}$ and $a_N = \kappa v^2 = \sqrt{a^2 - a_T^{\,2}} = \vec{\mathbf{a}} \bullet \hat{\mathbf{N}} = \dfrac{\left|\vec{\mathbf{v}} \times \vec{\mathbf{a}}\right|}{v}$

1.2 Lines of Force

The lines of force **r** associated with a vector field **F** are given by

$$\hat{\mathbf{T}} = \frac{d\vec{\mathbf{r}}}{ds} \parallel \vec{\mathbf{F}} \quad \Rightarrow \quad \frac{d\vec{\mathbf{r}}}{ds} = k\vec{\mathbf{F}}(s) \quad \Rightarrow \quad (k\,ds =) \quad \boxed{\frac{dx}{f_1} = \frac{dy}{f_2} = \frac{dz}{f_3}}$$

1.3 The Gradient Vector

The **directional derivative** of f in the direction of the unit vector $\hat{\mathbf{u}}$ is $D_{\hat{\mathbf{u}}} f = \vec{\nabla} f \bullet \hat{\mathbf{u}}$

$$\frac{df}{dt} = \vec{\nabla} f \bullet \frac{d\vec{\mathbf{r}}}{dt}, \quad \text{where} \quad \vec{\nabla} f = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right\rangle \quad \text{and} \quad \frac{d\vec{\mathbf{r}}}{dt} = \left\langle \frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_n}{dt} \right\rangle$$

Plane with normal vector $\mathbf{n} = \langle A, B, C \rangle$ passing through point $\mathbf{r}_o = \langle x_o, y_o, z_o \rangle$:

$$\vec{\mathbf{n}} \bullet \vec{\mathbf{r}} = \vec{\mathbf{n}} \bullet \vec{\mathbf{r}}_o \quad \Rightarrow \quad Ax + By + Cz + D = 0, \quad \text{where} \quad D = -(Ax_o + By_o + Cz_o)$$

Line with direction vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ passing through point $\mathbf{r}_o = \langle x_o, y_o, z_o \rangle$:

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_o + t\vec{\mathbf{v}} \quad \Rightarrow \quad \frac{x - x_o}{v_1} = \frac{y - y_o}{v_2} = \frac{z - z_o}{v_3} \quad (\text{except when a } v_i = 0)$$

Tangent plane to $f(x, y, z) = c$ at $P(x_o, y_o, z_o)$ is $\vec{\mathbf{n}} \bullet \vec{\mathbf{r}} = \vec{\mathbf{n}} \bullet \vec{\mathbf{r}}_o$, where $\vec{\mathbf{n}} = \vec{\nabla} f \big|_P$.

1.4 Divergence and Curl

$$\vec{\nabla} \times \vec{\nabla} f = \text{curl grad } f = \vec{\mathbf{0}}$$
$$\vec{\nabla} \bullet \vec{\nabla} \times \vec{\mathbf{F}} = \text{div curl } \vec{\mathbf{F}} = 0$$

$$\text{Laplacian of } V = \nabla^2 V = \vec{\nabla} \bullet (\vec{\nabla} V) = \text{div grad } V$$

If $\vec{\mathbf{v}}(x, y) = u(x, y)\hat{\mathbf{i}} + v(x, y)\hat{\mathbf{j}}$ and div $\vec{\mathbf{v}} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$, then the stream function $\psi(x, y)$ exists such that $\frac{\partial \psi}{\partial x} = v$ and $\frac{\partial \psi}{\partial y} = -u$. **Streamlines** are $\psi(x, y) = c$.

1.5 Conversions between Coordinate Systems

To convert a vector expressed in Cartesian components $v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$ into the equivalent vector expressed in **cylindrical polar** coordinates $v_\rho \hat{\boldsymbol{\rho}} + v_\phi \hat{\boldsymbol{\phi}} + v_z \hat{\mathbf{k}}$, express the Cartesian components v_x, v_y, v_z in terms of (ρ, ϕ, z) using

$x = \rho \cos \phi, \quad y = \rho \sin \phi, \quad z = z$; then evaluate

1.5 Conversions between Coordinate Systems (continued)

$$\begin{bmatrix} v_\rho \\ v_\phi \\ v_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Use the inverse matrix [= transpose] to transform back to Cartesian coordinates.

To convert a vector expressed in Cartesian components $v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}$ into the equivalent vector expressed in **<u>spherical polar</u>** <u>coordinates</u> $v_r \hat{\mathbf{r}} + v_\theta \hat{\boldsymbol{\theta}} + v_\phi \hat{\boldsymbol{\phi}}$, express the Cartesian components v_x, v_y, v_z in terms of (r, θ, ϕ) using

$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta;$ then evaluate

$$\begin{bmatrix} v_r \\ v_\theta \\ v_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

Use the inverse matrix [= transpose] to transform back to Cartesian coordinates.

1.6 <u>Basis Vectors in Other Coordinate Systems</u>

<u>Cylindrical Polar</u>:

$$\frac{d}{dt}\hat{\boldsymbol{\rho}} = \frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d}{dt}\hat{\boldsymbol{\phi}} = -\frac{d\phi}{dt}\hat{\boldsymbol{\rho}}$$

$$\frac{d}{dt}\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{r}} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{k}}$$

$$\Rightarrow \quad \vec{\mathbf{v}} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\mathbf{e}}_z$$

<u>Spherical Polar</u>:

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \sin\theta\,\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}} + \cos\theta\,\frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$

$$\frac{d\hat{\boldsymbol{\phi}}}{dt} = -\left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)\frac{d\phi}{dt}$$

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad \Rightarrow \quad \vec{\mathbf{v}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}}$$

1.7 Gradient Operator in Other Coordinate Systems

Gradient operator $$\vec{\nabla} = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial}{\partial u_3}$$

Gradient $$\vec{\nabla} V = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial V}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial V}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial V}{\partial u_3}$$

Divergence $$\vec{\nabla} \bullet \vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial\left(h_2 h_3 f_1\right)}{\partial u_1} + \frac{\partial\left(h_3 h_1 f_2\right)}{\partial u_2} + \frac{\partial\left(h_1 h_2 f_3\right)}{\partial u_3} \right)$$

Curl $$\vec{\nabla} \times \vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{\mathbf{e}}_1 & h_2 \hat{\mathbf{e}}_2 & h_3 \hat{\mathbf{e}}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix}$$

Laplacian $$\nabla^2 V = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial V}{\partial u_1} \right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial V}{\partial u_2} \right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial V}{\partial u_3} \right) \right)$$

$$dV = h_1 h_2 h_3 \; du_1 du_2 du_3 = \left| \frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} \right| du_1 du_2 du_3$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix} du_1 du_2 du_3$$

Cartesian: $h_x = h_y = h_z = 1$.

Cylindrical polar: $h_\rho = h_z = 1, \quad h_\phi = \rho$.

Spherical polar: $h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta$.

2.1 Line Integrals

The location $\langle \vec{\mathbf{r}} \rangle$ of the centre of mass of a wire is $\langle \vec{\mathbf{r}} \rangle = \dfrac{\vec{\mathbf{M}}}{m}$, where

$$\vec{\mathbf{M}} = \int_{t_0}^{t_1} \rho\, \vec{\mathbf{r}} \frac{ds}{dt}\, dt\,, \quad m = \int_{t_0}^{t_1} \rho \frac{ds}{dt}\, dt \quad \text{and} \quad \frac{ds}{dt} = \left| \frac{d\vec{\mathbf{r}}}{dt} \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,.$$

2.2 Green's Theorem

For a simple closed curve C enclosing a finite region D of $\mathbb{R}^2$ and for any vector function $\vec{\mathbf{F}} = \langle f_1, f_2 \rangle$ that is differentiable everywhere on C and everywhere in D,

$$\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \iint_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dA$$

2.3 Path Independence

When a vector field **F** is defined on a simply connected domain Ω, these statements are all equivalent (that is, **all** of them are true or all of them are false):

- $\vec{\mathbf{F}} = \vec{\nabla} \phi$ for some scalar field ϕ that is differentiable everywhere in Ω;
- **F** is conservative;
- $\int_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}}$ is path-independent (has the same value no matter which path within Ω is chosen between the two endpoints, for any two endpoints in Ω);
- $\int_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = \phi_{\text{end}} - \phi_{\text{start}}$ (for any two endpoints in Ω);
- $\oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} = 0$ for all closed curves C lying entirely in Ω;
- $\dfrac{\partial f_2}{\partial x} = \dfrac{\partial f_1}{\partial y}$ everywhere in Ω; and
- $\vec{\nabla} \times \vec{\mathbf{F}} = \vec{\mathbf{0}}$ everywhere in Ω (so that the vector field **F** is irrotational).

There must be no singularities anywhere in the domain Ω in order for the above set of equivalencies to be valid.

2.4 Surface Integrals - Projection Method

For surfaces $z = f(x, y)$, $\vec{\mathbf{N}} = \left\langle -\dfrac{\partial f}{\partial x}, -\dfrac{\partial f}{\partial y}, +1 \right\rangle$ and

$$\iint_S g(\vec{\mathbf{r}})\, dS = \iint_D g(\vec{\mathbf{r}}) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA \quad (\text{where } dA = dx\, dy)$$

2.5 Surface Integrals - Surface Method

With a coordinate grid (u, v) on the surface S, $\iint_S g(\vec{\mathbf{r}})\, dS = \iint_S g(\vec{\mathbf{r}}) \left| \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v} \right| du\, dv$

The total flux of a vector field $\vec{\mathbf{F}}$ through a surface S is

$$\Phi = \iint_S \vec{\mathbf{F}} \cdot \mathbf{d}\vec{\mathbf{S}} = \iint_S \vec{\mathbf{F}} \cdot \hat{\mathbf{N}}\, dS = \iint_S \vec{\mathbf{F}} \cdot \frac{\partial \vec{\mathbf{r}}}{\partial u} \times \frac{\partial \vec{\mathbf{r}}}{\partial v}\, du\, dv$$

Some Common Parametric Nets

1) The circular plate $(x - x_o)^2 + (y - y_o)^2 \le a^2$ in the plane $z = z_o$.

Let the parameters be r, θ where $0 < r \le a$, $0 \le \theta < 2\pi$

$x = x_o + r\cos\theta, \quad y = y_o + r\sin\theta, \quad z = z_o$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial r} \times \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\right) = \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \pm r\,\hat{\mathbf{k}}$$

2) The circular cylinder $(x - x_o)^2 + (y - y_o)^2 = a^2$ with $z_o \le z \le z_1$.

Let the parameters be z, θ where $z_o \le z \le z_1$, $0 \le \theta < 2\pi$

$x = a\cos\theta, \; y = a\sin\theta, \; z = z$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial z} \times \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\right) = \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ -a\sin\theta & a\cos\theta & 0 \end{vmatrix} = \pm\left(-a\cos\theta\,\hat{\mathbf{i}} - a\sin\theta\,\hat{\mathbf{j}}\right)$$

Outward normal: $\vec{\mathbf{N}} = a\cos\theta\,\hat{\mathbf{i}} + a\sin\theta\,\hat{\mathbf{j}}$

3) The frustrum of the circular cone $w - w_o = a\sqrt{(u - u_o)^2 + (v - v_o)^2}$ where $w_1 \le w \le w_2$ and $w_o \le w_1$. Let the parameters here be r, θ where

$\frac{w_1 - w_o}{a} \le r \le \frac{w_2 - w_o}{a}, \qquad 0 \le \theta < 2\pi$

$x = u = u_o + r\cos\theta, \quad y = v = v_o + r\sin\theta, \quad z = w = w_o + ar$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial r} \times \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\right) = \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & a \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= \pm\left[(-ar\cos\theta)\,\hat{\mathbf{i}} + (-ar\sin\theta)\,\hat{\mathbf{j}} + r\,\hat{\mathbf{k}}\right]$$

Outward normal: $\vec{\mathbf{N}} = ar\cos\theta\,\hat{\mathbf{i}} + ar\sin\theta\,\hat{\mathbf{j}} - r\,\hat{\mathbf{k}}$

4) The portion of <u>the elliptic paraboloid</u>

$$z - z_o = a^2 (x - x_o)^2 + b^2 (y - y_o)^2 \quad \text{with} \quad z_o \le z_1 \le z \le z_2$$

Let the parameters here be r, θ where

$$\sqrt{\frac{z_1 - z_o}{a^2 \cos^2\theta + b^2 \sin^2\theta}} \le r \le \sqrt{\frac{z_2 - z_o}{a^2 \cos^2\theta + b^2 \sin^2\theta}} , \quad 0 \le \theta < 2\pi$$

$x = x_o + r \cos\theta$, $y = y_o + r \sin\theta$, $z = z_o + r^2 (a^2 \cos^2\theta + b^2 \sin^2\theta)$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial r} \times \frac{\partial \vec{\mathbf{r}}}{\partial \theta}\right) = \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \cos\theta & \sin\theta & 2r\left(a^2\cos^2\theta + b^2\sin^2\theta\right) \\ -r\sin\theta & r\cos\theta & 2r^2\left(b^2 - a^2\right)\sin\theta\cos\theta \end{vmatrix}$$

$$= \pm\left[\left(-2a^2r^2\cos\theta\right)\hat{\mathbf{i}} + \left(-2b^2r^2\sin\theta\right)\hat{\mathbf{j}} + r\,\hat{\mathbf{k}}\right]$$

Outward normal: $\vec{\mathbf{N}} = \left(2a^2r^2\cos\theta\right)\hat{\mathbf{i}} + \left(2b^2r^2\sin\theta\right)\hat{\mathbf{j}} - r\,\hat{\mathbf{k}}$

5) The <u>surface of the sphere</u> $(x - x_\circ)^2 + (y - y_\circ)^2 + (z - z_\circ)^2 = a^2$.

Let the parameters here be θ, ϕ where $0 \le \theta \le \pi$, $0 \le \phi < 2\pi$

$x = x_o + a \sin\theta \cos\phi$, $y = y_o + a \sin\theta \sin\phi$, $z = z_o + a \cos\theta$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial \theta} \times \frac{\partial \vec{\mathbf{r}}}{\partial \phi}\right) = \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a\cos\theta\cos\phi & a\cos\theta\sin\phi & -a\sin\theta \\ -a\sin\theta\sin\phi & a\sin\theta\cos\phi & 0 \end{vmatrix}$$

$$= \pm a^2 \sin\theta \left[\left(\sin\theta\cos\phi\right)\hat{\mathbf{i}} + \left(\sin\theta\sin\phi\right)\hat{\mathbf{j}} + \left(\cos\theta\right)\hat{\mathbf{k}}\right]$$

Outward normal: $\vec{\mathbf{N}} = a^2 \sin\theta \left[\left(\sin\theta\cos\phi\right)\hat{\mathbf{i}} + \left(\sin\theta\sin\phi\right)\hat{\mathbf{j}} + \left(\cos\theta\right)\hat{\mathbf{k}}\right]$

6) The part of <u>the plane</u> $A(x - x_\circ) + B(y - y_\circ) + C(z - z_\circ) = 0$ in the first octant with $A, B, C > 0$ and $Ax_o + By_o + Cz_o > 0$.

Let the parameters be x, y where

$$0 \le x \le \frac{Ax_\circ + By_\circ + Cz_\circ - By}{A}; \qquad 0 \le y \le \frac{Ax_\circ + By_\circ + Cz_\circ}{B}$$

$$\vec{\mathbf{N}} = \pm\left(\frac{\partial \vec{\mathbf{r}}}{\partial x} \times \frac{\partial \vec{\mathbf{r}}}{\partial y}\right) = \pm\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -A/C \\ 0 & 1 & -B/C \end{vmatrix} = \pm\left[\frac{A}{C}\hat{\mathbf{i}} + \frac{B}{C}\hat{\mathbf{j}} + \hat{\mathbf{k}}\right]$$

2.6 Theorems of Gauss and Stokes; Potential Functions

Gauss' divergence theorem: $\displaystyle \oiint_S \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{S}} \;=\; \iiint_V \vec{\nabla} \bullet \vec{\mathbf{F}}\, dV$ on a simply-connected domain.

Gauss' law for the net flux through any smooth simple closed surface S, in the presence of a point charge q, is:

$$\oiint_S \vec{\mathbf{E}} \bullet \mathbf{d}\vec{\mathbf{S}} \;=\; \begin{cases} \dfrac{q}{\varepsilon} & \text{if } S \text{ encloses } O \\ 0 & \text{otherwise} \end{cases}$$

Stokes' theorem: $\displaystyle \oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} \;=\; \iint_S \vec{\nabla} \times \vec{\mathbf{F}} \bullet \widehat{\mathbf{N}}\, dS \;=\; \iint_S \left(\text{curl}\, \vec{\mathbf{F}}\right) \bullet \mathbf{d}\vec{\mathbf{S}}$

On a simply-connected domain Ω the following statements are either all true or all false:

- $\mathbf{F}$ is conservative.
- $\mathbf{F} \equiv \nabla\phi$
- $\nabla \times \mathbf{F} \equiv \mathbf{0}$
- $\displaystyle \int_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} \;=\; \phi\left(P_{\text{end}}\right) - \phi\left(P_{\text{start}}\right)$ - independent of the path between the two points.
- $\displaystyle \oint_C \vec{\mathbf{F}} \bullet \mathbf{d}\vec{\mathbf{r}} \;=\; 0 \quad \forall C \subset \Omega$

ϕ is the potential function for $\mathbf{F}$, so that $\displaystyle \left\langle \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right\rangle \;=\; \left\langle F_1, F_2, F_3 \right\rangle$.

3. Fourier Series

The **Fourier series** of $f(x)$ on the interval $(-L, L)$ is

$$f(x) \;=\; \frac{a_0}{2} \;+\; \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) \;+\; b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

where

$$a_n \;=\; \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx\,, \quad \left(n = 0, 1, 2, 3, \ldots\right)$$

and

$$b_n \;=\; \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx\,, \quad \left(n = 1, 2, 3, \ldots\right)$$

If $f(x)$ is even ($f(-x) = +f(x)$ for all x), then $b_n = 0$ for all n, leaving a Fourier cosine series (and perhaps a constant term) only for $f(x)$.

If $f(x)$ is odd ($f(-x) = -f(x)$ for all x), then $a_n = 0$ for all n, leaving a Fourier sine series only for $f(x)$.

4.1 Major Classifications of Common PDEs

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$$

$D = B^2 - 4AC$
Hyperbolic, wherever (x, y) is such that $D > 0$;
Parabolic, wherever (x, y) is such that $D = 0$;
Elliptic, wherever (x, y) is such that $D < 0$.

4.2 d'Alembert Solution

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = r(x,y)$$

A.E.: $A\lambda^2 + B\lambda + C = 0$

C.F.: $u_C(x, y) = f(y+\lambda_1 x) + g(y+\lambda_2 x)$, [except when $D = 0$]

where $\lambda_1 = \dfrac{-B-\sqrt{D}}{2A}$ and $\lambda_2 = \dfrac{-B+\sqrt{D}}{2A}$ and $D = B^2 - 4AC$

When $D = 0$, $u_C(x, y) = f(y+\lambda x) + h(x,y)\,g(y+\lambda x)$,

where $h(x, y)$ is a linear function that is neither zero nor a multiple of $(y+\lambda x)$.

P.S.: if RHS = n^{th} order polynomial in x and y, then try an $(n+2)^{\text{th}}$ order polynomial.

4.3 The Wave Equation – Vibrating Finite String

$\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2}$ for $0 < x < L$ and $t > 0$ with $y(0, t) = y(L, t) = 0$ for $t \geq 0$,

$y(x, 0) = f(x)$ for $0 \leq x \leq L$ and $\left.\dfrac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$ for $0 \leq x \leq L$

Substitute $y(x, t) = X(x)\,T(t)$ into the PDE. ... leads, via Fourier series, to

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$
$$+ \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

4.4 The Maximum-Minimum Principle

Let m and M be the minimum and maximum values respectively of u on the boundary of the domain Ω.

If $\nabla^2 u \geq 0$ in Ω, then u is **subharmonic** and $u(\vec{\mathbf{r}}) < M$ or $u(\vec{\mathbf{r}}) \equiv M$ $\forall \vec{\mathbf{r}}$ in Ω

If $\nabla^2 u \leq 0$ in Ω, then u is **superharmonic** and $u(\vec{\mathbf{r}}) > m$ or $u(\vec{\mathbf{r}}) \equiv m$ $\forall \vec{\mathbf{r}}$ in Ω

If $\nabla^2 u = 0$ in Ω, then u is **harmonic** (both subharmonic and superharmonic) and u is either constant on $\overline{\Omega}$ or $m < u < M$ everywhere on Ω.

4.5 The Heat Equation

If the temperature $u(x, t)$ in a bar is $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ together with the boundary conditions $u(0, t) = T_1$ and $u(L, t) = T_2$ and the initial condition $u(x, 0) = f(x)$, then

$u(x, t) = X(x)\, T(t)$ ... leads to $u(x,t) = v(x,t) + \left(\frac{T_2 - T_1}{L}\right) x + T_1$ where

$$v(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left(\int_0^L \left(f(z) - \frac{T_2 - T_1}{L} z - T_1 \right) \sin\left(\frac{n\pi z}{L}\right) dz \right) \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2 \pi^2 k t}{L^2}\right)$$