ENGI 9420 Engineering Analysis
Assignment 4 Questions

2012 Fall

due in class on 2012 October 22 (Monday)

[Eigenvalues; stability analysis; Chapter 4]

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1. For the matrix [20]

$$A = \begin{bmatrix} -9 & 3 \\ -8 & 1 \end{bmatrix}$$

(a) Find the eigenvalues.

(b) Find the unit eigenvectors associated with each eigenvalue.

(c) Construct the 2×2 matrix X whose columns are the unit eigenvectors found in part (b) and evaluate the product $X^{-1}AX$.

(d) Use part (a) to determine the nature and stability of the critical point of the linear system

$$\frac{dx}{dt} + 9x = 3y \ , \quad \frac{dy}{dt} + 8x = y$$

(e) Sketch the orbits near the critical point; and

(f) Find the general solution of the linear system.

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2. In a system with two or more states, (for example, the energy levels of the electron shells in an atom), probabilities can be assigned for a transition from one state to another. The simplest example is a two-state model, with states 'A' and 'B'. In any time step, there is a probability a that an object in state A will stay in state A (and therefore the complementary probability $(1 - a)$ that the object will move to state B). There is a probability b that an object in state B will stay in state B (and therefore the complementary probability $(1 - b)$ that the object will move to state A). Both a and b are numbers strictly between 0 and 1. These probabilities can be expressed in a transition matrix M (also known as a Markov matrix): [20]

$$M = \begin{matrix} & \text{from state A} & \text{B} \\ \text{to state A} & a & 1-b \\ \text{to state B} & 1-a & b \end{matrix} \quad M = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix}$$

Note how each column of M has entries that add up to 1 exactly, no matter what choices are made for a and b.

Let the state vector $\vec{\mathbf{x}}_n$ represent the proportion of objects that are in each state at time step n:

$$\vec{\mathbf{x}}_n = \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \begin{matrix} \text{proportion in state A at time } n \\ \text{proportion in state B at time } n \end{matrix}$$

Again, the entries in the column add up to 1: $\alpha_n + \beta_n = 1$ ($\vec{\mathbf{x}}_n$ is a Markov vector).

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2. (continued)

The proportion of objects in each state in the next time step is related to the present proportions by

$$\vec{\mathbf{x}}_{n+1} = \mathrm{M}\,\vec{\mathbf{x}}_n \quad \Rightarrow \quad \begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix} \cdot \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}$$

As the system evolves through a long sequence of time steps, the system settles down to a steady state

$$\vec{\mathbf{x}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \text{ such that } \mathrm{M}\vec{\mathbf{x}} = \vec{\mathbf{x}} \Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

(a) Show that one of the eigenvalues of the transition matrix M is $\lambda = 1$, no matter what the values of the transition probabilities a and b may be.

(b) Show that there is exactly one eigenvector of M for the eigenvalue $\lambda = 1$, whose entries add up to 1 (a Markov eigenvector) and express it in terms of a and b.

(c) Show that there is **no** Markov eigenvector of M for the other eigenvalue.

(d) In the particular case where an object in state A has a 40% chance of staying in state A, but an object in state B has an 80% chance of staying in state B, find the steady-state proportions of objects in each state.

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3. For the non-linear system [15]

$$\frac{dx}{dt} = 3x + y + 7, \quad \frac{dy}{dt} = 2x^2 + 3y + 3$$

i. determine the critical point(s);
ii. find the linear system associated with each critical point;
iii. determine the nature and stability of the critical point(s);
iv. sketch the orbits near the critical point(s); and
v. sketch the orbits on a diagram that includes all critical point(s).

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4. For the non-linear system [15]

$$\frac{d^2x}{dt^2} + x - \frac{x^3}{3} = 0$$

i. determine the critical point(s);
ii. find the linear system associated with each critical point;
iii. determine the nature and stability of the critical point(s);
iv. sketch the orbits near the critical point(s); and
v. sketch the orbits on a diagram that includes all critical point(s).

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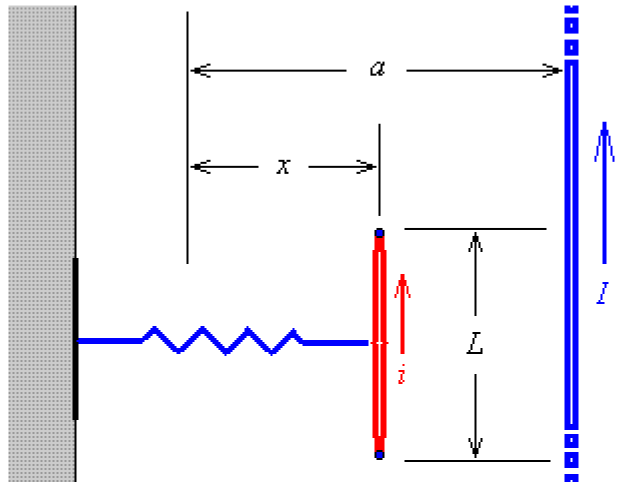
5. A mutual force of attraction is exerted between parallel current carrying wires.



The infinite wire carries current I. The finite wire of length L carries current i in the same direction and is restrained by a spring. According to the Biot-Savart law, the mutual force of attraction is

$$\frac{2IiL}{(\text{separation})} = \frac{2IiL}{a-x}$$

where $x = 0$ is the position at which the spring force is zero. The mass of the finite wire is m and the restoring constant of the spring is k. The equation of motion of the restrained wire is

$$\ddot{x} + \frac{k}{m}\left(x - \frac{b}{a-x}\right) = 0 \quad \text{where} \quad b = \frac{2IiL}{k}$$

For each of the cases $b < a^2/4$, $b = a^2/4$ and $b > a^2/4$, [30]

(a) Locate and classify the singularities (using $dx/dt = y$).

(b) Sketch the phase portrait.

(c) Where it exists, find the equation of the separatrix.
[The separatrix is the orbit that separates closed orbits from open orbits.
It usually passes through at least one singularity.]

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