4. Stability Analysis for Non-linear Ordinary Differential Equations

A pair of simultaneous first order homogeneous linear ordinary differential equations for two functions $x(t)$, $y(t)$ of one independent variable t,

$$\dot{x} = \frac{dx}{dt} = ax + by$$

$$\dot{y} = \frac{dy}{dt} = cx + dy$$

may be represented by the matrix equation $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

A single second order linear homogeneous ordinary differential equation for $x(t)$ with constant coefficients,

$$\frac{d^2x}{dt^2} + p\frac{dx}{dt} + qx = 0 \qquad \Rightarrow \quad \frac{dy}{dt} + py + qx = 0$$

may be re-written as a linked pair of first order homogeneous ordinary differential equations, by introducing a second dependent variable:

$$\frac{dx}{dt} = y \qquad \Rightarrow \quad \frac{dy}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2}$$

$$\frac{dy}{dt} = -qx - py$$

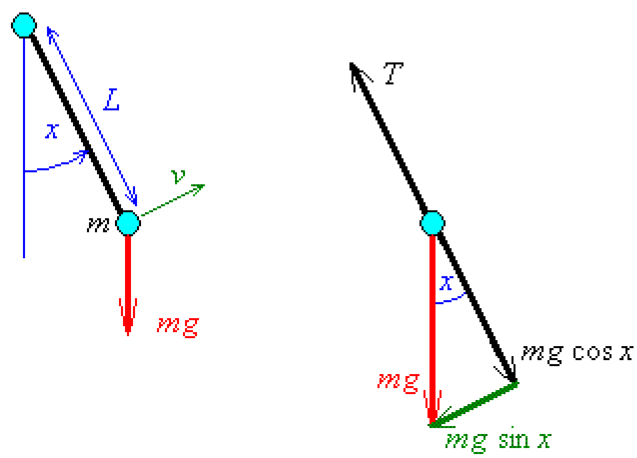
and may also be represented in matrix form $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

The general solution for (x, y) in either case can be displayed graphically as a set of contour curves (or level curves) in a **phase space**.

Sections in this chapter:

4.01 Motion of a Pendulum

Consider a pendulum, moving under its own weight, without friction.
The pendulum bob has mass m, the shaft has length L and negligible mass, and the angle of the shaft with the vertical is x. The tension along the shaft is T. The acceleration due to gravity is g ($\approx 9.81 \text{ m s}^{-2}$).



Resolving forces radially (centripetal force) $\quad -T + mg\cos x = -mL\,\dot{x}^2$

Resolving forces transverse to the pendulum: $\quad -mg\sin x = mL\,\ddot{x}$

$$\Rightarrow \quad \ddot{x} + k^2 \sin x = 0, \qquad \text{where} \quad k^2 = \frac{g}{L} \qquad \textbf{(1)}$$

The Maclaurin series expansion of $\sin x$ is:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Provided the oscillations of the pendulum are small, ($x << 1$), $\sin x \approx x$ and the ordinary differential equation governing the motion of the pendulum is, to a good approximation,

$$\frac{d^2x}{dt^2} + k^2 x = 0 \qquad \textbf{(2)}$$

(which is the ODE of simple harmonic motion)

Let the angular velocity of the pendulum be $v = \dot{x} = \dfrac{dx}{dt}$.

Then, using the chain rule of differentiation,

$$\ddot{x} = \dot{v} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx}$$

The ODE **(2)** becomes

$$v \frac{dv}{dx} + k^2 x = 0 \qquad \Rightarrow \quad v\,dv + k^2 x\,dx = 0$$

$$\Rightarrow \quad \int v\,dv + k^2 \int x\,dx = 0 \qquad \Rightarrow \quad \frac{v^2}{2} + k^2 \frac{x^2}{2} = \frac{c}{2} \qquad \Rightarrow \quad v^2 + k^2 x^2 = c \qquad \textbf{(3)}$$

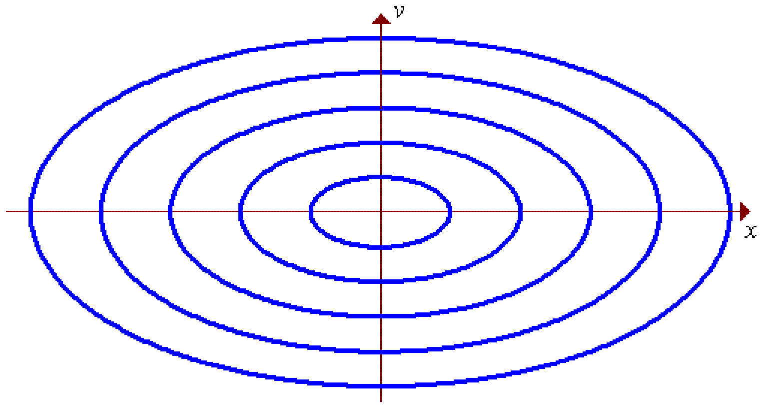
If at time $t = 0$ the pendulum is passing through its equilibrium position with angular speed v_o, then the initial conditions are

$$x(0) = 0\,, \qquad \left.\frac{dx}{dt}\right|_{t=0} = v(0) = v_o$$

Substituting the initial conditions into **(2)**, $v_o^{\,2} + k^2 \times 0 = c$, which leads to a complete solution for v as an implicit function of x,

$$v^2 + k^2 x^2 = v_o^{\,2} \qquad \textbf{(4)}$$

A plot of this solution for various choices of v_o generates a family of concentric ellipses.



Recall that this solution is valid only for small displacements *x*.

The *x*-*v* plane is called the **phase plane**.

Returning to the more general case

$$\ddot{x} + k^2 \sin x = 0, \qquad \text{where} \quad k^2 = \frac{g}{L} \qquad \textbf{(1)}$$

and again using

$$\ddot{x} = \dot{v} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx}$$

the ODE can be re-written as

$$v \frac{dv}{dx} + k^2 \sin x = 0 \quad \Rightarrow \quad v\,dv + k^2 \sin x\,dx = 0$$

$$\Rightarrow \int v\,dv + k^2 \int \sin x\,dx = 0 \quad \Rightarrow \quad \frac{v^2}{2} - k^2 \cos x = \frac{c}{2} \qquad \textbf{(5)}$$

$$\Rightarrow \underbrace{\frac{1}{2} m L^2 v^2}_{K.E.} + \underbrace{\left(-mL^2 k^2 \cos x\right)}_{P.E.} = \underbrace{\frac{1}{2} m L^2 c}_{\text{Total energy}}$$

However, the kinetic energy is $\frac{1}{2} m (Lv)^2$. In the absence of friction, the sum of kinetic and potential energy is constant, so that the potential energy of the pendulum must be $-mL^2k^2 \cos x$ (= $-mg\,L \cos x$, which makes sense upon examining the diagram on page 4.02). Each value of total energy $E = \frac{1}{2} m L^2 c$ generates an **orbit** (or energy curve).

The relationship between total energy and initial angular velocity is obtained from substituting the initial conditions ($x = 0$ and $v = v_o$ when $t = 0$) into **(5)**:

$$\frac{{v_o}^2}{2} - k^2 = \frac{c}{2} \quad \Rightarrow \quad c = {v_o}^2 - 2k^2$$

$$\Rightarrow \quad v^2 - 2k^2 \cos x = {v_o}^2 - 2k^2 \qquad \textbf{(6)}$$

$$\Rightarrow \quad {v_o}^2 - v^2 = 2k^2 - 2k^2 \cos x = 2k^2 (1 - \cos x) \geq 0$$

$$\Rightarrow \quad {v_o}^2 - v^2 \geq 0 \quad \Rightarrow \quad {v_o}^2 \geq v^2 \quad \Rightarrow \quad |v| \leq |v_o|$$

The maximum angular speed of the pendulum, v_o, occurs when $x = 0$.

Also, using $\cos 2\theta = 1 - 2\sin^2\theta \;\Rightarrow\; 2\sin^2\theta = 1 - \cos 2\theta$,

$$\frac{{v_o}^2 - v^2}{2k^2} = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right)$$

$$\therefore \quad \frac{{v_o}^2 - v^2}{4k^2} = \sin^2\left(\frac{x}{2}\right)$$

Recall that the angular velocity is just $v = \dfrac{dx}{dt}$.

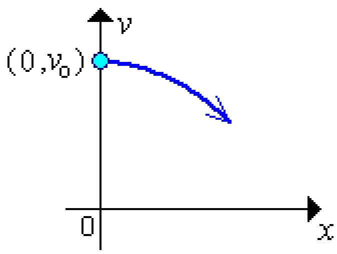
Differentiating the complete solutions **(6)**: $v^2 - 2k^2 \cos x = v_o^{\,2} - 2k^2$ implicitly with respect to time, we obtain

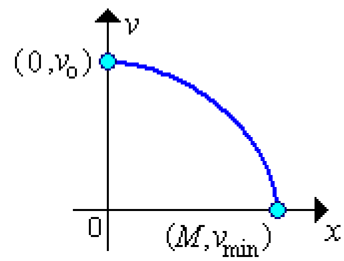
$$2v\frac{dv}{dt} + 2k^2 \sin x \frac{dx}{dt} = 0 \qquad \Rightarrow \quad \frac{dv}{dt} = -k^2 \sin x$$

This expression can also be derived directly from the ODE $\ddot{x} + k^2 \sin x = 0$ **(1)**.

When $0 \le x \le \pi$ and $v > 0$, $\dfrac{dx}{dt} = v > 0$ and $\dfrac{d^2 x}{dt^2} = \dfrac{dv}{dt} = -k^2 \sin x < 0$.

Therefore, in the phase plane, as x increases from the starting point $(0, v_o)$, v decreases in the first quadrant, until the maximum value of x (label that maximum value of x as M).

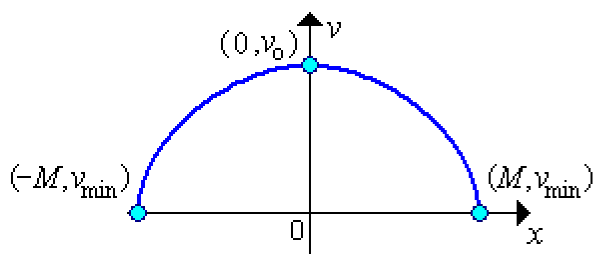




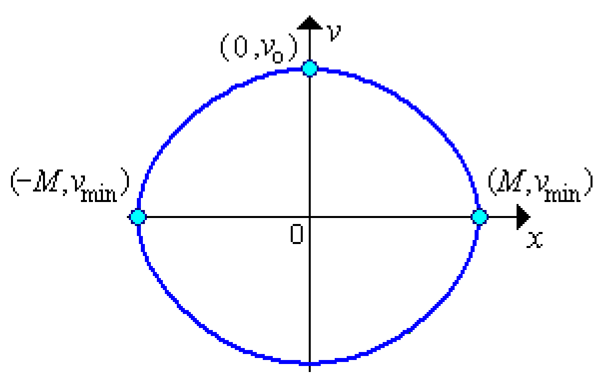
Tracking back into the second quadrant, before $(0, v_o)$,

$$-\pi \le x \le 0 \text{ and } v > 0 \;\Rightarrow\; \frac{dx}{dt} = v > 0 \text{ and } \frac{d^2 x}{dt^2} = \frac{dv}{dt} = -k^2 \sin x > 0 .$$

The orbit increases from $x = -M$ to a maximum at $(0, v_o)$, then decreases until $x = +M$.
This tracks the motion of the pendulum on its complete swing from left to right.



By symmetry, the swing in the opposite direction should generate a mirror image in the x axis of the phase plane, to complete the orbit.



(6) $\Rightarrow \quad v^2 - 2k^2 \cos x = {v_o}^2 - 2k^2$

$$\Rightarrow \quad v^2 = {v_o}^2 - 2k^2 (1 - \cos x) = {v_o}^2 - 2k^2 \left(2\sin^2 \left(\frac{x}{2} \right) \right)$$

$$\Rightarrow \quad v^2 = {v_o}^2 - \left(2k \sin\left(\frac{x}{2} \right) \right)^2 \qquad \textbf{(7)}$$

Three cases arise:

$|v_o| < 2k$:

v will decrease to zero:

$$v = 0 \quad \Rightarrow \quad 0 = {v_o}^2 - \left(2k \sin\left(\frac{x}{2} \right) \right)^2 \quad \Rightarrow \quad \sin\left(\frac{x}{2} \right) = \pm \frac{v_o}{2k}$$

In the first quadrant of the phase plane, the orbit will move right and down to an intercept on the x axis at $(M, 0)$, where $\sin\left(\frac{M}{2} \right) = +\frac{v_o}{2k}$ and $0 < M < \pi$. Extending to the other three quadrants, the orbits resemble ellipses, centred on the origin.

$|v_o| = 2k$:

v will just barely decrease to zero:

$$v = 0 \quad \Rightarrow \quad 0 = (2k)^2 - \left(2k \sin\left(\frac{x}{2} \right) \right)^2 = (2k)^2 \left(\cos\left(\frac{x}{2} \right) \right)^2 \quad \Rightarrow \quad M = \pm \pi$$

The pendulum swings all the way to the upside-down position and comes to rest there, before either swinging back or continuing on in the same direction.

$|v_o| > 2k$:

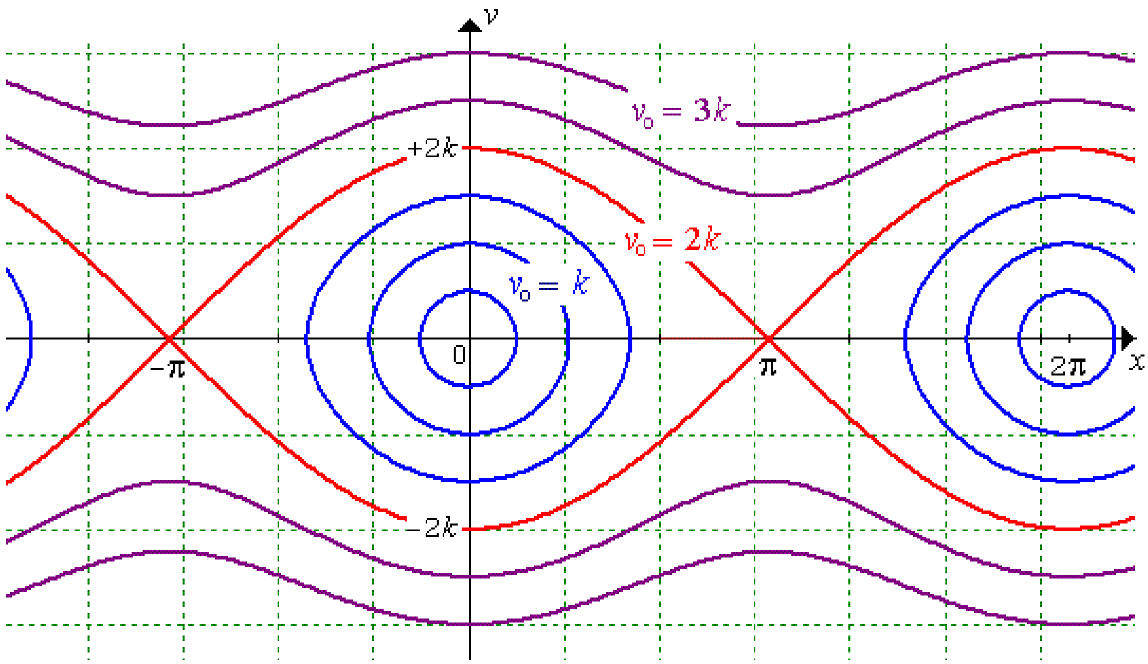
The pendulum will never come to rest, reaching a non-zero minimum speed as it passes through the upside-down position:

$$\sin\left(\frac{x}{2} \right) = \pm \frac{v_o}{2k} \text{ has no real solution for } x \text{ when } |v_o| > 2k.$$

If it is swinging anticlockwise, then the orbit stays in the first and second quadrants.
If it is swinging clockwise, then the orbit stays in the third and fourth quadrants.
These orbits are **not** closed and extend beyond the range $-\pi \le x \le \pi$.

$$v_{\min} = {v_o}^2 - \left(2k \sin\left(\frac{\pi}{2} \right) \right)^2 = {v_o}^2 - 4k^2$$

We can then generate the full set of orbits in the phase plane for the general pendulum problem.



As time progresses, one moves along an orbit to the right above the x axis, but to the left below the x axis $\left(\because \quad v = \dfrac{dx}{dt} \right)$.

The relationship **(7)** between angular velocity v and angle x is itself a first order non-linear ordinary differential equation for x as a function of the time t:

$$\left(\frac{dx}{dt}\right)^2 = v_o^{\,2} - \left(2k \sin\left(\frac{x}{2}\right)\right)^2 \quad \Rightarrow \quad \frac{dx}{dt} = \pm\sqrt{v_o^{\,2} - \left(2k \sin\left(\frac{x}{2}\right)\right)^2}$$

$$\Rightarrow \quad \pm \int \frac{dx}{\sqrt{v_o^{\,2} - 4k^2 \sin^2\left(\frac{x}{2}\right)}} = t + C$$

For the case of closed orbits ($|v_o| < 2k$), the time to complete one orbit (the period T of the pendulum) can be shown to be

$$T = \frac{4}{k}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - b^2 \sin^2 \theta}}, \quad \text{where} \quad b = \sin\frac{M}{2} = \frac{v_o}{2k} \quad \text{and} \quad k = \sqrt{\frac{g}{L}}$$

This is a complete elliptic integral of the first kind, which has no analytic solution in terms of finite combinations of algebraic functions, (except for special choices of v_o and k). As $v_o \to 2k$, the period T diverges to infinity – it takes forever for the zero energy pendulum to reach the upside-down position.

4.02 Stability of Stationary Points

Consider the (generally non-linear) system of simultaneous first order ordinary differential equations

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y) \qquad \textbf{(1)}$$

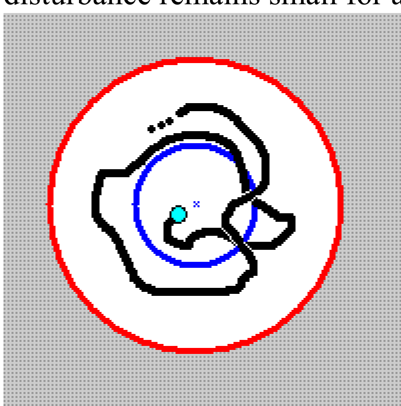
Using the chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{\dot{y}}{\dot{x}} = \dfrac{Q(x,y)}{P(x,y)}$.

This can be integrated with respect to x to obtain a solution for y as an implicit function of x, provided $P(x, y) \neq 0$. At points where $P(x, y) = 0$ but $Q(x, y) \neq 0$, one may integrate $\dfrac{dx}{dy} = \dfrac{P(x,y)}{Q(x,y)}$ instead.

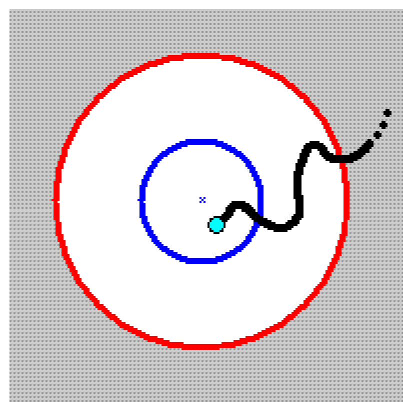
Points on the phase plane where $P(x, y) = Q(x, y) = 0$ are **singular points**.
A unique slope does not exist at such points.

Alternative names for singular points are **equilibrium** points or **stationary** points (because both x and y do not [instantaneously] change with time there) or **critical** points or **fixed** points.

A singular point is **stable** (and is called an "**attractor**") if the response to a small disturbance remains small for all time.



Stable singular point:
all paths starting inside the inner circle stay closer than the outer circle forever.



Unstable singular point.
(or “source”)

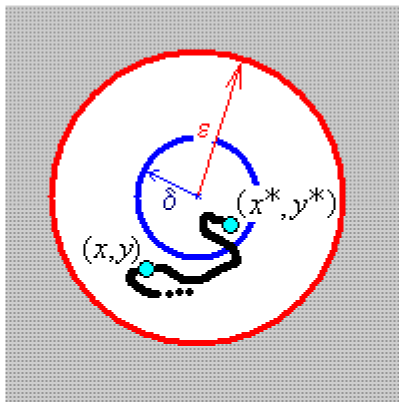
Consider the system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \quad x(0) = x^*, \quad y(0) = y^*, \quad P(0,0) = Q(0,0) = 0 \qquad \textbf{(2)}$$

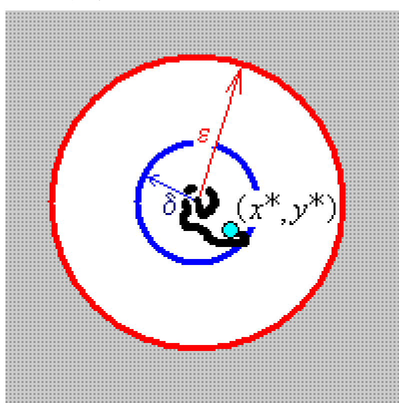
which has a stationary point at the origin.
Let $x(t; x^*)$, $y(t; y^*)$ be the complete solution to this system.

The stationary point at the origin is stable if and only if,
for every $\varepsilon > 0$ (however small), there exists a $\delta(\varepsilon)$ such that whenever the point $(x^*, y^*) = (x(0; x^*), y(0; y^*))$ is closer than δ to the origin, the point $(x(t; x^*), y(t; y^*))$ remains closer than ε to the origin for all time, or

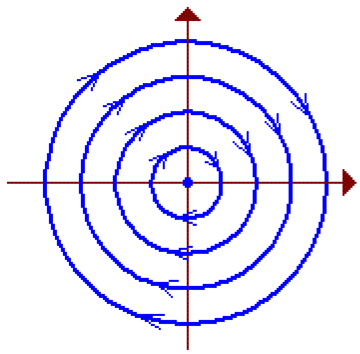
$$\sqrt{(x^*)^2 + (y^*)^2} < \delta \quad \Rightarrow \quad \sqrt{x^2(t;x^*) + y^2(t;y^*)} < \varepsilon \quad \forall t$$



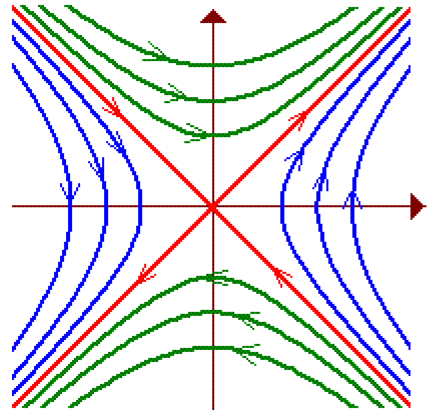
A stationary point is **asymptotically stable** (also known as a "sink") if it is stable and any disturbance ultimately vanishes: $\lim\limits_{t\to\infty}\left[x^2(t;x^*) + y^2(t;y^*)\right] = 0$.



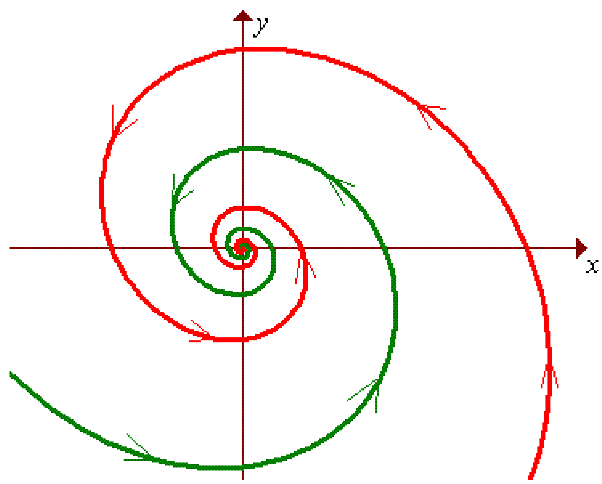
Here are three types of stationary points with nearby orbits:



Stable centre
(but *not* asymptotically stable)



Unstable saddle point
[all saddle points are unstable]



Asymptotically stable focus
(or spiral sink)

4.03 Linear Approximation to a System of Non-Linear ODEs (1)

The Taylor series of any function $f(x, y)$ about the point (x_0, y_0) is

$$f(x,y) = f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} \cdot (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} \cdot (y-y_0) + \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_0,y_0)} \frac{(x-x_0)^2}{2!} + 2\left.\frac{\partial^2 f}{\partial x\,\partial y}\right|_{(x_0,y_0)} \frac{(x-x_0)(y-y_0)}{2!} + \left.\frac{\partial^2 f}{\partial y^2}\right|_{(x_0,y_0)} \frac{(y-y_0)^2}{2!} + \ldots \qquad \textbf{(1)}$$

provided that the series converges to $f(x, y)$.

This allows us to create a linear approximation to the non-linear system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \quad x(0) = x^*, \quad y(0) = y^*, \quad P(0,0) = Q(0,0) = 0. \qquad \textbf{(2)}$$

$$P(x,y) = P(0,0) + \left.\frac{\partial P}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial P}{\partial y}\right|_{(0,0)} y + P_1(x,y) \qquad \textbf{(3)}$$

where $\displaystyle\lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = 0$, (because $P_1(x, y)$ is at least second order in x, y)

and similarly for $Q(x, y)$, so that the system becomes

$$\begin{aligned} \dot{x} &= ax + by + P_1(x,y) \\ \dot{y} &= cx + dy + Q_1(x,y) \end{aligned} \qquad \textbf{(4)}$$

where a, b, c, d are all constants.

In the neighbourhood of the singular point (0, 0), this system can be modelled by the linear system

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \qquad \textbf{(5)}$$

where $\displaystyle a = \left.\frac{\partial P}{\partial x}\right|_{(0,0)}, \quad b = \left.\frac{\partial P}{\partial y}\right|_{(0,0)}, \quad c = \left.\frac{\partial Q}{\partial x}\right|_{(0,0)}, \quad d = \left.\frac{\partial Q}{\partial y}\right|_{(0,0)}.$

4.04 Reminder of Linear Ordinary Differential Equations

To find the general solution of the homogeneous second order linear ODE

$$\frac{d^2 y}{dx^2} + p \frac{dy}{dx} + q\, y = 0,$$

with constant real coefficients p and q,
form the **auxiliary equation** or **characteristic equation**

$$\lambda^2 + p\,\lambda + q = 0$$

and evaluate the discriminant $D = p^2 - 4q$ and the roots $\lambda_1, \lambda_2 = \dfrac{-p \pm \sqrt{D}}{2}$.

Three cases arise.

$D > 0$: The characteristic equation has a pair of distinct real roots λ_1, λ_2.
The general solution is $y = A e^{\lambda_1 x} + B e^{\lambda_2 x}$.

$D = 0$: The characteristic equation has a pair of equal real roots λ.
The general solution is $y = (Ax + B) e^{\lambda x}$.

$D < 0$: The characteristic equation has a complex conjugate pair of roots $\lambda_1, \lambda_2 = a \pm bj$.
The general solution is $y = e^{ax} (A \cos bx + B \sin bx)$,

where A, B are arbitrary constants of integration.

To find the general solution of the system of simultaneous first order linear ODEs

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + dy$$

substitute the trial solution $\left(x(t), y(t)\right) = \left(\alpha e^{\lambda t}, \beta e^{\lambda t}\right)$ into the ODE, to obtain

$$\alpha \lambda \cancel{e^{\lambda t}} = a\alpha \cancel{e^{\lambda t}} + b\beta \cancel{e^{\lambda t}}$$

$$\beta \lambda \cancel{e^{\lambda t}} = c\alpha \cancel{e^{\lambda t}} + d\beta \cancel{e^{\lambda t}}$$

or, in matrix form,

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\alpha = \beta = 0$ is a solution (the trivial solution) for any choice of a, b, c, d and λ.
Non-trivial solutions exist when the determinant of the matrix of coefficients is zero:

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \quad \lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

which is the **characteristic equation** of the system.

The solutions to the characteristic equation are the **eigenvalues** of the coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and, for each eigenvalue λ, a non-zero vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ that satisfies the equation

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is an **eigenvector** for that eigenvalue.

The general solution to the system of ODEs is a linear combination of the solutions arising from each eigenvalue:

$$\left(x(t), y(t)\right) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\ c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)$$

unless the eigenvalues are equal, in which case the general solution is

$$\left(x(t), y(t)\right) = \left((c_1\alpha_1 + c_2\alpha_2 t)e^{\lambda t},\ (c_1\beta_1 + c_2\beta_2 t)e^{\lambda t}\right)$$

(where, in this case, (α_1, β_1) is not necessarily an eigenvector).

4.05 Stability Analysis for a Linear System

In the case where (0, 0) is the only critical point of the system

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

it follows that the characteristic equation $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$
has only non-zero roots and that $\det A = ad - bc \neq 0$.

Proof:

If $\lambda = 0$ then at least one eigenvalue of the coefficient matrix A is zero, from which it follows immediately that

$$\begin{vmatrix} a-0 & b \\ c & d-0 \end{vmatrix} = \det A = ad - bc = 0$$

Both roots non-zero $\Rightarrow$ $ad - bc \neq 0$.

If (0, 0) is the only critical point of the system, then no other choice of (x, y) satisfies both equations

$$\begin{matrix} ax + by = 0 \\ cx + dy = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} (ad-bc)x = 0 \\ (ad-bc)y = 0 \end{matrix}$$

from which it follows immediately that $ad - bc = \det A \neq 0$.

If the roots are both non-zero and (x, y) is a critical point of the system, then

$$\begin{matrix} ax + by = 0 \\ cx + dy = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} (ad-bc)x = 0 \\ (ad-bc)y = 0 \end{matrix}$$

But $\lambda_1, \lambda_2 \neq 0 \Rightarrow ad - bc \neq 0 \Rightarrow$ (0, 0) is the only solution to this pair of simultaneous linear equations.

Therefore (0, 0) is the only critical point of the system if and only if both roots of the characteristic equation are non-zero.

Let (α_i, β_i) be the eigenvector associated with the eigenvalue λ_i of the coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let c_1, c_2 be arbitrary constants.

Case of **real, distinct, negative eigenvalues** (with $\lambda_2 < \lambda_1 < 0$):

Two linearly independent solutions are

$$(x(t), y(t)) = \left(\alpha_1 e^{\lambda_1 t},\ \beta_1 e^{\lambda_1 t}\right) \quad \text{and} \quad (x(t), y(t)) = \left(\alpha_2 e^{\lambda_2 t}, \beta_2 e^{\lambda_2 t}\right)$$

The general solution is

$$(x(t), y(t)) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\ c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)$$

One can see that $\lim_{t\to\infty} (x(t), y(t)) = (0, 0)$.
All orbits therefore terminate at the critical point at the origin.
The system is **asymptotically stable**.

If both arbitrary constants are zero, then we have the trivial solution ($x = y = 0$ for all t).

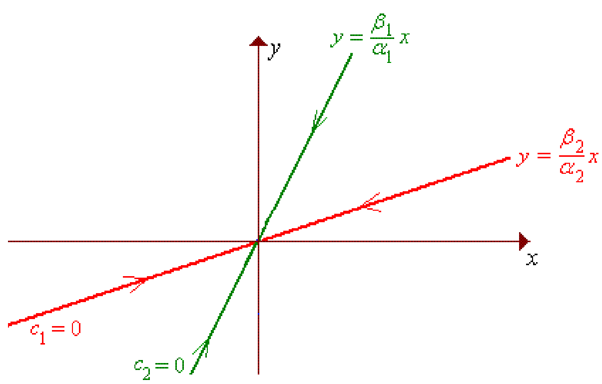
If one of the arbitrary constants is zero (say c_1), then

$$(x(t), y(t)) = \left(c_2\alpha_2 e^{\lambda_2 t},\ c_2\beta_2 e^{\lambda_2 t}\right) \quad \Rightarrow \quad y(t) = \frac{\beta_2}{\alpha_2} x(t)$$

which is a straight line through the origin, of slope $\dfrac{\beta_2}{\alpha_2}$.

[The situation is similar if c_2 is zero.]

We therefore obtain straight-line trajectories ending at the singular point, when exactly one of the arbitrary constants is zero.



If neither arbitrary constant is zero, then

$$\frac{y(t)}{x(t)} = \frac{c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}}{c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t}} = \frac{c_1\beta_1 + c_2\beta_2 e^{(\lambda_2-\lambda_1)t}}{c_1\alpha_1 + c_2\alpha_2 e^{(\lambda_2-\lambda_1)t}} = \frac{c_1\beta_1 e^{-(\lambda_2-\lambda_1)t} + c_2\beta_2}{c_1\alpha_1 e^{-(\lambda_2-\lambda_1)t} + c_2\alpha_2}$$

Because $\lambda_2 < \lambda_1 < 0$,

$$\lim_{t\to-\infty} \frac{y(t)}{x(t)} = \lim_{t\to-\infty} \frac{c_1\beta_1 e^{-(\lambda_2-\lambda_1)t} + c_2\beta_2}{c_1\alpha_1 e^{-(\lambda_2-\lambda_1)t} + c_2\alpha_2} = \frac{\beta_2}{\alpha_2}$$

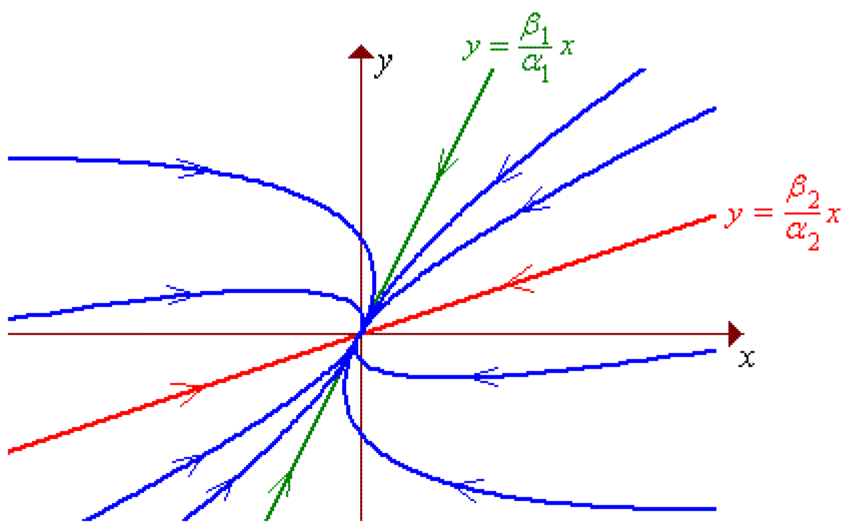
and

$$\lim_{t\to\infty} \frac{y(t)}{x(t)} = \lim_{t\to\infty} \frac{c_1\beta_1 + c_2\beta_2 e^{(\lambda_2-\lambda_1)t}}{c_1\alpha_1 + c_2\alpha_2 e^{(\lambda_2-\lambda_1)t}} = \frac{\beta_1}{\alpha_1}$$

All orbits therefore come in from infinity parallel to the line $y = \frac{\beta_2}{\alpha_2}x$.

All orbits share the same tangent at the origin, $y = \frac{\beta_1}{\alpha_1}x$.

We obtain a **stable node** that is also asymptotically stable.



[The case illustrated here is $\alpha_1 = 1$, $\alpha_2 = 3$, $\beta_1 = 2$, $\beta_2 = 1$, $\lambda_1 = -5$, $\lambda_2 = -10$, which is generated from $A = \begin{bmatrix} -11 & +3 \\ -2 & -4 \end{bmatrix}$.]

Case of **real, distinct, positive eigenvalues** (with $\lambda_2 > \lambda_1 > 0$):

The analysis leads to the same phase space, except that the arrows are reversed.
The result is an **unstable node**.

Case of **real, distinct eigenvalues of opposite sign** (with $\lambda_2 < 0 < \lambda_1$):

The general solution is

$$\left(x(t), y(t)\right) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\ c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)$$

$\lambda_2 < 0 < \lambda_1 \quad \Rightarrow \quad \lim_{t \to -\infty}\left(x(t), y(t)\right)$ and $\lim_{t \to \infty}\left(x(t), y(t)\right)$ do not exist (infinite),
(with the exception of the orbit for $c_1 = 0$).
All orbits (except $c_1 = 0$) therefore move away from the critical point at the origin.
The system is **unstable**.

If both arbitrary constants are zero, then we have the trivial solution ($x = y = 0$ for all t).

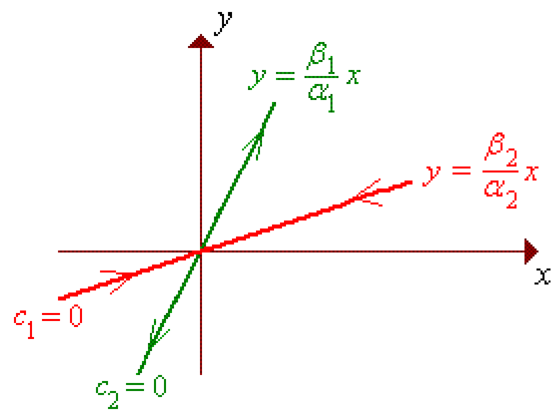
If one of the arbitrary constants is zero (say c_1), then

$$\left(x(t), y(t)\right) = \left(c_2\alpha_2 e^{\lambda_2 t},\ c_2\beta_2 e^{\lambda_2 t}\right) \qquad \Rightarrow \quad y(t) = \frac{\beta_2}{\alpha_2}x(t)$$

which is a straight line through the origin, of slope $\dfrac{\beta_2}{\alpha_2}$.

[The situation is similar if c_2 is zero.]

We therefore obtain straight-line trajectories when one of the arbitrary constants is zero. One of them ($c_1 = 0$) ends at the singular point while the other begins there.



If neither arbitrary constant is zero, then

$$\frac{y(t)}{x(t)} = \frac{c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}}{c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t}} = \frac{c_1\beta_1 + c_2\beta_2 e^{(\lambda_2 - \lambda_1)t}}{c_1\alpha_1 + c_2\alpha_2 e^{(\lambda_2 - \lambda_1)t}} = \frac{c_1\beta_1 e^{-(\lambda_2 - \lambda_1)t} + c_2\beta_2}{c_1\alpha_1 e^{-(\lambda_2 - \lambda_1)t} + c_2\alpha_2}$$

Because $\lambda_2 < 0 < \lambda_1$,

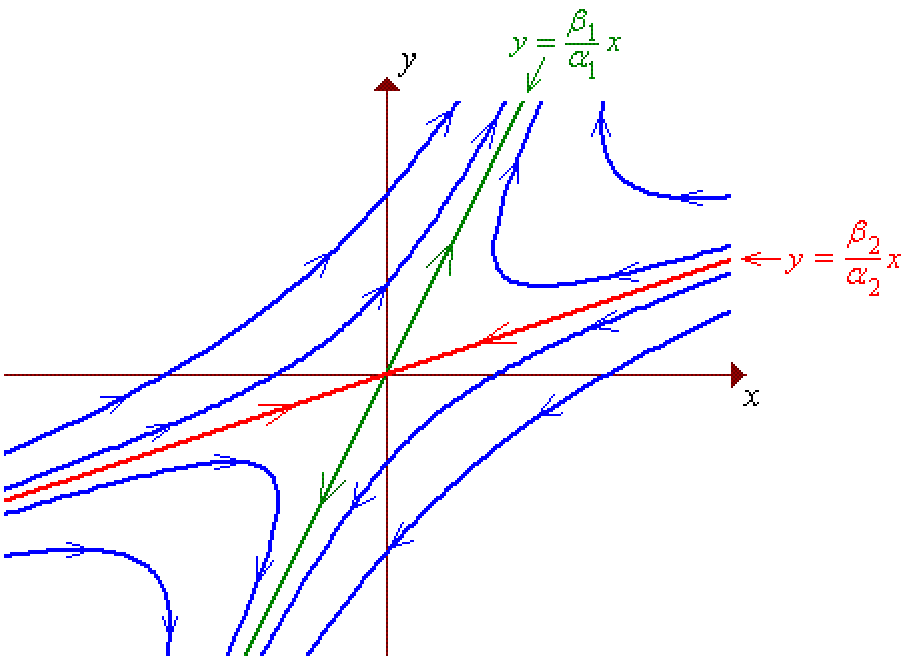
$$\lim_{t \to -\infty}\frac{y(t)}{x(t)} = \lim_{t \to -\infty}\frac{c_1\beta_1 e^{-(\lambda_2 - \lambda_1)t} + c_2\beta_2}{c_1\alpha_1 e^{-(\lambda_2 - \lambda_1)t} + c_2\alpha_2} = \frac{\beta_2}{\alpha_2}$$

and

$$\lim_{t \to \infty}\frac{y(t)}{x(t)} = \lim_{t \to \infty}\frac{c_1\beta_1 + c_2\beta_2 e^{(\lambda_2 - \lambda_1)t}}{c_1\alpha_1 + c_2\alpha_2 e^{(\lambda_2 - \lambda_1)t}} = \frac{\beta_1}{\alpha_1}$$

All orbits therefore share the same asymptotes, $y = \frac{\beta_2}{\alpha_2} x$ (incoming) and $y = \frac{\beta_1}{\alpha_1} x$ (outgoing).

We obtain a **saddle point**, which is an unstable critical point.



[The case illustrated here is $\alpha_1 = 1,\ \alpha_2 = 3,\ \beta_1 = 2,\ \beta_2 = 1,\ \lambda_1 = +5,\ \lambda_2 = -5$, which is generated from $A = \begin{bmatrix} 7 & -6 \\ 4 & -7 \end{bmatrix}$.]

Case of **real, equal, negative eigenvalues** ($\lambda_1 = \lambda_2 < 0$) **and** $b = c = 0$:

The system is uncoupled:

$$\frac{dx}{dt} = ax$$
$$\frac{dy}{dt} = dy$$

and equal eigenvalues now require $a = d = \lambda$.
The general solution is $\left(x(t), y(t)\right) = \left(c_1 e^{\lambda t}, c_2 e^{\lambda t}\right)$.

$\lambda < 0 \quad \Rightarrow \quad \lim_{t \to -\infty} \left(|x(t)|, |y(t)|\right) = (\infty, \infty)$ and $\lim_{t \to \infty} \left(x(t), y(t)\right) = (0, 0)$.
All orbits therefore terminate at the critical point at the origin.
The system is asymptotically stable.

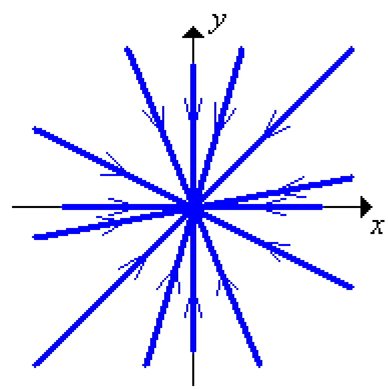
If both arbitrary constants are zero, then we have the trivial solution ($x = y = 0$ for all t).

$c_1 \neq 0 \quad \Rightarrow \quad \dfrac{y(t)}{x(t)} = \dfrac{c_2}{c_1} \quad \forall t$

and $c_1 = 0, c_2 \neq 0 \quad \Rightarrow \quad x(t) = 0 \quad \forall t$

The orbits are straight lines
ending at the critical point at the origin.

The critical point is an **asymptotically stable star-shaped node**.



Additional Note:

The eigenvalues of *any* triangular matrix are the diagonal entries of that matrix:

The characteristic equation of $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ 0 & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) = 0 \qquad \Rightarrow \quad \lambda = a \text{ or } d$$

Case of **real, equal, negative eigenvalues** ($\lambda_1 = \lambda_2 < 0$) **and** b, c not both zero:

The characteristic equation $\lambda^2 - (a+d)\lambda + (ad-bc) = 0$
has the discriminant $(a+d)^2 - 4(ad-bc) = (a-d)^2 + 4bc = 0$.

The solution of the characteristic equation simplifies to $\lambda = \dfrac{a+d}{2}$.

The general solution is $(x(t), y(t)) = \left((c_1\alpha_1 + c_2\alpha_2 t)e^{\lambda t},\ (c_1\beta_1 + c_2\beta_2 t)e^{\lambda t}\right)$.

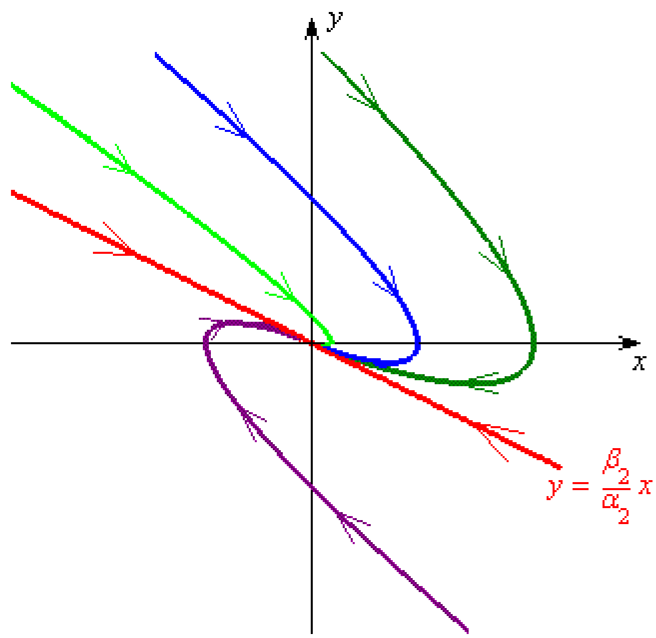
$\lambda < 0 \quad \Rightarrow \quad \lim\limits_{t\to-\infty}(|x(t)|, |y(t)|) = (\infty, \infty)$ and $\lim\limits_{t\to\infty}(x(t), y(t)) = (0,0)$.

All orbits therefore terminate at the critical point at the origin.
The system is **asymptotically stable**.

If both arbitrary constants are zero, then we have the trivial solution ($x = y = 0$ for all t).

If $c_2 \neq 0$, then $\dfrac{y(t)}{x(t)} = \dfrac{c_1\beta_1 + c_2\beta_2 t}{c_1\alpha_1 + c_2\alpha_2 t} \to \dfrac{\beta_2}{\alpha_2}$ as $t \to \pm\infty$

All orbits (except for $c_2 = 0$) therefore come in from infinity parallel to the line $y = \dfrac{\beta_2}{\alpha_2}x$, which is also a tangent at the origin. It can be shown that $\dfrac{\beta_1}{\alpha_1} = \dfrac{\beta_2}{\alpha_2}$ when $c_2 = 0$, so that the trajectories for $c_1 = 0$ and $c_2 = 0$ are both $y = \dfrac{\beta_2}{\alpha_2}x$.



Neither eigenvalue can be zero, otherwise (0, 0) is not the only critical point (as shown on page 4.14).

Case of **real, equal, positive eigenvalues** ($\lambda_1 = \lambda_2 > 0$)

The analysis leads to the same phase planes as in the case of real equal negative eigenvalues, but the signs of the arrows are reversed and the result is an **unstable node**.

Case of **complex conjugate pair of eigenvalues with negative real part**

The eigenvalues (roots of the characteristic equation) are

$$\lambda_1 = a + jb, \quad \lambda_2 = a - jb, \quad (a < 0).$$

The general solution has the form

$$x(t) = \left[c_1 \left(A_1 \cos bt - A_2 \sin bt \right) + c_2 \left(A_1 \sin bt + A_2 \cos bt \right) \right] e^{at}$$

$$y(t) = \left[c_1 \left(B_1 \cos bt - B_2 \sin bt \right) + c_2 \left(B_1 \sin bt + B_2 \cos bt \right) \right] e^{at}$$

Using the definitions

$$A = \sqrt{\left(c_2 A_1 - c_1 A_2\right)^2 + \left(c_1 A_1 + c_2 A_2\right)^2}, \quad B = \sqrt{\left(c_2 B_1 - c_1 B_2\right)^2 + \left(c_1 B_1 + c_2 B_2\right)^2}$$

$$\cos\alpha = \frac{c_1 A_1 + c_2 A_2}{A}, \quad \sin\alpha = \frac{c_2 A_1 - c_1 A_2}{A}, \quad \cos\beta = \frac{c_1 B_1 + c_2 B_2}{B}, \quad \sin\beta = \frac{c_2 B_1 - c_1 B_2}{B}$$

the general solution can be written more compactly as

$$\left(x(t), y(t)\right) = \left(A\, e^{at} \cos(bt + \alpha),\ B\, e^{at} \cos(bt + \beta) \right)$$

$$a < 0 \quad \Rightarrow \quad \lim_{t \to -\infty} \left(|x(t)|, |y(t)| \right) = (\infty, \infty) \quad \text{and} \quad \lim_{t \to \infty} \left(x(t), y(t) \right) = (0, 0).$$

$$\text{If } \ x(t) = 0 \quad \text{then} \quad bt + \alpha = \frac{\pi}{2} + n\pi \quad (n \in \mathbb{Z})$$

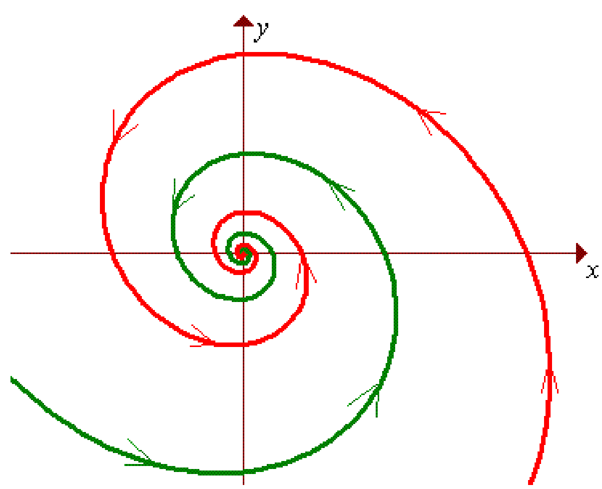
$$\text{If } \ y(t) = 0 \quad \text{then} \quad bt + \beta = \frac{\pi}{2} + n\pi \quad (n \in \mathbb{Z})$$

$$\frac{y(t)}{x(t)} = \frac{B \cos(bt + \beta)}{A \cos(bt + \alpha)}$$

$\dfrac{y(t)}{x(t)}$ is periodic, with period $\dfrac{2\pi}{b}$.

The orbits spiral in to the origin.

We have an asymptotically stable spiral, also known as a **stable focus**.



Case of **complex conjugate pair of eigenvalues with positive real part**

The analysis leads to the same phase planes as in the case of negative real part, but the signs of the arrows are reversed and the result is an **unstable focus**.

Case of **complex conjugate pair of eigenvalues with zero real part** (pure imaginary)

The eigenvalues (roots of the characteristic equation) are

$$\lambda_1 = -jb, \quad \lambda_2 = +jb.$$

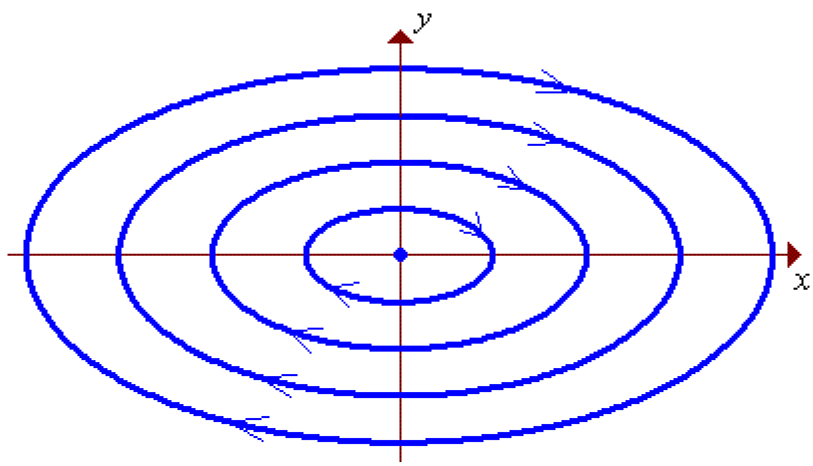
The general solution has the compact form

$$\big(x(t), y(t)\big) = \big(A\cos(bt+\alpha),\ B\cos(bt+\beta)\big)$$

If $\alpha = 0$ and $\beta = -\dfrac{\pi}{2}$, then

$$\big(x(t), y(t)\big) = \big(A\cos bt,\ B\sin bt\big) \quad \Rightarrow \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

so that the orbits are ellipses, centred on the critical point at the origin.
This is a **stable centre**.



Other choices of α and β also lead to concentric sets of ellipses, but rotated with respect to the coordinates axes.

Note that this is the only case of a stable critical point that is ***not*** asymptotically stable.

Summary for the Linear System

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy, \qquad (a, b, c, d = \text{constants})$$

Characteristic equation:

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Discriminant

$$D = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc$$

Roots of characteristic equation (= eigenvalues of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$):

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2}$$

Cases:

$a + d$	D	other condition	λ	Type of point
$a + d < 0$	$D > 0$	$ad - bc > 0$	real, distinct negative	**Stable** node
$a + d < 0$	$D = 0$	$b = c = 0$	real, equal negative	**Stable** star shape
$a + d < 0$	$D = 0$	b, c not both 0	real, equal negative	**Stable** node
$a + d < 0$	$D < 0$		complex conjugate pair	**Stable** focus [spiral]
$a + d = 0$	$D < 0$		Pure imaginary pair	**Stable** centre
$a + d > 0$	$D > 0$	$ad - bc > 0$	real, distinct positive	**Unstable** node
(*any*)	$D > 0$	$ad - bc < 0$	real, distinct opposite signs	**Unstable** saddle point
$a + d > 0$	$D = 0$	$b = c = 0$	real, equal positive	**Unstable** star shape
$a + d > 0$	$D = 0$	b, c not both 0	real, equal positive	**Unstable** node
$a + d > 0$	$D < 0$		complex conjugate pair	**Unstable** focus [spiral]

Note that $ad - bc = \det A$ and that $a + d$ = the trace of the matrix A.

In brief, if the real parts of both eigenvalues are negative (or both zero), then the origin is stable. Otherwise it is unstable.

[See also the example at "`www.engr.mun.ca/~ggeorge/9420/demos/phases.html`".]

Example 4.05.1

Find the nature of the critical point of the system

$$\frac{dx}{dt} = 4x - 3y\,, \quad \frac{dy}{dt} = 5x - 4y$$

and find the general solution.

The coefficient matrix is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 5 & -4 \end{pmatrix}$.

trace(A) $= a + d = 4 + (-4) = 0$

$D = (a - d)^2 + 4bc = (4 + 4)^2 + 4(-3)(5) = 64 - 60 = +4 > 0$

$\det A = \begin{vmatrix} 4 & -3 \\ 5 & -4 \end{vmatrix} = -16 + 15 < 0$

$D > 0$ and $ad - bc < 0 \Rightarrow \lambda$ are real with opposite signs and the critical point is a **saddle point** (**unstable**).

Solving the system:

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{0 \pm \sqrt{4}}{2} = \pm 1$$

$$\big(x(t), y(t)\big) = \left(c_1\alpha_1 e^{-t} + c_2\alpha_2 e^{t},\ c_1\beta_1 e^{-t} + c_2\beta_2 e^{t}\right)$$

where $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -1$

and $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = +1$.

To find the eigenvectors, find non-zero solutions to the equation

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

At $\lambda = -1$:

$$\begin{pmatrix} 4+1 & -3 \\ 5 & -4+1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $5\alpha - 3\beta = 0$ will provide an eigenvector.

Select $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Example 4.05.1 (continued)

At $\lambda = +1$:

$$\begin{pmatrix} 4-1 & -3 \\ 5 & -4-1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $\alpha - \beta = 0$ will provide an eigenvector.

Select $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The general solution is

$$\boxed{\left(x(t), y(t)\right) = \left(3c_1e^{-t} + c_2e^t,\ 5c_1e^{-t} + c_2e^t\right)}$$

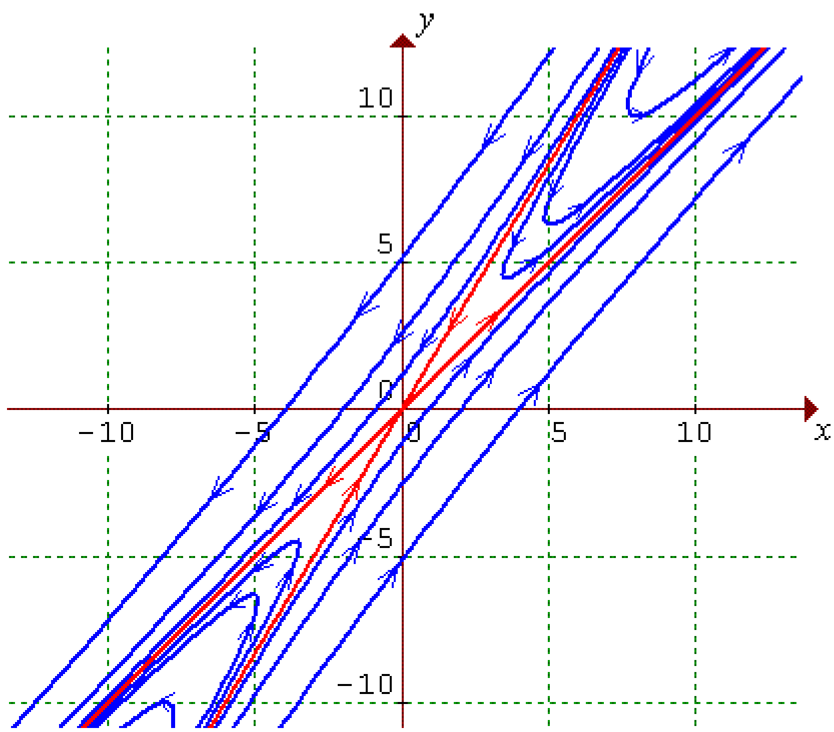
[It is simple to check that $(4x - 3y, 5x - 4y)$ is indeed equal to $\left(\dot{x}, \dot{y}\right)$].

Also note that

$$\frac{y(t)}{x(t)} = \frac{5c_1e^{-t} + c_2e^t}{3c_1e^{-t} + c_2e^t} \Rightarrow \lim_{t\to-\infty}\frac{y(t)}{x(t)} = \frac{5}{3} \quad (c_1 \neq 0) \quad \text{and} \quad \lim_{t\to+\infty}\frac{y(t)}{x(t)} = 1 \quad (c_2 \neq 0)$$

so that all orbits for which both c_1 and c_2 are non-zero share the same asymptotes,
$3y = 5x$ (which is the incoming orbit, when $c_2 = 0$) and
$y = x$ (which is the outgoing orbit, when $c_1 = 0$).

A few representative orbits and the two asymptotes are plotted in this phase space diagram:



Example 4.05.2

Find the nature of the critical point of the system

$$\frac{dx}{dt} = -2x + y\,, \quad \frac{dy}{dt} = x - 2y$$

and find the general solution.

The coefficient matrix is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

trace(A) $= a + d = -2 + -2 = -4 < 0$.

$D = (a-d)^2 + 4bc = (-2+2)^2 + 4(1)(1) = 0 + 4 = 4 > 0$

$\Rightarrow$ λ are real, distinct and negative and

$\det A = ad - bc = 4 - 1 = 3 > 0$ $\Rightarrow$ the critical point is a **stable node**.

Solving the system:

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-4 \pm \sqrt{4}}{2} = -2 \pm 1 = -3, -1$$

$$\big(x(t), y(t)\big) = \left(c_1\alpha_1 e^{-3t} + c_2\alpha_2 e^{-t},\ c_1\beta_1 e^{-3t} + c_2\beta_2 e^{-t}\right)$$

where $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -3$

and $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -1$.

To find the eigenvectors, find non-zero solutions to the equation

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

At $\lambda = -3$:

$$\begin{pmatrix} -2+3 & 1 \\ 1 & -2+3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $\alpha + \beta = 0$ will provide an eigenvector.

Select $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

At $\lambda = -1$:

$$\begin{pmatrix} -2+1 & 1 \\ 1 & -2+1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $-\alpha + \beta = 0$ will provide an eigenvector.

Select $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Example 4.05.2 (continued)

The general solution is

$$\boxed{\left(x(t), y(t)\right) = \left(c_1 e^{-3t} + c_2 e^{-t},\ -c_1 e^{-3t} + c_2 e^{-t}\right)}$$

[It is simple to check that ($-2x + y,\ x - 2y$) is indeed equal to $\left(\dot{x}, \dot{y}\right)$].

Also note that

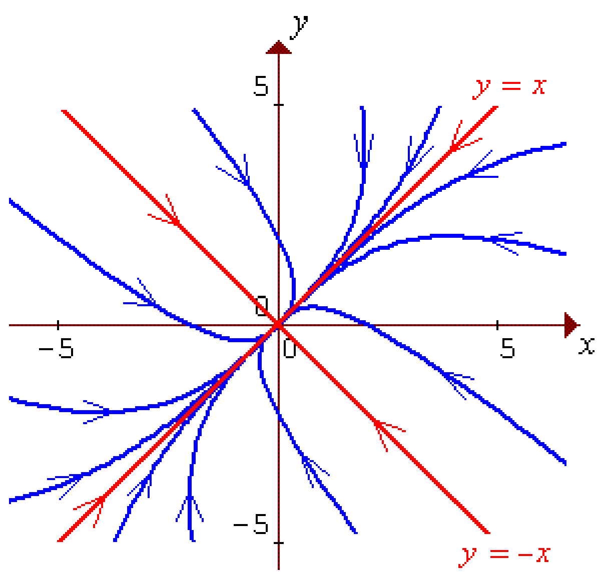
$$\frac{y(t)}{x(t)} = \frac{-c_1 e^{-3t} + c_2 e^{-t}}{c_1 e^{-3t} + c_2 e^{-t}} = \frac{-c_1 e^{-2t} + c_2}{c_1 e^{-2t} + c_2} = \frac{-c_1 + c_2 e^{2t}}{c_1 + c_2 e^{2t}}$$

$$\Rightarrow \lim_{t \to -\infty} \frac{y(t)}{x(t)} = -1 \quad \left(c_1 \neq 0\right) \quad \text{and} \quad \lim_{t \to +\infty} \frac{y(t)}{x(t)} = 1 \quad \left(c_2 \neq 0\right)$$

and $\lim\limits_{t \to \infty} \left(x(t), y(t)\right) = \lim\limits_{t \to \infty} \left(c_1 e^{-3t} + c_2 e^{-t},\ -c_1 e^{-3t} + c_2 e^{-t}\right) = (0,0)$

so that all orbits for which both c_1 and c_2 are non-zero come in from a direction parallel to $y = -x$ (which is the orbit when $c_2 = 0$) and share the same tangent at the origin, $y = x$ (which is the orbit when $c_1 = 0$).

A few representative orbits and the common tangent are plotted in this phase space diagram:



Example 4.05.3

Find the nature of the critical point of the system

$$\frac{dx}{dt} = x - 5y\,, \quad \frac{dy}{dt} = x - 3y$$

and find the general solution.

The coefficient matrix is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$.

trace(A) $= a + d = 1 + -3 = -2 < 0$.

$D = (a-d)^2 + 4bc = (1+3)^2 + 4(-5)(1) = 16 - 20 = -4 < 0$

$\Rightarrow$ λ are a complex conjugate pair with negative real part
and the critical point is a **stable focus** (spiral).

Solving the system:

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm j$$

$$\big(x(t), y(t)\big) = \left(c_1\alpha_1 e^{(-1-j)t} + c_2\alpha_2 e^{(-1+j)t},\ c_1\beta_1 e^{(-1-j)t} + c_2\beta_2 e^{(-1+j)t}\right)$$

where $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -1 - j$

and $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -1 + j$.

To find the eigenvectors, find non-zero solutions to the equation

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

At $\lambda = -1 - j$:

$$\begin{pmatrix} 1-(-1-j) & -5 \\ 1 & -3-(-1-j) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2+j & -5 \\ 1 & -2+j \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $\alpha = (2-j)\,\beta$ will provide an eigenvector.
Note that the two rows of the matrix equation are equivalent:

$$(2+j)\alpha - 5\beta = 0 \quad \Rightarrow \quad \alpha = \frac{5\beta}{2+j} = \frac{5\beta}{2+j}\cdot\frac{2-j}{2-j} = \frac{5(2-j)\beta}{4+1} = (2-j)\beta$$

Select $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 2-j \\ 1 \end{pmatrix}$.

At $\lambda = -1 + j$:

$$\begin{pmatrix} 1-(-1+j) & -5 \\ 1 & -3-(-1+j) \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2-j & -5 \\ 1 & -2-j \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $\alpha = (2+j)\,\beta$ will provide an eigenvector.

Example 4.05.3 (continued)

Again the two rows of the matrix equation are equivalent.

Select $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 2+j \\ 1 \end{pmatrix}$.

The general solution is

$$\big(x(t), y(t)\big) = \left(c_1(2-j)e^{(-1-j)t} + c_2(2+j)e^{(-1+j)t},\ c_1 e^{(-1-j)t} + c_2 e^{(-1+j)t}\right)$$

But $c_1(2-j)e^{(-1-j)t} + c_2(2+j)e^{(-1+j)t}$

$$= e^{-t}\Big(c_1(2-j)(\cos t - j\sin t) + c_2(2+j)(\cos t + j\sin t)\Big)$$

$$= e^{-t}\begin{pmatrix} c_1\big((2\cos t - \sin t) - j(\cos t + 2\sin t)\big) \\ +\ c_2\big((2\cos t - \sin t) + j(\cos t + 2\sin t)\big) \end{pmatrix}$$

$$= e^{-t}\Big((c_1+c_2)(2\cos t - \sin t) + j(-c_1+c_2)(\cos t + 2\sin t)\Big)$$

$$= e^{-t}\Big(c_3(2\cos t - \sin t) + c_4(\cos t + 2\sin t)\Big)$$

where new real arbitrary constants c_3, c_4 are defined in terms of the complex arbitrary constants c_1, c_2 by $c_3 = c_1 + c_2$, $c_4 = j(-c_1 + c_2)$

Similarly, $c_1 e^{(-1-j)t} + c_2 e^{(-1+j)t} = e^{-t}\Big(c_1(\cos t - j\sin t) + c_2(\cos t + j\sin t)\Big)$

$$= e^{-t}\Big((c_1+c_2)\cos t + j(-c_1+c_2)\sin t\Big)$$

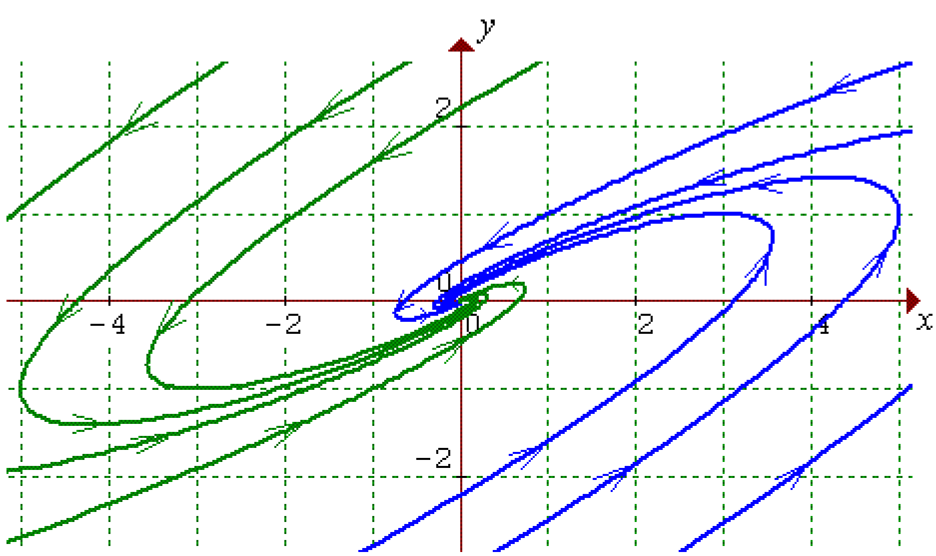
Therefore, in terms of purely real quantities, the general solution is

$$\boxed{\big(x(t), y(t)\big) = e^{-t}\Big(c_3(2\cos t - \sin t) + c_4(\cos t + 2\sin t),\ c_3\cos t + c_4\sin t\Big)}$$

One can show that this solution does satisfy the original system of ODEs

and $\lim_{t\to\infty}\big(x(t), y(t)\big) = \lim_{t\to\infty}\big(e^{-t} \times [\text{finite vector}]\big) = (0,0)$.

The orbits spiral in to the origin. A few representative orbits are plotted in this phase space diagram:



General Form for the General Solution

From the linear system of ODEs

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

calculate the discriminant

$$D = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc$$

If $D > 0$ then the general solution is

$$\boxed{(x(t), y(t)) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\ c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)},$$

where

$$\lambda_1 = \frac{(a+d) - \sqrt{D}}{2}, \quad \lambda_2 = \frac{(a+d) + \sqrt{D}}{2},$$

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix} \dfrac{(a-d) - \sqrt{D}}{2} \\ c \end{pmatrix},$$

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix} \dfrac{(a-d) + \sqrt{D}}{2} \\ c \end{pmatrix}$$ and c_1, c_2 are arbitrary constants.

[An exception occurs if $c = 0$: use $\begin{pmatrix} b \\ \dfrac{(d-a) \pm \sqrt{D}}{2} \end{pmatrix}$ instead.]

If $D < 0$ then the general solution is

$$\boxed{\begin{aligned}&(x(t), y(t)) = \\ &e^{ut}\left(c_3\left((u-d)\cos vt - v\sin vt\right) + c_4\left(v\cos vt + (u-d)\sin vt\right),\ c\left(c_3\cos vt + c_4\sin vt\right)\right)\end{aligned}}$$

where $u = \dfrac{a+d}{2}$ $\left(\Rightarrow\ u - d = \dfrac{a-d}{2}\right)$ and $v = \dfrac{\sqrt{-(a-d)^2 - 4bc}}{2} = \dfrac{\sqrt{-D}}{2}$

and c_3, c_4 are [real] arbitrary constants.

[The derivation of this general result follows steps similar to those of Example 4.05.3.]

The situation for $D = 0$ is more complicated.
The general solution is

$$\boxed{\left(x(t), y(t)\right) = \left(\left(c_1\left(\frac{a-d}{2}\right) + c_2\left(1 + \left(\frac{a-d}{2}\right)(1+t)\right)\right)e^{\lambda t},\ c\left(c_1 + c_2(1+t)\right)e^{\lambda t}\right)},$$

unless $a = d$ and $c = 0$ but $b \neq 0$, in which case

$$\boxed{\left(x(t), y(t)\right) = \left((c_1 + c_2 t)e^{at},\ \frac{c_2}{b}e^{at}\right)}$$

or the decoupled system $a = d$ and $b = c = 0$, in which case

$$\boxed{\left(x(t), y(t)\right) = \left(c_1 e^{at},\ c_2 e^{at}\right)}$$

where the sole distinct eigenvalue and eigenvector are

$$\lambda = \frac{(a+d)}{2},$$

$$\begin{pmatrix}\alpha_1 \\ \beta_1\end{pmatrix} = \text{any non-zero multiple of} \begin{pmatrix}\frac{a-d}{2} \\ c\end{pmatrix} \quad \left(\text{or } \begin{pmatrix}1 \\ 0\end{pmatrix} \text{ if } a = d \text{ and } c = 0\right).$$

Outline derivation of the general solution:

The one eigenvalue and eigenvector generate part of the complementary function:

$$\left(x_1(t), y_1(t)\right) = \left(\alpha_1 e^{\lambda t},\ \beta_1 e^{\lambda t}\right)$$

x_2 and y_2 must be of the form $e^{\lambda t}$ multiplied by a linear function of t :

$$\left(x_2(t), y_2(t)\right) = \left((\alpha_2 + \alpha_3 t)e^{\lambda t},\ (\beta_2 + \beta_3 t)e^{\lambda t}\right)$$

But, upon substituting (x_2, y_2) into the system of ODEs, we find that $\begin{pmatrix}\alpha_3 \\ \beta_3\end{pmatrix} = \begin{pmatrix}\alpha_1 \\ \beta_1\end{pmatrix}$

and we obtain the singular linear system

$$\begin{pmatrix}a-\lambda & b \\ c & d-\lambda\end{pmatrix}\begin{pmatrix}\alpha_2 \\ \beta_2\end{pmatrix} = \begin{pmatrix}\alpha_1 \\ \beta_1\end{pmatrix}$$

so that

$$\begin{pmatrix}\alpha_2 \\ \beta_2\end{pmatrix} = \text{any non-zero multiple of} \begin{pmatrix}\lambda - d + 1 \\ c\end{pmatrix} \quad \text{or} \quad \begin{pmatrix}b \\ \lambda - a + 1\end{pmatrix}.$$

4.06 Linear Approximation to a System of Non-Linear ODEs (2)

From sections 4.02 and 4.03, the non-linear system

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y) \qquad \textbf{(1)}$$

with critical point at (0, 0) may be expressed as

$$\begin{aligned} \dot{x} &= ax + by + P_1(x,y) \\ \dot{y} &= cx + dy + Q_1(x,y) \end{aligned} \qquad \textbf{(2)}$$

where a, b, c, d are all constants and

$$\lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{Q_1(x,y)}{\sqrt{x^2+y^2}} = 0 .$$

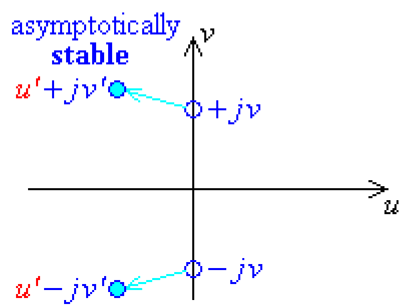
Near the critical point (0, 0), this system may be approximated by the linear system

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \qquad \textbf{(3)}$$

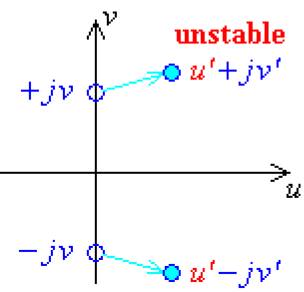
Effect of Small Perturbations

Small perturbations in the values of the coefficients a, b, c, d are reflected in small changes in the eigenvalues λ.

If the eigenvalues are a pure imaginary pair, $\lambda = \pm jv$, then the critical point is a centre. The effect of small changes in the coefficients will change the eigenvalues to the complex conjugate pair $\lambda' = u' \pm jv'$, where u' is small in magnitude and v' is close to v.

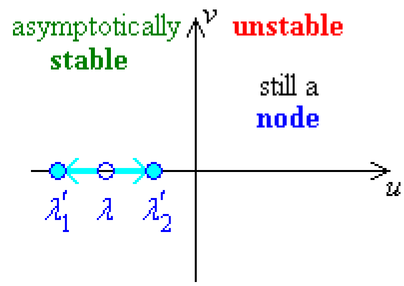


The trajectories of the new system are likely to be spirals. The critical point will be an asymptotically stable focus if $u' < 0$, a stable (but not asymptotically stable) centre if $u' = 0$, but it will be an unstable focus if $u' > 0$.

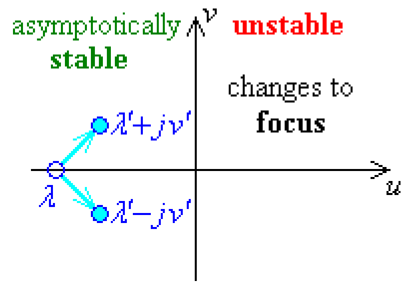


Therefore small perturbations in a linear system with pure imaginary eigenvalues are likely to result in radical changes in the trajectories (orbits) and may change the stable system into an unstable system.

If the eigenvalues are a real equal pair, $\lambda_1 = \lambda_2$, then a slight perturbation is likely to separate the roots into distinct values. If those values are still real, then the critical point remains a node.



If the perturbed eigenvalues are a complex conjugate pair, then the nature of the trajectories will change into spirals and the critical point changes from a node into a focus.



However, in both cases, an asymptotically stable critical point remains asymptotically stable after a small perturbation, while an unstable critical point remains unstable.

In all other cases, a slight perturbation leaves the sign of the real part of both eigenvalues unchanged and affects neither the type of critical point nor the overall type of the orbits.

These results are summarized in the two following theorems.

Poincaré's Theorem:

The singularities of the non-linear system **(2)** are identical to the singularities of the linear system **(3)**, except for the cases

$D < 0$ and $a + d = 0$, which is a centre in the linear case, but may be a centre or a focus in the non-linear case; and
$D = 0$, which is either a node or a focus in the non-linear case.

Theorem on stability of the singularity at (0, 0):

Linear approximation	Non-linear system
asymptotically stable	asymptotically stable
unstable	unstable
stable but not asymptotically stable	any (unstable, stable or asymptotically stable)

Example 4.06.1

Perform a stability analysis on the system
$$\frac{dx}{dt} = x - x^2 + xy\,, \quad \frac{dy}{dt} = 2y - xy - 6y^2$$

Find the critical points:

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \quad \Rightarrow \quad x - x^2 + xy = 0 \quad \text{and} \quad 2y - xy - 6y^2 = 0$$

$$\Rightarrow \quad x(1 - x + y) = 0 \quad \text{and} \quad y(2 - x - 6y) = 0$$

This generates four solutions:

$$x = 0 \quad \text{and} \quad y = 0 \quad \Rightarrow \quad (x, y) = (0, 0)$$

$$x = 0 \quad \text{and} \quad 2 - x - 6y = 0 \quad \Rightarrow \quad (x, y) = \left(0, \tfrac{1}{3}\right)$$

$$1 - x + y = 0 \quad \text{and} \quad y = 0 \quad \Rightarrow \quad (x, y) = (1, 0)$$

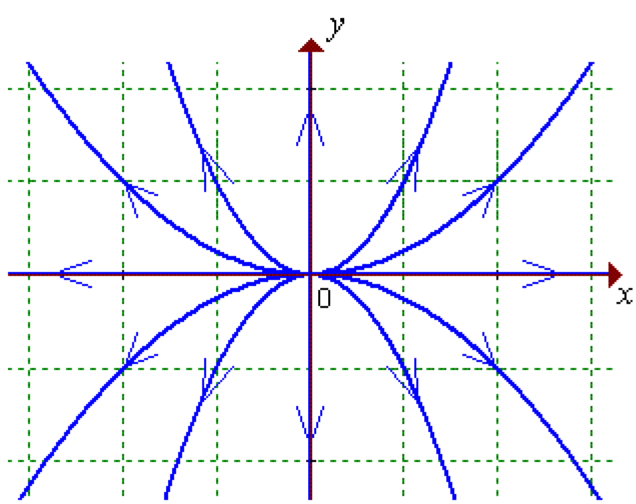
$$1 - x + y = 0 \quad \text{and} \quad 2 - x - 6y = 0 \quad \Rightarrow \quad (x, y) = \left(\tfrac{8}{7}, \tfrac{1}{7}\right)$$

Linearize the system near each critical point.

Near the critical point (0, 0), it is obvious that the linear approximation to the system is
$$\frac{dx}{dt} = x\,, \quad \frac{dy}{dt} = 2y$$

This is an uncoupled system of ODEs, whose general solution is quickly found (by direct integration or by the results on page 4.30) to be
$$\big(x(t), y(t)\big) = \left(c_1 e^t, c_2 e^{2t}\right)$$

The critical point (0, 0) is an **unstable node**.



Also note that $y = k\,x^2$, so that the solutions are parabolas all sharing the same vertex at the origin, except for the lines $x = 0$ (from $c_1 = 0$) and $y = 0$ (from $c_2 = 0$).

The orbits all begin at the origin and move away, (a hallmark of an unstable node).

This diagram is valid for the linear approximation everywhere, but is valid only in the immediate neighbourhood of (0, 0) for the non-linear system.

Example 4.06.1 (continued)

$$P(x,y) = x - x^2 + xy, \qquad Q(x,y) = 2y - xy - 6y^2$$

$$\Rightarrow \quad \frac{\partial P}{\partial x} = 1 - 2x + y \qquad \frac{\partial P}{\partial y} = x$$

and $\dfrac{\partial Q}{\partial x} = -y \qquad \dfrac{\partial Q}{\partial y} = 2 - x - 12y$

Near a critical point (a, b), the linear approximation to the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \left.\dfrac{\partial P}{\partial x}\right|_{(a,b)} & \left.\dfrac{\partial P}{\partial y}\right|_{(a,b)} \\ \left.\dfrac{\partial Q}{\partial x}\right|_{(a,b)} & \left.\dfrac{\partial Q}{\partial y}\right|_{(a,b)} \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix} = \begin{pmatrix} 1-2a+b & a \\ -b & 2-a-12b \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

Near the critical point $\left(0, \frac{1}{3}\right)$, the linear approximation to the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{3} & 0 \\ -\frac{1}{3} & 2-\frac{12}{3} \end{pmatrix} \begin{pmatrix} x \\ y-\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 0 \\ -\frac{1}{3} & -2 \end{pmatrix} \begin{pmatrix} x \\ y-\frac{1}{3} \end{pmatrix}$$

A simple change of variables to $(x,z) = \left(x, y-\frac{1}{3}\right)$ leads to

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 0 \\ -\frac{1}{3} & -2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

The matrix is triangular, allowing us to identify the eigenvalues immediately.
The eigenvalues are of opposite sign (+4/3 and –2).
Therefore the critical point at $\left(0, \frac{1}{3}\right)$ is an **unstable saddle point**.

From page 4.30,

$$D = (a-d)^2 + 4bc = \left(\tfrac{4}{3}+2\right)^2 + 0 = \left(\tfrac{10}{3}\right)^2$$

The eigenvector corresponding to eigenvalue $\lambda = -2$ is

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \text{ any non-zero multiple of } \begin{pmatrix} \dfrac{(a-d)-\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{\frac{10}{3}-\frac{10}{3}}{2} \\ -\dfrac{1}{3} \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

The eigenvector corresponding to eigenvalue $\lambda = +4/3$ is

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \text{ any non-zero multiple of } \begin{pmatrix} \dfrac{(a-d)+\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{\frac{10}{3}+\frac{10}{3}}{2} \\ -\dfrac{1}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 10 \\ -1 \end{pmatrix}$$

Example 4.06.1 (continued)

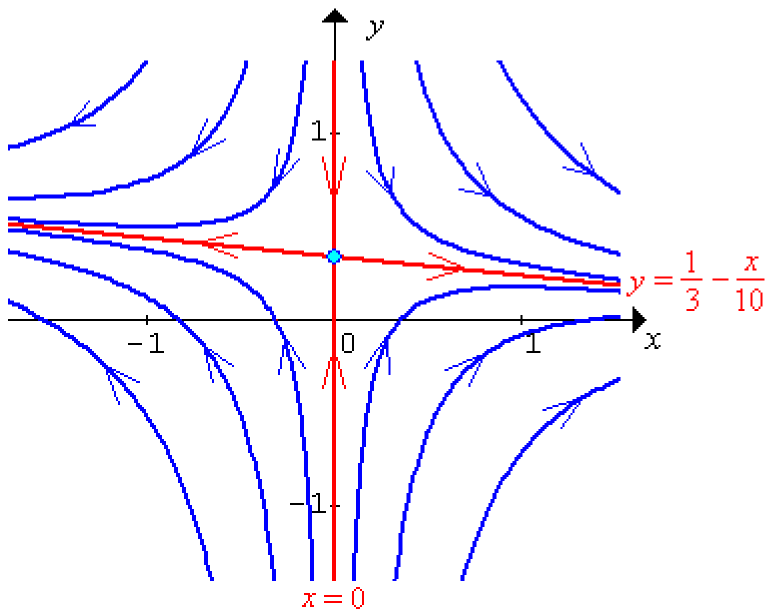
The asymptotes are therefore $x = 0$ (corresponding to $c_1 = 0$; inward because $\lambda_1 < 0$) and $x = -10\left(y - \frac{1}{3}\right)$, (corresponding to $c_2 = 0$; outward because $\lambda_2 > 0$). This is sufficient to sketch the phase portrait, even without the general solution.

The general solution near the critical point $\left(0, \frac{1}{3}\right)$ is

$$\boxed{\left(x(t), z(t)\right) = \left(x(t), y(t) - \tfrac{1}{3}\right) = \left(10c_1 e^{\frac{4}{3}t},\ -c_1 e^{\frac{4}{3}t} + c_2 e^{-2t}\right)}$$

where c_1, c_2 are arbitrary constants.

All trajectories (except for $c_2 = 0$) therefore come in from infinity near the asymptote $x = 0$ (where $c_1 = 0$)
and
all trajectories (except for $c_1 = 0$) return to infinity near the asymptote
$y - \frac{1}{3} = -\frac{1}{10}x$
(where $c_2 = 0$).



The diagram is valid for the linear approximation everywhere, but is valid only in the immediate neighbourhood of $\left(0, \frac{1}{3}\right)$ for the non-linear system.

Example 4.06.1 (continued)

Near the critical point (1, 0), the linear approximation to the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1-2 & 1 \\ 0 & 2-1 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix}$$

Change variables to $(w, y) = (x-1, y)$. Then

$$\begin{pmatrix} \dot{w} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix}$$

The matrix is triangular, with eigenvalues of opposite sign (–1 and +1).
Therefore the critical point at (1, 0) is also a[n **unstable**] **saddle point**.

$$D = (a-d)^2 + 4bc = (-1-1)^2 + 0 = 4$$

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix} \dfrac{(a-d)-\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{-2-2}{2} \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$c = 0$, so the alternative form is needed for the other eigenvector:

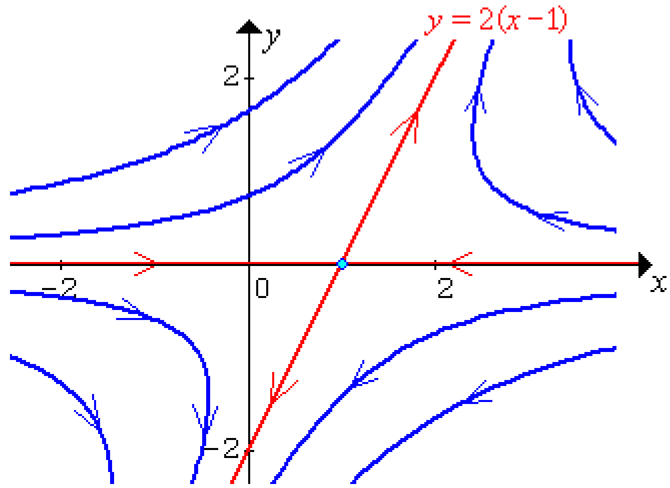
$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix} b \\ \dfrac{(d-a)+\sqrt{D}}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{2+2}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The general solution near the critical point (1, 0) is

$$\boxed{(w(t), y(t)) = (x(t)-1, y(t)) = \left(c_1 e^{-t} + c_2 e^t,\ 2c_2 e^t\right)}$$

where c_1, c_2 are arbitrary constants.

All trajectories (except for $c_1 = 0$) therefore come in from infinity near the asymptote $y = 0$ (where $c_2 = 0$)
and
all trajectories (except for $c_2 = 0$) return to infinity near the asymptote $y = 2(x-1)$ (where $c_1 = 0$).



Again, the diagram is valid for the linear approximation everywhere, but is valid only in the immediate neighbourhood of (1, 0) for the non-linear system.

Example 4.06.1 (continued)

Near the critical point $\left(\frac{8}{7},\frac{1}{7}\right)$, the linear approximation to the system is

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\frac{7-16+1}{7} & \frac{8}{7}\\ \frac{-1}{7} & \frac{14-8-12}{7}\end{pmatrix}\begin{pmatrix}x-\frac{8}{7}\\ y-\frac{1}{7}\end{pmatrix} = \begin{pmatrix}\frac{-8}{7} & \frac{8}{7}\\ \frac{-1}{7} & \frac{-6}{7}\end{pmatrix}\begin{pmatrix}x-\frac{8}{7}\\ y-\frac{1}{7}\end{pmatrix}$$

Change variables to $(w,z) = \left(x-\frac{8}{7}, y-\frac{1}{7}\right)$

$$\begin{pmatrix}\dot{w}\\ \dot{z}\end{pmatrix} = \frac{1}{7}\begin{pmatrix}-8 & 8\\ -1 & -6\end{pmatrix}\begin{pmatrix}w\\ z\end{pmatrix}$$

$$D = (a-d)^2 + 4bc = \left(\frac{-8+6}{7}\right)^2 + 4\left(\frac{8}{7}\right)\left(\frac{-1}{7}\right) = -\frac{28}{49} = -\frac{4}{7}$$

$D<0$ and $(a+d)<0$ $\Rightarrow$ the critical point is an **asymptotically stable focus**.

The eigenvalues, which are the solutions to $\det\begin{pmatrix}-\frac{8}{7}-\lambda & \frac{8}{7}\\ -\frac{1}{7} & -\frac{6}{7}-\lambda\end{pmatrix} = 0$, are

$$\lambda = \frac{(a+d)\pm\sqrt{D}}{2} = \frac{-14\pm\sqrt{-28}}{14} = -1\pm j\frac{\sqrt{7}}{7}$$

Using the formula on page 4.30, $u=-1,\ v=\frac{1}{\sqrt{7}}=\frac{\sqrt{7}}{7},\ u-d=-1+\frac{6}{7}=-\frac{1}{7}$.

$$(w(t),z(t)) = \left(x(t)-\tfrac{8}{7}, y(t)-\tfrac{1}{7}\right) =$$

$$e^{-t}\left(c_3\left(-\frac{1}{7}\cos\frac{t}{\sqrt{7}}-\frac{1}{\sqrt{7}}\sin\frac{t}{\sqrt{7}}\right)+c_4\left(\frac{1}{\sqrt{7}}\cos\frac{t}{\sqrt{7}}-\frac{1}{7}\sin\frac{t}{\sqrt{7}}\right),\ -\frac{1}{7}\left(c_3\cos\frac{t}{\sqrt{7}}+c_4\sin\frac{t}{\sqrt{7}}\right)\right)$$

Redefining the arbitrary constants as $c_5=-\frac{1}{7}c_3$ and $c_6=-\frac{1}{7}c_4$,

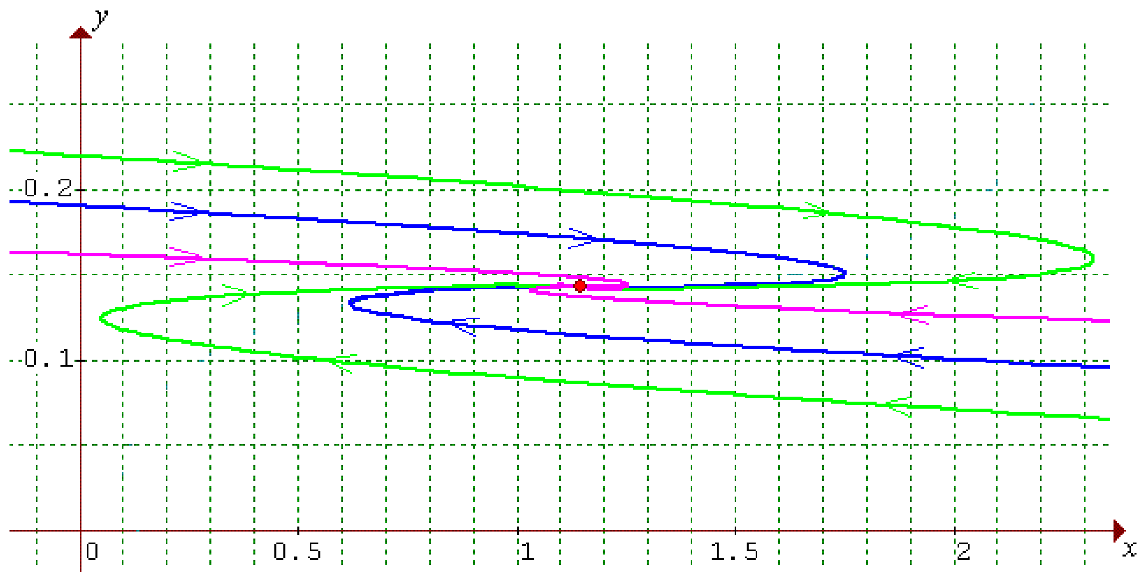
the general solution near the critical point $\left(\frac{8}{7},\frac{1}{7}\right)$ is

$$\boxed{\begin{gathered}(w(t),z(t)) = \left(x(t)-\tfrac{8}{7}, y(t)-\tfrac{1}{7}\right) = \\ e^{-t}\left(c_5\left(\cos\frac{t}{\sqrt{7}}+\sqrt{7}\sin\frac{t}{\sqrt{7}}\right)+c_6\left(-\sqrt{7}\cos\frac{t}{\sqrt{7}}+\sin\frac{t}{\sqrt{7}}\right),\ \left(c_5\cos\frac{t}{\sqrt{7}}+c_6\sin\frac{t}{\sqrt{7}}\right)\right)\end{gathered}}$$

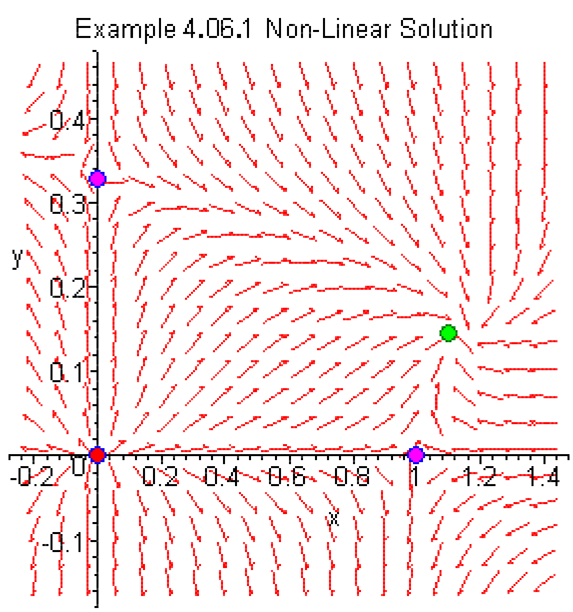
where c_5, c_6 are arbitrary constants.

All trajectories spiral in to the critical point [a phase portrait for the linear approximation is on the next page].

Example 4.06.1 (continued)

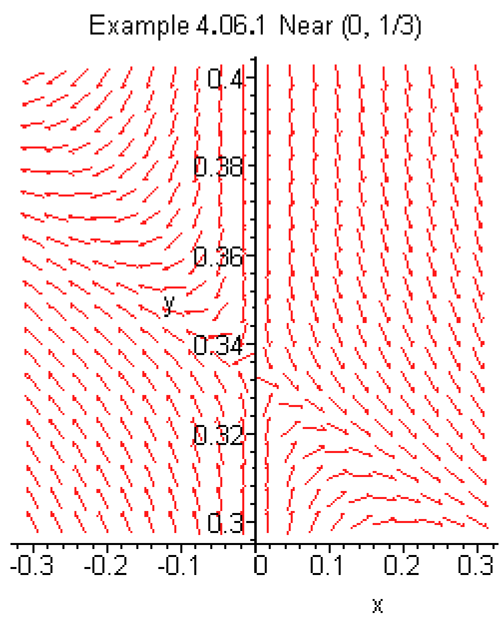


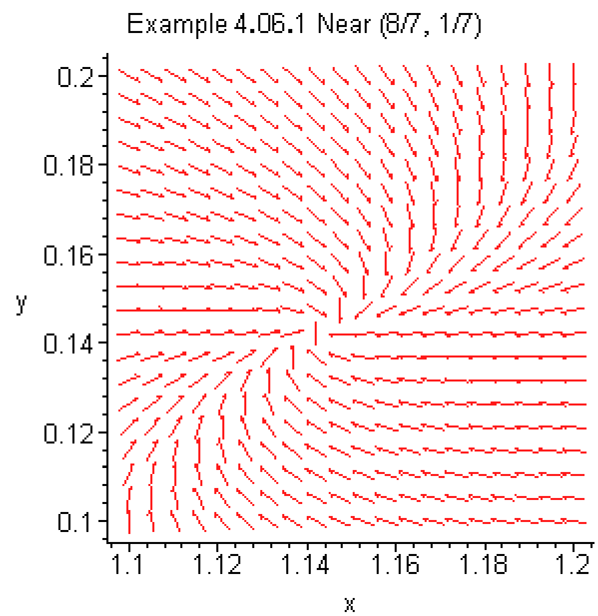
Maple produces the following direction field plots:



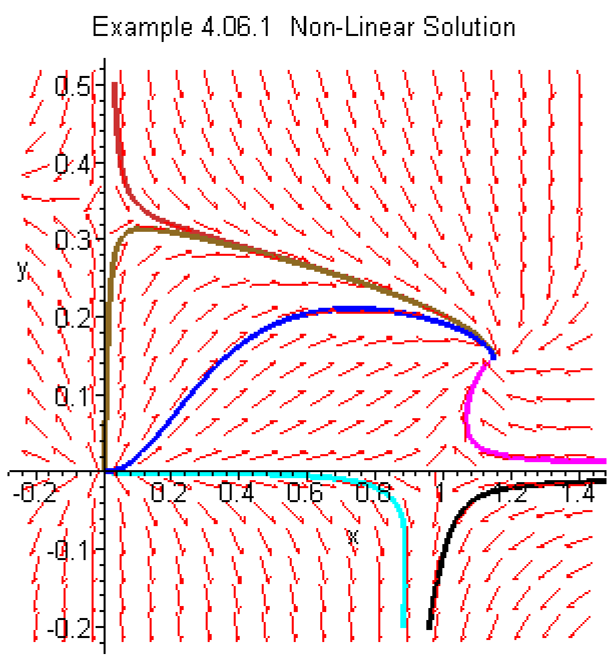
One can clearly see trajectories flowing outward from the unstable node at the origin in all directions. The natures of the other critical points are somewhat less obvious. Zooming in to the neighbourhood of one of the saddle points and to the neighbourhood of the stable focus produces the next pair of diagrams:

Example 4.06.1 (continued)





Maple can also superimpose some trajectories on these phase portraits:



Maple commands for this plot:

```
> with(DEtools):

> phaseportrait(
[diff(x(t),t) =
x(t) - x(t)^2 + x(t)*y(t),
diff(y(t),t) =
2*y(t) - x(t)*y(t) - 6*y(t)^2],
[x(t), y(t)], t=-10..10,
[[x(0)=0.1, y(0)=0.02],
[x(0)=0.8, y(0)=-0.02],
[x(0)=1.2, y(0)=0.02],
[x(0)=0.05, y(0)=0.3],
[x(0)=0.05, y(0)=0.4],
[x(0)=1.2, y(0)=-0.02]],
x=-0.2..1.4, y=-0.2..0.5,
stepsize=.01, colour=red,
linecolour=[blue, cyan, magenta, sienna, orange, black],
title=`Example 4.06.1  Non-Linear Solution`);
```

Example 4.06.2

Perform a stability analysis on the system

$$\frac{dx}{dt} = y + x\left(1 - x^2 - y^2\right), \quad \frac{dy}{dt} = -x + y\left(1 - x^2 - y^2\right)$$

Find the critical points:

Clearly $(x, y) = (0, 0)$ satisfies $\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0$.

$$\frac{dx}{dt} = \frac{dy}{dt} = x = 0 \qquad \Rightarrow \quad y = 0$$

$$\frac{dx}{dt} = \frac{dy}{dt} = y = 0 \qquad \Rightarrow \quad x = 0$$

If $x \neq 0$ and $y \neq 0$, then at any critical point

$$y + x\left(1 - x^2 - y^2\right) = -x + y\left(1 - x^2 - y^2\right) = 0$$

$$\Rightarrow \quad y = -x\left(1 - x^2 - y^2\right) \quad \text{and} \quad x = y\left(1 - x^2 - y^2\right)$$

$$\Rightarrow \quad \left(1 - x^2 - y^2\right) = -\frac{x}{y} = \frac{y}{x} \qquad \Rightarrow \quad \left(\frac{y}{x}\right)^2 = -1$$

which has no solution for real (x, y).

Therefore the only critical point is (0, 0).

The linear approximation to the non-linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$D = (a - d)^2 + 4bc = (1 - 1)^2 + 4(1)(-1) = -4 < 0$$

$(a + d) = 2 > 0 \quad \Rightarrow$ the critical point is an **unstable focus**.

Using the formula on page 4.30,

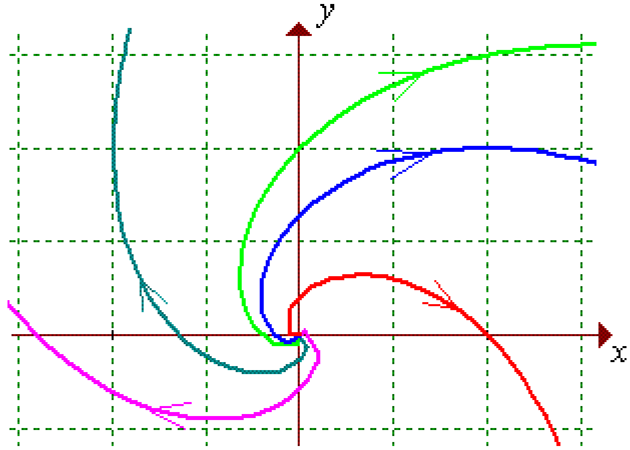
$$u = \frac{a + d}{2} = 1, \qquad u - d = \frac{a - d}{2} = 0, \quad v = \frac{\sqrt{-D}}{2} = 1$$

$$\boxed{\left(x(t), y(t)\right) = e^t \left(c_3 \sin t - c_4 \cos t,\ c_3 \cos t + c_4 \sin t\right)}$$

and c_3, c_4 are [real] arbitrary constants.

Example 4.06.2 (continued)

Solution in the neighbourhood of the only critical point (0, 0):



Now consider the distance r of any point (x, y) from the critical point (0, 0):

$$r^2 = x^2 + y^2 \quad \Rightarrow \quad 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad \text{[chain rule]}$$

From the original non-linear system:

$$r\frac{dr}{dt} = x\left(y + x\left(1-x^2-y^2\right)\right) + y\left(-x + y\left(1-x^2-y^2\right)\right)$$

$$= xy + x^2\left(1-r^2\right) - xy + y^2\left(1-r^2\right) = r^2\left(1-r^2\right)$$

$$\Rightarrow \quad \frac{dr}{dt} = r\left(1-r^2\right) \begin{cases} < 0 & (r>1) \\ > 0 & (r<1) \end{cases}$$

Therefore solutions starting closer than one unit to the critical point spiral out, but solutions starting further away than one unit approach the critical point. A solution on the circle $r = 1$ never changes its distance from the origin and stays on that circle, but is not stationary (because the only critical point is at (0, 0), not on that circle).

Note that $x^2 + y^2 = 1$ is a solution to the non-linear equation:

$$\frac{dx}{dt} = y + x\left(1-x^2-y^2\right) = y \quad \text{and} \quad \frac{dy}{dt} = -x + y\left(1-x^2-y^2\right) = -x$$

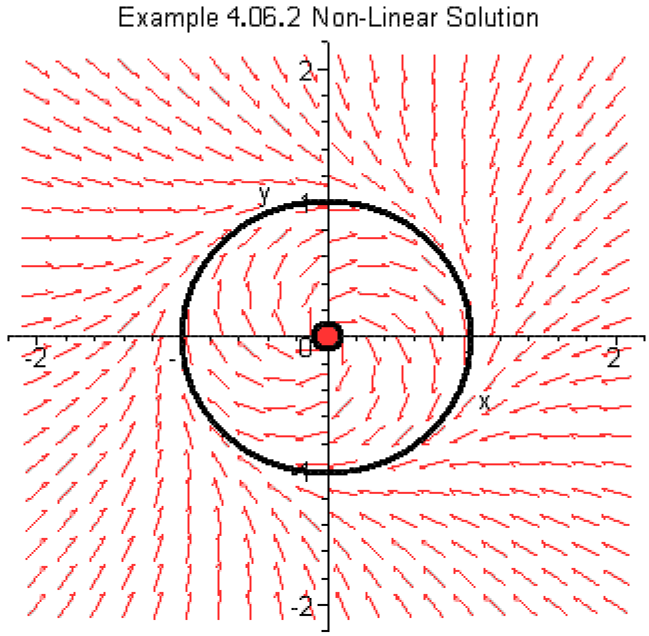
$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{x}{y}$$

But $x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2y\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

No other solution may cross this solution.
Therefore all non-trivial solutions approach the unit circle in the limit as $t \to \infty$.

The unit circle here is an example of a **limit cycle**.

Example 4.06.2 (continued)



Further notes:

Let $u = r^2$, then $du = 2r\,dr$ and

$$r\frac{dr}{dt} = r^2\left(1-r^2\right) \Rightarrow \frac{1}{2}\frac{du}{dt} = u\left(1-u\right) \Rightarrow \frac{du}{u\left(1-u\right)} = 2\,dt$$

$$\Rightarrow \int\left(\frac{1}{u} + \frac{1}{1-u}\right)du = 2\int 1\,dt \qquad \Rightarrow \ln u - \ln\left(1-u\right) = 2t + C$$

$$\ln\left(\frac{u}{1-u}\right) = 2t + C \qquad \Rightarrow \frac{u}{1-u} = e^{2t+C} = c_1 e^{2t}$$

$$\Rightarrow u = c_1 e^{2t} - u\,c_1 e^{2t} \Rightarrow \left(1 + c_1 e^{2t}\right)u = c_1 e^{2t}$$

$$\Rightarrow \left(c_2 e^{-2t} + 1\right)u = 1, \quad \text{where} \quad c_2 = \frac{1}{c_1}$$

$$\Rightarrow u = \frac{1}{1 + c_2 e^{-2t}} \qquad \Rightarrow r\left(t\right) = \frac{1}{\sqrt{1 + c_2 e^{-2t}}}$$

If $r(0) = r_o$, then

$$c_1 e^{2t} = \frac{u}{1-u} \qquad \Rightarrow c_1 = \frac{r_o^{\,2}}{1-r_o^{\,2}} \qquad \Rightarrow c_2 = \frac{1-r_o^{\,2}}{r_o^{\,2}}$$

Therefore

$$\boxed{r\left(t\right) = \frac{1}{\sqrt{1 + \left(\dfrac{1-r_o^{\,2}}{r_o^{\,2}}\right)e^{-2t}}}}$$

Example 4.06.2 (continued)

Also note that, from the polar coordinate system,

$$(x, y) = (r\cos\theta, r\sin\theta)$$

$$\Rightarrow \quad (\dot{x}, \dot{y}) = \left(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta},\ \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}\right)$$

$$\Rightarrow \quad x\dot{y} - y\dot{x} = (r\cos\theta)\dot{r}\sin\theta + (r\cos\theta)r\cos\theta\,\dot{\theta} - (r\sin\theta)\dot{r}\cos\theta + (r\sin\theta)r\sin\theta\,\dot{\theta}$$

$$= r^2\left(\cos^2\theta + \sin^2\theta\right)\dot{\theta} = r^2\dot{\theta}$$

But the non-linear system is

$$(\dot{x}, \dot{y}) = \left(y + x\left(1 - r^2\right),\ -x + y\left(1 - r^2\right)\right)$$

$$\Rightarrow \quad x\dot{y} - y\dot{x} = -x^2 + xy\left(1 - r^2\right) - y^2 - xy\left(1 - r^2\right) = -\left(x^2 + y^2\right) = -r^2$$

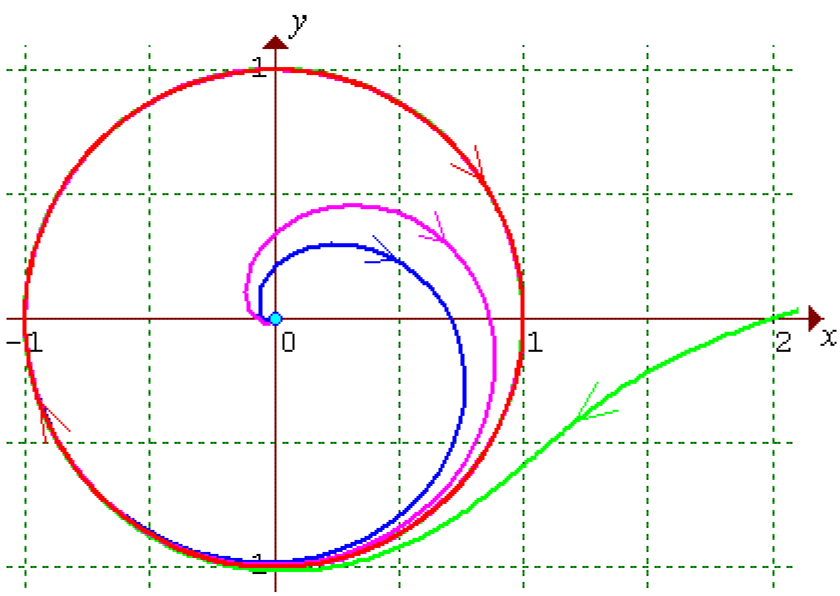
Therefore $\quad r^2\dot{\theta} = -r^2 \quad \Rightarrow \quad \dot{\theta} = -1 \quad \Rightarrow \quad \theta = -t + C$

All paths move clockwise with constant angular speed.

In Cartesian coordinates all orbits can therefore be described by

$$\boxed{\left(x(t), y(t)\right) = \frac{1}{\sqrt{1 + \left(\dfrac{1 - r_o^{\,2}}{r_o^{\,2}}\right)e^{-2t}}}\left(\cos\left(t - \theta_o\right),\ -\sin\left(t - \theta_o\right)\right)}$$

where $(x(0), y(0)) = r_o\,(\cos\,\theta_o,\ \sin\,\theta_o)$



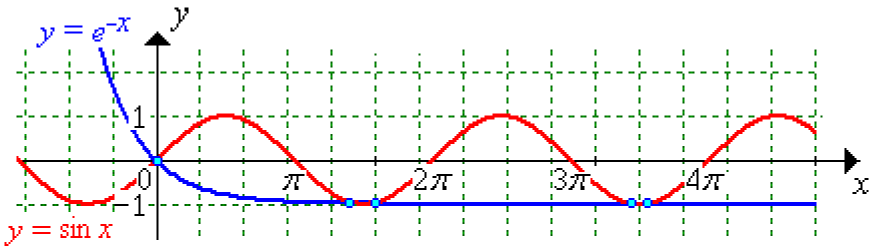
Example 4.06.3 (A more challenging and tedious case, for reference only)

Find and determine the nature of all critical points of the system

$$\frac{dx}{dt} = e^{-x} - y - 1, \qquad \frac{dy}{dt} = y - \sin x \qquad \textbf{(1)}$$

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \quad \Rightarrow \quad y = e^{-x} - 1 \quad \text{and} \quad y = \sin x$$

Critical points occur where the graphs of $y = e^{-x} - 1$ and $y = \sin x$ intersect.



The critical points are (0, 0) and (x_i, y_i), where

$$x_i = (4i-1)\tfrac{\pi}{2} - \delta_i,\ (4i-1)\tfrac{\pi}{2} + \varepsilon_i\ ; \quad (i = 1, 2, 3, \ldots)$$

Linearize for the critical point at (0, 0):

$$\left(e^{-x} - y - 1\right)\Big|_{\text{near }(0,0)} \approx \left.\frac{\partial P}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial P}{\partial y}\right|_{(0,0)} y = \left(-e^{-0}\right)x + (-1)\,y$$

$$\left(y - \sin x\right)\Big|_{\text{near }(0,0)} \approx \left.\frac{\partial Q}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial Q}{\partial y}\right|_{(0,0)} y = \left(-\cos 0\right)x + (1)\,y$$

Therefore the linear system that models the non-linear system **(1)** near (0, 0) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \textbf{(2)}$$

Finding the eigenvalues:

$$\det\left(A - \lambda I\right) = \begin{vmatrix} -1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = -(1+\lambda)(1-\lambda) - (-1)(-1) = \lambda^2 - 1 - 1$$

$$\det\left(A - \lambda I\right) = 0 \quad \Rightarrow \quad \lambda^2 = 2 \quad \Rightarrow \quad \lambda = \pm\sqrt{2}$$

The eigenvalues are real and of opposite sign.
The critical point (0, 0) of **(2)** [and therefore also of **(1)**] is an **unstable saddle point**.

Example 4.06.3 (continued)

Using the results on page 4.30,

$$D = (a-d)^2 + 4bc = (-1-1)^2 + 4(-1)(-1) = 4+4 = 8$$

The eigenvectors are

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix} \dfrac{(a-d)-\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{-2-2\sqrt{2}}{2} \\ -1 \end{pmatrix} \quad \text{for} \quad \lambda = -\sqrt{2}$$

and

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix} \dfrac{(a-d)+\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{-2+2\sqrt{2}}{2} \\ -1 \end{pmatrix} \quad \text{for} \quad \lambda = +\sqrt{2}$$

Choose a multiple of -1 in both cases.

The general solution of **(2)** is therefore

$$\boxed{\left(x(t), y(t)\right) = \left(c_1\left(1+\sqrt{2}\right)e^{-\sqrt{2}t} + c_2\left(1-\sqrt{2}\right)e^{\sqrt{2}t},\ c_1 e^{-\sqrt{2}t} + c_2 e^{\sqrt{2}t}\right)}$$

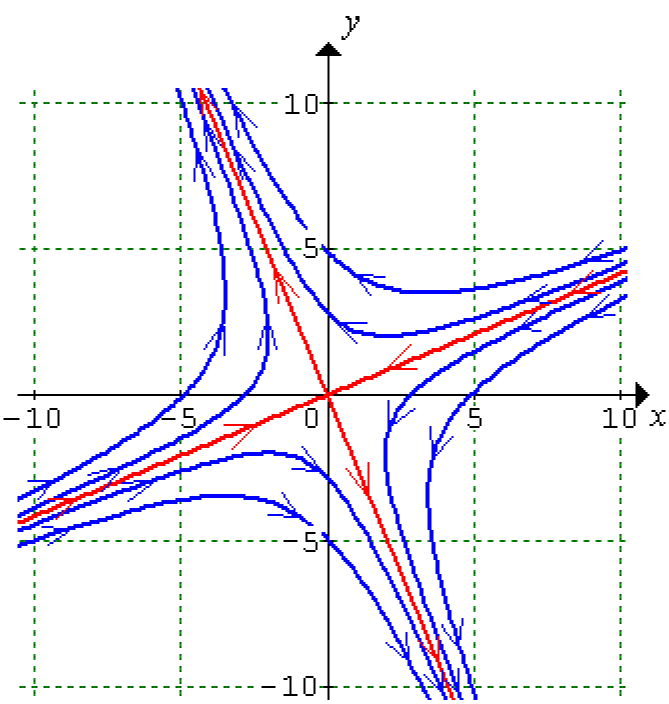
$$\lim_{t\to-\infty} \frac{y(t)}{x(t)} = \frac{1}{1+\sqrt{2}} = \frac{1-\sqrt{2}}{1-2} = \sqrt{2}-1 > 0 \quad \text{and}$$

$$\lim_{t\to+\infty} \frac{y(t)}{x(t)} = \frac{1}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{1-2} = -1-\sqrt{2} < 0$$

The trajectories (except for $c_1 = 0$) therefore come in from infinity along the asymptote $y = \left(\sqrt{2}-1\right)x$.

The trajectories (except for $c_2 = 0$) return to infinity along the asymptote $y = -\left(1+\sqrt{2}\right)x$.

The phase space diagram for this solution (completely valid only for the linear system **(2)**) is



The phase space for the non-linear system **(1)** resembles this diagram only in the immediate neighbourhood of the critical point (0, 0).

Example 4.06.3 (continued)

At other critical points (k, l),

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \quad \Rightarrow \quad l = \sin k \quad \Rightarrow \quad e^{-k} - \sin k - 1 = 0$$

Linearizing (Taylor's series for $P(x, y)$ about $(x, y) = (k, l)$):

$$\left(e^{-x} - y - 1\right)\Big|_{\text{near }(k,l)} \approx \left.\frac{\partial P}{\partial x}\right|_{(k,l)} (x-k) + \left.\frac{\partial P}{\partial y}\right|_{(k,l)} (y-l) = \left(-e^{-k}\right)(x-k) + (-1)(y-l)$$

$$\left(y - \sin x\right)\Big|_{\text{near }(k,l)} \approx \left.\frac{\partial Q}{\partial x}\right|_{(k,l)} (x-k) + \left.\frac{\partial Q}{\partial y}\right|_{(k,l)} (y-l) = (-\cos k)(x-k) + (1)(y-l)$$

Therefore the linear system that models the non-linear system **(1)** near (k, l) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -e^{-k} & -1 \\ -\cos k & 1 \end{pmatrix} \begin{pmatrix} x-k \\ y-l \end{pmatrix} \qquad \textbf{(3)}$$

Finding the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -e^{-k} - \lambda & -1 \\ -\cos k & 1-\lambda \end{vmatrix} = -\left(e^{-k} + \lambda\right)(1-\lambda) - (-\cos k)(-1)$$

$$= \lambda^2 - \left(1 - e^{-k}\right)\lambda - \left(e^{-k} + \cos k\right) = 0$$

$$\Rightarrow \quad \lambda = \frac{\left(1-e^{-k}\right) \pm \sqrt{\left(1-e^{-k}\right)^2 + 4\left(e^{-k} + \cos k\right)}}{2}$$

$$\Rightarrow \quad \lambda = \frac{\left(1-e^{-k}\right) \pm \sqrt{\left(1+e^{-k}\right)^2 + 4\cos k}}{2}$$

Now $k > 0 \quad \Rightarrow \quad 0 < e^{-k} < 1 \quad \Rightarrow \quad 0 < 1 - e^{-k} < 1$

Recall that $k = x_i = (4i-1)\frac{\pi}{2} - \delta_i,\ (4i-1)\frac{\pi}{2} + \varepsilon_i\ ; \quad (i = 1, 2, 3, \ldots)$

Examining the right-hand critical point in each pair,

$k = (4i-1)\frac{\pi}{2} + \varepsilon_i\ , \quad (0 < \varepsilon_i \ll 1)$

$$\Rightarrow \quad \cos k = \cos\left((4i-1)\tfrac{\pi}{2} + \varepsilon_i\right) = \cos\left(\varepsilon_i - \tfrac{\pi}{2}\right) = \cos\left(\tfrac{\pi}{2} - \varepsilon_i\right) = \sin \varepsilon_i \approx \varepsilon_i > 0$$

Therefore $\sqrt{\left(1-e^{-k}\right)^2 + 4\left(e^{-k} + \cos k\right)} > \left(1 - e^{-k}\right) > 0$

and the two eigenvalues are real and of opposite sign.
These critical points are therefore all unstable **saddle points**.

Example 4.06.3 (continued)

Examining the left-hand critical point in each pair,
$k = (4i-1)\frac{\pi}{2} - \delta_i\,, \quad (0 < \delta_i \ll 1)$

$\Rightarrow \quad \cos k = \cos\left((4i-1)\frac{\pi}{2} - \delta_i\right) = \cos\left(-\delta_i - \frac{\pi}{2}\right) = \cos\left(\delta_i + \frac{\pi}{2}\right) = -\sin\delta_i$

But k is the solution to $e^{-k} - \sin k - 1 = 0$

$\Rightarrow \quad e^{-k} = 1 + \sin k = 1 + \sin\left((4i-1)\frac{\pi}{2} - \delta_i\right)$

$= 1 + \sin\left(-\delta_i - \frac{\pi}{2}\right) = 1 - \sin\left(\delta_i + \frac{\pi}{2}\right) = 1 - \cos\delta_i$

$\Rightarrow \quad e^{-k} + \cos k \approx \left(1 - \cos\delta_i\right) - \sin\delta_i \approx 1 - \left(1 - \frac{\delta_i^{\,2}}{2}\right) - \delta_i < 0 \quad$ (δ_i is small)

$\Rightarrow \quad \sqrt{\left(1 - e^{-k}\right)^2 + 4\left(e^{-k} + \cos k\right)} < 1 - e^{-k}$

But $\lambda = \dfrac{\left(1 - e^{-k}\right) \pm \sqrt{\left(1 - e^{-k}\right)^2 + 4\left(e^{-k} + \cos k\right)}}{2}$

Therefore the eigenvalues are a real distinct positive pair and the singularity is an **unstable node**.

The locations and nature of the first five critical points are listed here.

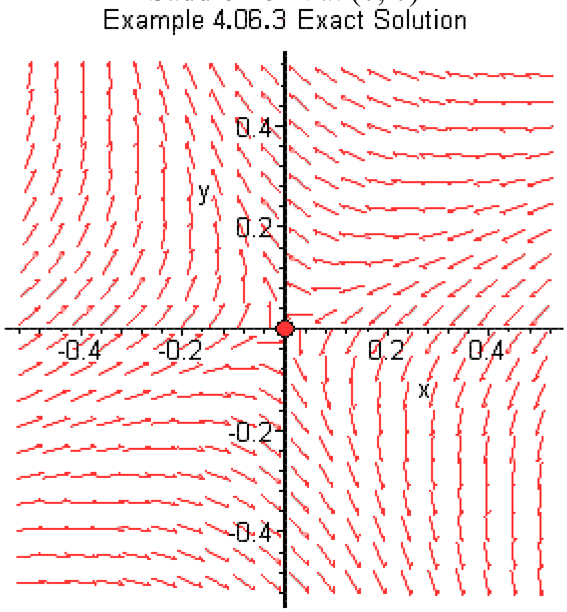
x	y	λ_1	λ_2	Type
0	0	–1.4142...	+1.4142...	unstable saddle point
4.56820...	–0.98962...	+0.1608...	+0.8287...	unstable node
4.83833...	–0.99208...	–0.1200...	+1.1121...	unstable saddle point
10.98977...	–0.99998...	+0.0058...	+0.9941...	unstable node
11.00135...	–0.99998...	–0.0057...	+1.0057...	unstable saddle point

Here is a Maple session to create direction field plots for the first three critical points of the non-linear system.

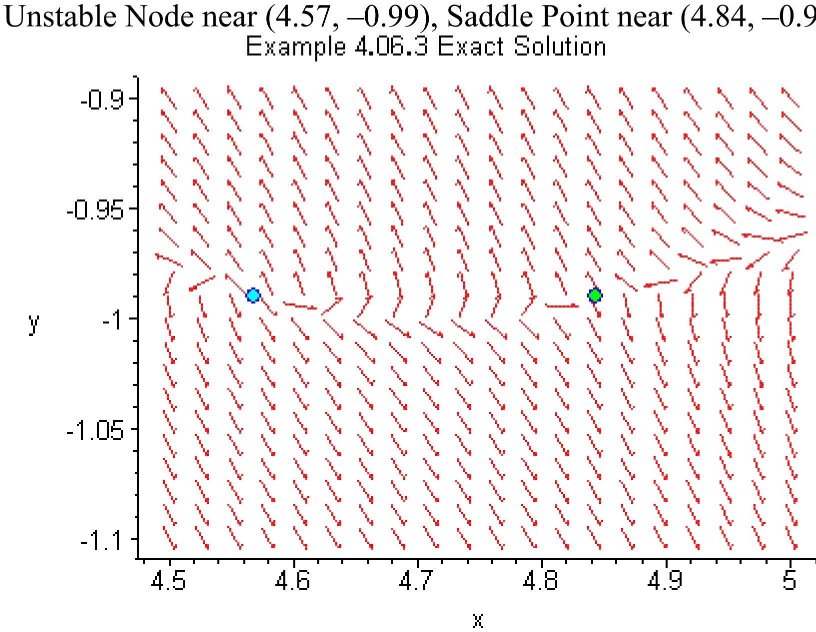
```
> with(DEtools):
> DEplot([diff(x(t),t) = -y(t) - 1 + exp(-x(t)),
  diff(y(t),t) = y(t) - sin(x(t))],
  [x(t),y(t)], t=-1..1, x=-0.5..0.5, y=-0.5..0.5,
  title=`Example 4.06.3 Exact Solution`);

> DEplot([diff(x(t),t) = -y(t) - 1 + exp(-x(t)),
  diff(y(t),t) = y(t) - sin(x(t))],
  [x(t),y(t)], t=-1..1, x=4.5..5, y=-1.1..-0.9,
  title=`Example 4.06.3 Exact Solution`);
```

Example 4.06.3 (continued)

Saddle Point at (0, 0)



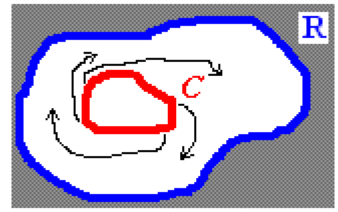
Unstable Node near (4.57, –0.99), Saddle Point near (4.84, –0.99)

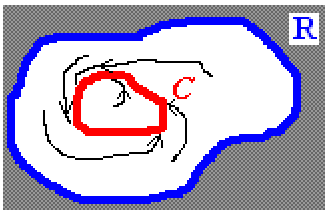


Solution curves can be traced by following the arrows (which at every location point in the direction of $\frac{dy}{dx}$).

4.07 Limit Cycles

If, in some region, all trajectories begin on a closed curve inside that region, then that curve is an unstable limit cycle.





If all trajectories terminate on the curve, then it is a stable limit cycle.

More formally,
Let R be a bounded region in the xy plane.
Let C be a closed curve composed of interior points of R and bounding a region A.
Let C be a solution curve of the system

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y) \qquad \textbf{(1)}$$

where $P(x, y)$ and $Q(x, y)$ are differentiable with respect to x and y at all points of R.
C is a **limit cycle** of **(1)** if no other closed solution curve is close to C and if all orbits sufficiently near it approach it asymptotically as $t \to -\infty$ (unstable) or as $t \to +\infty$ (stable).

Bendixon Non-existence Theorem:

For system **(1)**, if the expression $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ does not change sign or vanish identically in a simply connected (= "no holes") region D inside R, then no closed trajectory can exist entirely within D.

The contrapositive statement is:
If C is a closed solution curve of **(1)** in R, then $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ must vanish for some subset of R.

Proof:

If C is a closed curve in R with interior region A, then Green's theorem in two dimensions states

$$\int_C (P\,dy - Q\,dx) = \iint_A \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\,dy \qquad \textbf{(2)}$$

But, for all points on C, (which is a solution curve of **(1)**),

$$P\,dy - Q\,dx = \left(P\frac{dy}{dx} - Q \right) dx = \left(P\frac{\dot{y}}{\dot{x}} - Q \right) dx = \left(P\frac{Q}{P} - Q \right) dx \equiv 0$$

It then follows that

$$\oint_C (P\,dy - Q\,dx) = \iint_A \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\,dy = 0$$

This is not possible unless the integrand $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ changes sign or is identically zero inside region A.

Poincaré-Bendixon Theorem (Existence Theorem for Limit Cycles)

If the solution curve C of the system **(1)** is in and remains in a bounded region R for $t > t_o$ without approaching singular points and if $P(x, y)$ and $Q(x, y)$ are differentiable with respect to x and y at all points of R, then a limit cycle exists in R and either C is a limit cycle or it approaches a limit cycle as $t \to +\infty$.

4.08 Van der Pol's Equation

During an investigation of the properties of vacuum tubes, Van der Pol developed a second order non-linear ordinary differential equation to model the circuit:

$$\frac{d^2 x}{dt^2} - \mu\left(1-x^2\right)\frac{dx}{dt} + x = 0 \quad , \qquad (\mu > 0) \qquad \textbf{(1)}$$

The linear form resembles the linear ODE for the RLC circuit:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \qquad \textbf{(2)}$$

The resistance term in **(2)** provides damping provided $R > 0$.
If $R < 0$, then the solution is unstable and the current would have an ever increasing amplitude, which is what the linear form of **(1)** predicts, ($-\mu < 0$).

However, experimental evidence suggests that, after some initial increase in amplitude, a periodic solution is attained. This is an indication that a limit cycle may exist for **(1)**.

The "resistance" term $-\mu(1-x^2)$ in Van der Pol's equation is negative if $|x| < 1$, but is positive for $|x| > 1$. The non-linear term must be retained in order to find the periodic steady state solution.

Introduce a new variable y to Van der Pol's equation:

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \mu\left(1-x^2\right)y - x \end{aligned} \qquad \textbf{(3)}$$

The linear version of **(3)** is:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \textbf{(4)}$$

Finding the critical points of **(3)**:

$$\frac{dx}{dt} = 0 \quad \Rightarrow \quad y = 0\,, \qquad \frac{dy}{dt} = y = 0 \quad \Rightarrow \quad x = 0$$

Thus (0, 0) is the only critical point of **(3)**.

Applying the formulae from page 4.30:

$D = (a-d)^2 + 4bc = (0-\mu)^2 + 4(1)(-1) = \mu^2 - 4$

so that $D < 0$ for $0 < \mu < 2$ and $D > 0$ for $\mu > 2$

$(a+d) = \mu > 0$

$0 < \mu < 2 \quad \Rightarrow$ the critical point (0, 0) is an **unstable focus**.

$\mu > 2 \quad \Rightarrow$ (0, 0) is an **unstable node**.

The eigenvalues are

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} \quad \Rightarrow \quad \operatorname{Re}(\lambda) > 0$$

so that (0, 0) is unstable for all $\mu > 0$.

Searching for limit cycles:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}\left(-x + \mu\left(1-x^2\right)y\right) = \mu\left(1-x^2\right)$$

$$|x| < 1 \quad \Rightarrow \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} > 0$$

Because $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ does not change sign anywhere in the region $|x| < 1$, there are no limit cycles contained entirely in that region (by the Bendixon Non-existence Theorem). There may be a limit cycle in a region that includes $x = -1$ and/or $x = +1$.

Transforming **(3)** to polar coordinates,

$r^2 = x^2 + y^2$

$$\Rightarrow \quad r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = x(y) + y\left(\mu\left(1-x^2\right)y - x\right) = \mu\left(1-x^2\right)y^2$$

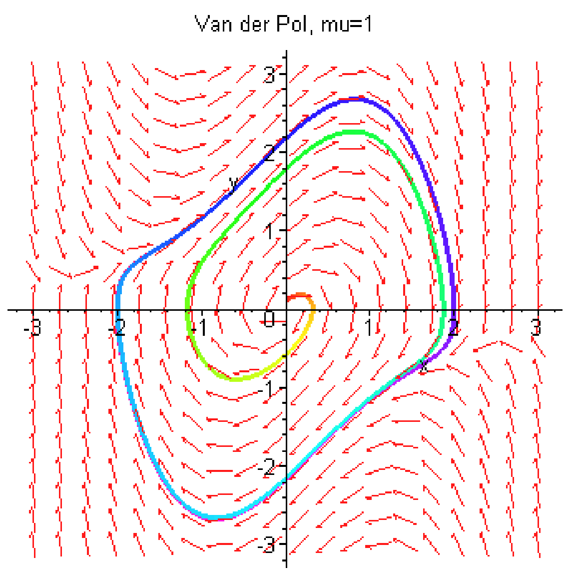
so that r is increasing with time for $|x| < 1$, but decreasing for $|x| > 1$ and not changing when $x = \pm 1$.

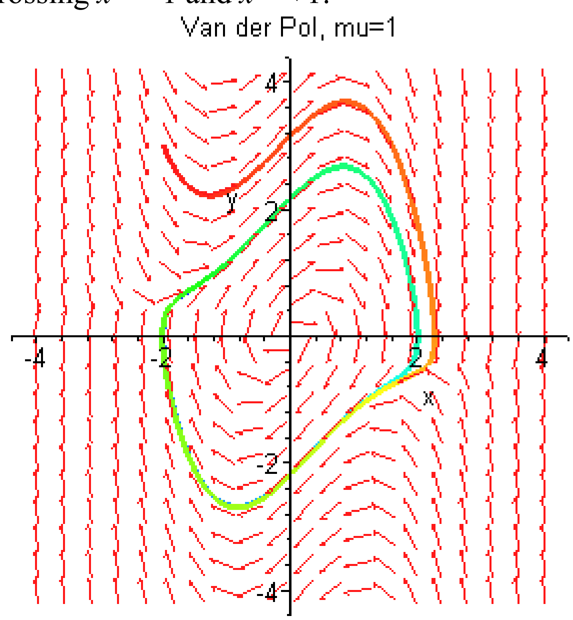
This suggests that a region extending to a sufficiently large x may contain a limit cycle.
When $0 < \mu < 2$ a closed periodic solution is possible and a limit cycle occurs.
When $\mu \geq 2$ a closed periodic solution is impossible and there is no limit cycle.

A Maple session that produces solution curves for the Van der Pol equation with $\mu = 1$ for two choices of starting point (one inside the limit cycle, one outside) is presented here.

```
> with(DEtools):
> phaseportrait([diff(x(t),t) = y(t), diff(y(t),t) =
y(t)*(1 - (x(t))^2) - x(t)],
[x(t),y(t)], t=0..20, [[x(0)=0,y(0)=0.1]], x=-3..3,
y=-3..3, stepsize=0.05, linecolour=t/2, title=`Van der Pol,
mu=1`);
> phaseportrait([diff(x(t),t) = y(t), diff(y(t),t) =
y(t)*(1 - (x(t))^2) - x(t)],
[x(t),y(t)], t=0..20, [[x(0)=-2,y(0)=3]], x=-4..4, y=-4..4,
stepsize=0.03, linecolour=t/2, title=`Van der Pol, mu=1`);
```

with output, clearly illustrating the limit cycle crossing $x = -1$ and $x = +1$:



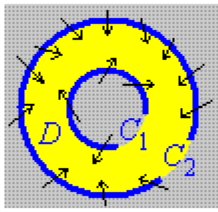


Again note how the trajectories move away from the origin only in the region $-1 < x < 1$.

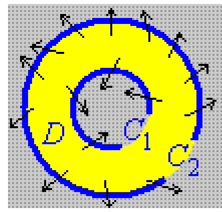
4.09 Theorem for Limit Cycles

Theorem (Extension of the Poincaré-Bendixon theorem):

Let D be an annular region between closed curves C_1 and C_2.



[stable]



[unstable]

If solution curves of the system

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y) \qquad \textbf{(1)}$$

enter D at every point of C_1 and C_2 (or leave at every point of C_1 and C_2), and there are no singularities of **(1)** in D or on C_1 or C_2, then a limit cycle exists in D.

It also follows that a closed curve cannot be a limit cycle unless it encloses a singularity.

Example 4.09.1

Determine whether a limit cycle exists for the second order ODE $\frac{d^2x}{dt^2} + x^2 + 1 = 0$.

The ODE can be rewritten as the first order non-linear system

$$\dot{x} = y$$
$$\dot{y} = -x^2 - 1$$

But $-x^2 - 1 < 0$ for all real x.
No critical point exists for real (x, y).
But a limit cycle must enclose a singularity.
Therefore no limit cycle exists for this system.

Example 4.09.2

Perform a stability analysis and determine whether a limit cycle exists for the system

$$\begin{aligned}\frac{dx}{dt} &= x\left(1-x^2-y^2\right) + 5y \\ \frac{dy}{dt} &= -5x + y\left(1-x^2-y^2\right)\end{aligned} \qquad \textbf{(1)}$$

One critical point occurs where $x = y = 0$.
Substitution of $x = 0$ into **(1)** leads to $y = 0$ and vice versa.
If $x \neq 0$ and $y \neq 0$, then **(1)** $\Rightarrow$ at a critical point

$$\left(1-x^2-y^2\right) = \frac{-5y}{x} = \frac{5x}{y} \qquad \Rightarrow \quad \left(\frac{y}{x}\right)^2 = -1$$

which has no real solution for (x, y). Therefore (0, 0) is the only critical point of **(1)**.

Near (0, 0), the linear approximation to **(1)** is

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}1 & 5\\ -5 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \qquad \textbf{(2)}$$

The eigenvalues may be found either by solving $\begin{vmatrix}1-\lambda & 5\\ -5 & 1-\lambda\end{vmatrix} = 0$ or by use of the formula on page 4.30:

$$D = (a-d)^2 + 4bc = (1-1)^2 + 4(-5)(5) = -100$$

$$\lambda = \frac{(a+d)\pm\sqrt{D}}{2} = \frac{2\pm\sqrt{-100}}{2} = 1\pm 5j$$

The eigenvalues are a complex conjugate pair with positive real part
$\Rightarrow$ the critical point of **(2)** (and therefore also of **(1)**) is an **unstable focus**.

Example 4.09.2 (continued)

Checking for a limit cycle,

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}\left(x\left(1-x^2-y^2\right)+5y\right) + \frac{\partial}{\partial y}\left(-5x+y\left(1-x^2-y^2\right)\right)$$

$$= 1-3x^2-y^2 + 1-x^2-3y^2 = 2\left(1-2\left(x^2+y^2\right)\right)$$

$$\Rightarrow \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \begin{cases} >0 & \left(x^2+y^2<\frac{1}{2}\right) \\ <0 & \left(x^2+y^2>\frac{1}{2}\right) \end{cases}$$

There may therefore be a limit cycle in a region bounded by $x^2+y^2=r^2$, where $r^2>\frac{1}{2}$, but it cannot exist entirely inside $x^2+y^2=\frac{1}{2}$.

Changing to polar coordinates,

$$r^2 = x^2 + y^2 \qquad \Rightarrow \quad 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

From the original non-linear system:

$$r\frac{dr}{dt} = x\left(x\left(1-x^2-y^2\right) + 5y\right) + y\left(-5x + y\left(1-x^2-y^2\right)\right)$$

$$= +5xy + x^2\left(1-r^2\right) - 5xy + y^2\left(1-r^2\right) = r^2\left(1-r^2\right)$$

$$\Rightarrow \quad \frac{dr}{dt} = r\left(1-r^2\right) \begin{cases} <\ 0 & (r>1) \\ >\ 0 & (r<1) \end{cases}$$

Therefore solutions that start closer than one unit to the critical point spiral out, but solutions that start further away than one unit approach the critical point. A solution on the circle $r=1$ never changes its distance from the origin and stays on that circle, but is not stationary.

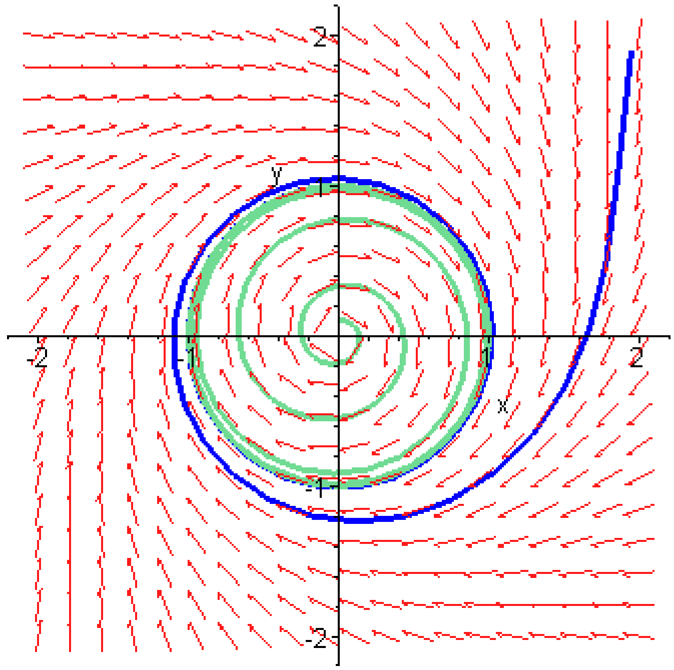
Therefore $x^2+y^2 = 1$ is the limit cycle.

Consider the region D bounded by the circles $x^2 + y^2 = 1/100$ and $x^2 + y^2 = 2$, inside which $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ and $\frac{dr}{dt}$ both change sign. All trajectories crossing the inner circle must be moving away from the origin into region D and all trajectories crossing the outer circle must be moving towards the origin, also into region D.

Thus, a solution path that enters D can never leave D.
There are no singularities in the region or its boundaries.
Therefore, by the Poincaré-Bendixon theorem, a limit cycle exists in the region.

Example 4.09.2 (continued)

Example 4.09.2 - Limit Cycle



Checking that $x^2 + y^2 = 1$ is a solution to the non-linear equation:

$$\frac{dx}{dt} = x\left(1 - x^2 - y^2\right) + 5y = 5y \quad \text{and} \quad \frac{dy}{dt} = -5x + y\left(1 - x^2 - y^2\right) = -5x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{x}{y}$$

But $x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2y\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

Therefore the limit cycle $x^2 + y^2 = 1$ is a solution to the non-linear system **(1)**.

4.10 Lyapunov Functions [for reference only - ***not*** examinable]

The equation of motion for an unforced damped elastic mass-spring system is

$$\frac{d^2x}{dt^2} + \varepsilon \frac{dx}{dt} + \mu x = 0 \qquad \mathbf{(1)}$$

Consider the case where the restoring force (per unit mass) coefficient $\mu = 1$ and the damping (per unit mass) coefficient ε is small and positive. The equivalent first order system is

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x - \varepsilon y \end{aligned} \qquad \mathbf{(2)}$$

The coefficient matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\varepsilon \end{pmatrix}$$

Using the results on page 4.30,

$D = (a-d)^2 + 4bc = (0+\varepsilon)^2 + 4(-1)(1) = \varepsilon^2 - 4 < 0$

$\lambda = \dfrac{(a+d) \pm \sqrt{D}}{2} = \dfrac{-\varepsilon \pm j\sqrt{4 - \varepsilon^2}}{2}$

[or solve the characteristic equation $\det(A - \lambda I) = 0$:

$(0-\lambda)(-\varepsilon-\lambda) - (1)(-1) = 0 \quad \Rightarrow \quad \lambda^2 - \varepsilon\lambda + 1 = 0$.]

The single critical point at the origin is therefore a **stable focus** (asymptotically stable).

The kinetic energy is $\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}my^2$.
The potential energy of a mass-spring system is proportional to the square of the extension x. Therefore the function $V(x,y) = \frac{1}{2}(x^2+y^2)$ is related to the total energy of the system. $V(x, y)$ has an absolute minimum value of 0 at the origin, which should therefore be a stable equilibrium point.

From the chain rule and **(2)**,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = x \cdot y + y(-x - \varepsilon y) = -\varepsilon y^2 \le 0 \quad \forall t$$

Therefore V decreases as t increases.
Also V decreases as the distance from the origin decreases.
Therefore the distance from the origin must decrease as t increases.
$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$. All orbits terminate at the origin.
Again, the origin is an asymptotically stable point.

Energy considerations and Stability:

For a system of differential equations that arises from the description of a physical system, if the total energy of the system is constant or decreasing and a critical point corresponds to a point of minimum potential energy of the system, then the critical point should be stable.

If the critical point corresponds to a maximum of potential energy (such as the upside-down position of the pendulum in section 4.01), then the critical point should be unstable.

If (0, 0) is an asymptotically stable critical point of the system

$$\frac{dx}{dt} = f(x,y) \quad , \quad \frac{dy}{dt} = g(x,y) \qquad \textbf{(3)}$$

then there must exist some domain D, containing (0, 0), such that all solutions in D must approach (0, 0) as $t \to \infty$.

Suppose that an energy function $V(x, y)$ exists such that $V(0, 0) = 0$ and $V(x, y) > 0$ everywhere else in D. Then, following any open orbit in D, V must decrease to zero as $t \to \infty$. The converse of these statements is more useful:
If V decreases to zero as $t \to \infty$ on every trajectory in D, then every trajectory in D must approach the origin as $t \to \infty$ and the origin is therefore asymptotically stable.

Definitions:

Let $V(x, y)$ be defined on some domain D that contains the origin.

V is **positive definite** on D if $V(0, 0) = 0$ and $V(x, y) > 0$ for all other points in D.
V is **negative definite** on D if $V(0, 0) = 0$ and $V(x, y) < 0$ for all other points in D.

V is **positive semi-definite** on D if $V(0, 0) = 0$ and $V(x, y) \geq 0$ for all other points in D.
V is **negative semi-definite** on D if $V(0, 0) = 0$ and $V(x, y) \leq 0$ for all other points in D.

A function $V(x, y)$ is a **Lyapunov function** for the system

$$\frac{dx}{dt} = f(x,y) \quad , \quad \frac{dy}{dt} = g(x,y) \qquad \textbf{(3)}$$

if there exists some neighbourhood of the origin in which

- V is a differentiable function of x and y;
- $V > 0$ except at the origin, where $V = 0$; and
- For any solution $(x(t), y(t))$ of **(3)** there exists a t_o such that $\frac{dV}{dt} \leq 0$ for all $t \geq t_o$.

Theorem:

If $V(x, y)$ is a Lyapunov function for the system **(3)**, then:

If $\frac{dV}{dt}$ is negative semidefinite, then $(0, 0)$ is stable.
If $\frac{dV}{dt}$ is negative definite, then $(0, 0)$ is asymptotically stable.
If $\frac{dV}{dt}$ is positive definite, then $(0, 0)$ is unstable.

Also note that, by the chain rule and **(3)**,

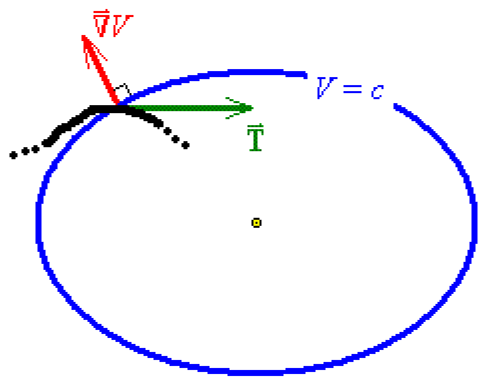
$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \frac{\partial V}{\partial x}\cdot f(x,y) + \frac{\partial V}{\partial y}\cdot g(x,y)$$

and that

$$\frac{dV}{dt} = \vec{\nabla} V \bullet \vec{\mathbf{T}}$$

where $\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}}$ is the gradient vector of the scalar function $V(x, y)$ and

$\vec{\mathbf{T}} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} = f(x,y)\hat{\mathbf{i}} + g(x,y)\hat{\mathbf{j}}$ is the **tangent vector** to the trajectory $(x(t), y(t))$.



If $\frac{dV}{dt}$ is negative definite, then the two vectors must point in directions more than 90° apart, everywhere in the region (except possibly at the origin). But the gradient vector points in the direction of increasing V, at right angles to the contours $V =$ constant.

V is positive definite, so its gradient vector points outward, away from the origin.

Therefore the trajectories must point inward, everywhere in the region where V is positive definite and $\frac{dV}{dt}$ is negative definite.

The general quadratic function

$$V(x,y) = ax^2 + bxy + cy^2$$

is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$

and

is negative definite if and only if $a < 0$ and $b^2 - 4ac < 0$

Example 4.10.1

The populations of a pair of competing species are modelled by the system

$$\frac{dx}{dt} = x(1 - x - y)$$
$$\frac{dy}{dt} = y(0.75 - y - 0.5x)$$

Investigate the stability of the critical point at (0.5, 0.5).

Transform the critical point to the origin with the change of coordinates

$$w = x - 0.5 \; ; \qquad z = y - 0.5$$

The system becomes

$$\frac{dw}{dt} = (w+0.5)\big(1 - (w+0.5) - (z+0.5)\big) = -0.5w - 0.5z - w^2 - wz$$
$$\frac{dz}{dt} = (z+0.5)\big(0.75 - (z+0.5) - 0.5(w+0.5)\big) = -0.25w - 0.5z - 0.5wz - z^2$$

There are many possible choices for a Lyapunov function, among the simplest of which is

$$V(w, z) = w^2 + z^2$$

V is clearly positive definite: $V(0, 0) = 0$ and $V(w, z) > 0$ everywhere else.

$$\frac{dV}{dt} = \frac{\partial V}{\partial w}\cdot\frac{dw}{dt} + \frac{\partial V}{\partial z}\cdot\frac{dz}{dt}$$
$$= 2w\left(-0.5w - 0.5z - w^2 - wz\right) + 2z\left(-0.25w - 0.5z - 0.5wz - z^2\right)$$
$$= -\left(w^2 + 1.5wz + z^2\right) - \left(2w^3 + 2w^2z + wz^2 + 2z^3\right)$$

In the quadratic expression $-\left(w^2 + 1.5wz + z^2\right)$, $a = c = -1$ and $b = -1.5$.

$a < 0$ and $b^2 - 4ac < 0$, so that $-\left(w^2 + 1.5wz + z^2\right)$ is negative definite.

The cubic terms can be of either sign, but sufficiently close to $(w, z) = (0, 0)$ they will be negligible compared to the quadratic terms. Therefore a region does exist around (0, 0) such that V is positive definite and $\frac{dV}{dt}$ is negative definite. The critical point must therefore be asymptotically stable.

By using a more complicated Lyapunov function and obtaining bounds on where its derivative is negative definite, one can estimate how far the region of asymptotic stability extends around the critical point.

Example 4.10.1 (continued)

Note that we can also investigate stability by finding the eigenvalues of the linear system that approximates the non-linear system near the critical point:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix} \begin{pmatrix} x-0.5 \\ y-0.5 \end{pmatrix}$$

The characteristic equation is

$\det(A - \lambda I) = 0 \quad \Rightarrow \quad (-0.5-\lambda)^2 - (-0.5)(-0.25) = 0$

$\Rightarrow \quad (\lambda+0.5)^2 = 0.125 \quad \Rightarrow \quad \lambda + 0.5 = \pm\sqrt{0.125} \quad \Rightarrow \quad \lambda = -0.5 \pm \sqrt{0.125}$

which is a real distinct negative pair.

The critical point is therefore an asymptotically stable node.

4.11 Duffing's Equation

Among the simplest models of damped non-linear forced oscillations of a mechanical or electrical system with a cubic stiffness term is Duffing's equation:

$$\frac{d^2 x}{dt^2} + a\frac{dx}{dt} + bx + cx^3 = d\cos\omega t \qquad \textbf{(1)}$$

In section 4.01, we considered the simple undamped pendulum:

$$\frac{d^2 x}{dt^2} + \frac{g}{L}\sin x = 0 \qquad \textbf{(2)}$$

When x is very small, $\sin x \approx x$ and **(2)** reduces to the ODE for simple harmonic motion.

The next order approximation is $\sin x \approx x - \frac{x^3}{6}$, so that **(2)** becomes

$$\frac{d^2 x}{dt^2} + \frac{g}{L}x - \frac{g}{L}\frac{x^3}{6} = 0 \qquad \textbf{(3)}$$

If we add a damping term $a\frac{dx}{dt}$ and a forcing function $d\cos\omega t$, then **(3)** becomes Duffing's equation **(1)**.

Special Case 1:

Conduct a stability analysis for the undamped unforced Duffing's equation

$$\frac{d^2x}{dt^2} + \omega^2 x + c\,x^3 = 0 \qquad \textbf{(4)}$$

The equivalent first order system is

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\omega^2 x - c\,x^3\end{aligned} \qquad \textbf{(5)}$$

Critical points:

$(y = 0)$ and $\left(x = 0 \quad \text{or} \quad x^2 = -\dfrac{\omega^2}{c}\right)$

Near (0, 0) the linear approximation is

$$\begin{pmatrix}\dot{x} \\ \dot{y}\end{pmatrix} = \begin{pmatrix}0 & 1 \\ -\omega^2 & 0\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} \qquad \textbf{(6)}$$

The characteristic equation is $\det(A - \lambda I) = 0 \;\Rightarrow\; \lambda^2 + \omega^2 = 0$
The eigenvalues are a pure imaginary pair
$\Rightarrow$ (0, 0) is a **centre**. It is stable but not asymptotically stable.
If $c \geq 0$, then this is the only critical point of **(4)**.

If $c < 0$, then there are two other critical points, at $\left(\pm\sqrt{\dfrac{\omega^2}{-c}},\, 0\right)$.

Special Case 1: (continued)

Near $\left(\pm\sqrt{\dfrac{\omega^2}{-c}}, 0 \right)$

$$\left.\frac{dx}{dt}\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} = \left.(y)\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} \approx \left.\frac{\partial P}{\partial x}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} \left(x \mp \sqrt{\frac{\omega^2}{-c}} \right) + \left.\frac{\partial P}{\partial y}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} y$$

$$= (0)\left(x \mp \sqrt{\frac{\omega^2}{-c}} \right) + (1)\,y = y$$

$$\left.\frac{dy}{dt}\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} = \left.\left(-\omega^2 x - c x^3\right)\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} \approx \left.\frac{\partial Q}{\partial x}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} \left(x \mp \sqrt{\frac{\omega^2}{-c}} \right) + \left.\frac{\partial Q}{\partial y}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} y$$

$$= \left(-\omega^2 - 3c\left(\frac{\omega^2}{-c}\right) \right)\left(x \mp \sqrt{\frac{\omega^2}{-c}} \right) + (0)\,y = 2\omega^2 \left(x \mp \sqrt{\frac{\omega^2}{-c}} \right)$$

The linear approximation to **(5)** near $\left(\pm\sqrt{\dfrac{\omega^2}{-c}}, 0 \right)$ is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x \mp \sqrt{\dfrac{\omega^2}{-c}} \\ y \end{pmatrix} \qquad \textbf{(7)}$$

The characteristic equation is $\det(A - \lambda I) = 0 \quad \Rightarrow \quad \lambda^2 - 2\omega^2 = 0$
The eigenvalues are a distinct real pair with opposite sign

$\Rightarrow \quad \left(\pm\sqrt{\dfrac{\omega^2}{-c}}, 0 \right)$ are **saddle points**. They are unstable.

Exact Solution of Special Case 1:

The system **(5),**

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -\omega^2 x - c\,x^3$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{-\omega^2 x - c\,x^3}{y}$$

$$\Rightarrow \quad y\,dy = \left(-\omega^2 x - c\,x^3\right) dx \qquad \Rightarrow \quad \frac{y^2}{2} = \frac{A}{2} - \frac{\omega^2 x^2}{2} - \frac{c\,x^4}{4}$$

Therefore the orbits in the phase space (x, y) are $y^2 = A - \omega^2 x^2 - \dfrac{c\,x^4}{2}$,

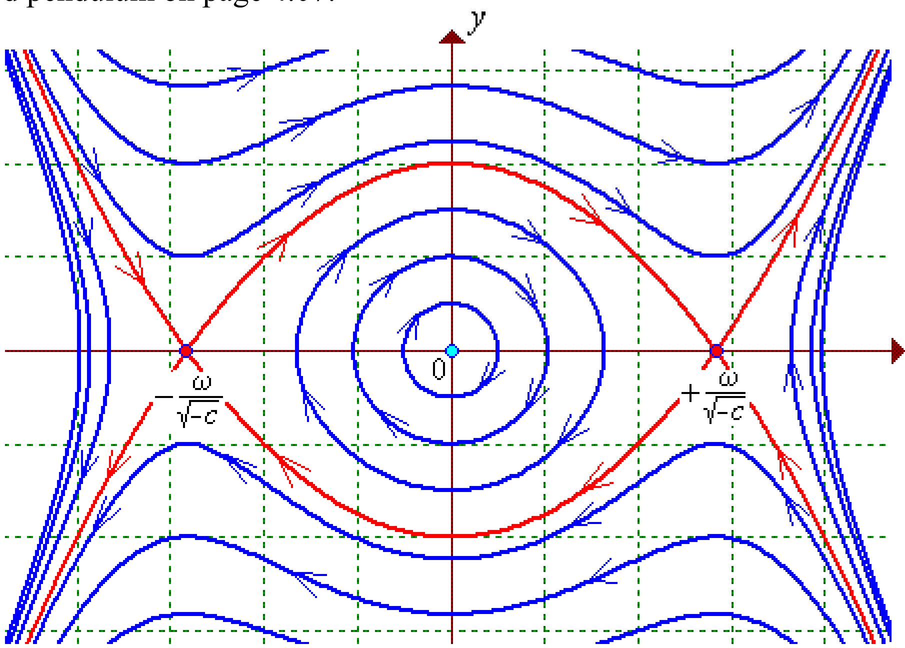
where A is an arbitrary constant.

If $c \geq 0$, then all orbits are closed, about the centre at (0, 0).

Special Case 1: (continued)

If $c < 0$, then those orbits far enough away from the centre are open, due to the influence of the saddle points at $\left(\pm\sqrt{\dfrac{\omega^2}{-c}}, 0 \right)$.

The part of the phase space between the two saddle points resembles that for the undamped pendulum on page 4.07:



The orbits passing through the saddle points separates closed orbits from open orbits and is called the **separatrix**.

The positive y axis intercept of each orbit is just the value of $\sqrt{A}$ for that orbit.
The separatrix has x axis intercepts at the saddle points. Therefore, for the separatrix,

$$A = \left(\omega^2 x^2 + \frac{cx^4}{2} \right)\Bigg|_{x = \pm\omega/\sqrt{-c}} = \omega^2 \left(\frac{\omega^2}{-c} \right) + \frac{c}{2}\left(\frac{\omega^4}{c^2} \right) = \frac{\omega^4}{-2c}$$

The equation of the separatrix is

$$y^2 = \frac{\omega^4}{2|c|} - \omega^2 x^2 + \frac{|c| x^4}{2}, \quad (c < 0)$$

Special Case 2:

Conduct a stability analysis for the damped unforced Duffing's equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + \omega^2 x + c\,x^3 = 0 \qquad \textbf{(8)}$$

The equivalent first order system is

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -\omega^2 x - c\,x^3 - a\,y \end{aligned} \qquad \textbf{(9)}$$

The critical points are the same as in special case 1:

$(y=0)$ and $\left(x=0 \quad \text{or} \quad x^2 = -\dfrac{\omega^2}{c}\right)$

Near (0, 0) the linear approximation is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \textbf{(10)}$$

The characteristic equation is $\det(A - \lambda I) = 0 \quad \Rightarrow \quad \lambda^2 + a\lambda + \omega^2 = 0$

$$\Rightarrow \quad \lambda = \frac{-a \pm \sqrt{a^2 - 4\omega^2}}{2}$$

The critical point is stable if $a > 0$ and unstable if $a < 0$.
It is a focus if $a^2 - 4\omega^2 < 0$ and a node otherwise.

If $c \geq 0$, then this is the only critical point of **(8)**.

If $c < 0$, then there are two other critical points, at $\left(\pm\sqrt{\dfrac{\omega^2}{-c}}, 0\right)$.

Special Case 2: (continued)

Near $\left(\pm\sqrt{\frac{\omega^2}{-c}},\, 0 \right)$

$$\left.\frac{dx}{dt}\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} = \left.(y)\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} \approx \left.\frac{\partial P}{\partial x}\right|_{\left(\pm\omega/\sqrt{-c},0\right)}\left(x \mp \frac{\omega}{\sqrt{-c}} \right) + \left.\frac{\partial P}{\partial y}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} y$$

$$= (0)\left(x \mp \frac{\omega}{\sqrt{-c}} \right) + (1)\,y = y$$

$$\left.\frac{dy}{dt}\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} = \left.\left(-\omega^2 x - cx^3 - ay\right)\right|_{\text{near}\left(\pm\omega/\sqrt{-c},0\right)} \approx \left.\frac{\partial Q}{\partial x}\right|_{\left(\pm\omega/\sqrt{-c},0\right)}\left(x \mp \frac{\omega}{\sqrt{-c}} \right) + \left.\frac{\partial Q}{\partial y}\right|_{\left(\pm\omega/\sqrt{-c},0\right)} y$$

$$= \left(-\omega^2 - 3c\left(\frac{\omega^2}{-c}\right) \right)\left(x \mp \frac{\omega}{\sqrt{-c}} \right) + (-a)\,y = 2\omega^2\left(x \mp \frac{\omega}{\sqrt{-c}} \right) - a\,y$$

The linear approximation to **(9)** near $\left(\pm\sqrt{\frac{\omega^2}{-c}},\, 0 \right)$ is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2\omega^2 & -a \end{pmatrix} \begin{pmatrix} x \mp \dfrac{\omega}{\sqrt{-c}} \\ y \end{pmatrix} \qquad \textbf{(11)}$$

The characteristic equation is $\det(A - \lambda I) = 0 \;\Rightarrow\; \lambda^2 + a\lambda - 2\omega^2 = 0$

$$\Rightarrow \quad \lambda = \frac{-a \pm \sqrt{a^2 + 8\omega^2}}{2}$$

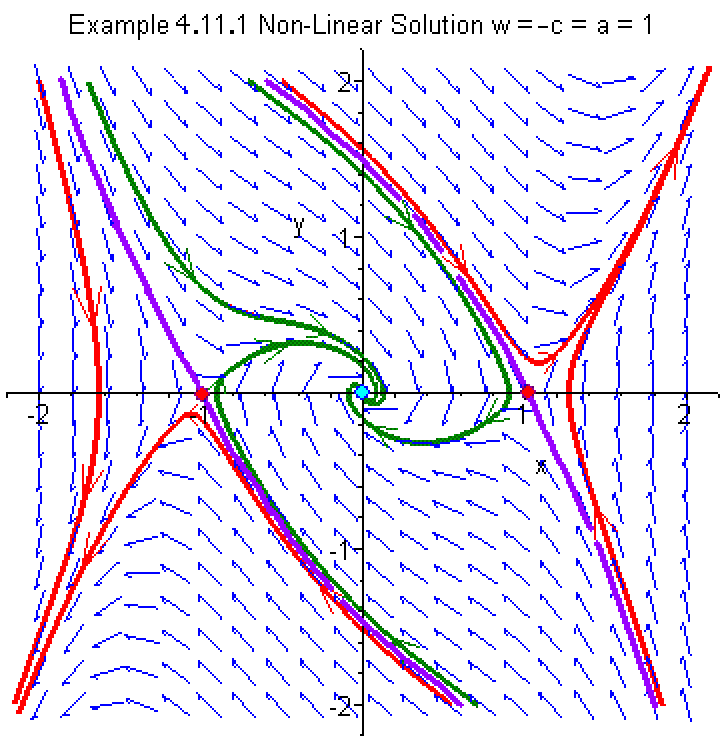
The eigenvalues are a distinct real pair with opposite sign

$\Rightarrow \left(\pm\sqrt{\frac{\omega^2}{-c}},\, 0 \right)$ are **saddle points**. They are unstable.

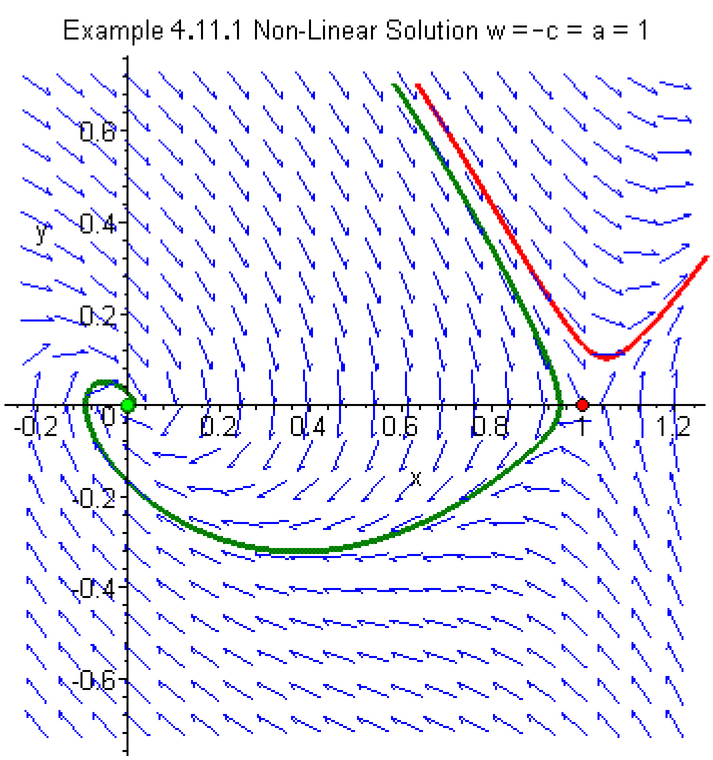
The presence of the damping term changes the centre into a stable focus (for physically reasonable values of a, ω and c, or, for particularly strong damping, a stable node). The form of the separatrix is more complicated, as trajectories leaving either saddle point in the direction of the origin are swept by the damping term into the focus (or node) instead of moving around the centre to the other saddle point. There are no closed orbits; just orbits that terminate at the origin or a saddle point and orbits that retreat to infinity.

Special Case 2: (continued)

An enhanced sample phase portrait plot from Maple is shown here:



and, zooming in,



4.12 More Examples

Example 4.12.1

Examine the stability of the linear second order differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \left(4\pi^2+1\right)x = 0$$

and find the complete solution for the initial conditions
$x(0) = 0\,, \quad y(0) = \dot{x}(0) = 2\pi\,.$

The system can be rewritten as the first order system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\left(4\pi^2+1\right) & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

The only critical point is at the origin.

$$D = (a-d)^2 + 4bc = (0+2)^2 + 4(1)\left(-\left(4\pi^2+1\right)\right) = -16\pi^2$$

$D < 0$ and $(a + d) < 0$ $\Rightarrow$ the critical point is an **asymptotically stable focus**.

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-2 \pm \sqrt{-16\pi^2}}{2} = -1 \pm 2\pi j$$

Using the formula on page 4.30, $u = -1,\ v = 2\pi\,,\ u - d = -1 + 2 = 1\,,\ c = -\left(4\pi^2+1\right)$.
The general solution is

$$x(t) = e^{-t}\left(c_3\left(\cos 2\pi t - 2\pi \sin 2\pi t\right) + c_4\left(2\pi \cos 2\pi t + \sin 2\pi t\right)\right)$$

$$y(t) = -e^{-t}\left(4\pi^2+1\right)\left(c_3 \cos 2\pi t + c_4 \sin 2\pi t\right)$$

[and one can check that $\frac{dx}{dt} = y$ is indeed true.]

$$\left(x(0), y(0)\right) = (0, 2\pi) \quad \Rightarrow \quad \left(c_3 + 2\pi c_4,\ -c_3\right) = \left(0, \frac{2\pi}{4\pi^2+1}\right)$$

$$\Rightarrow \quad \left(c_3,\ c_4\right) = \left(\frac{-2\pi}{4\pi^2+1}, \frac{1}{4\pi^2+1}\right)$$

Example 4.12.1 (continued)

The complete solution is

$$\big(x(t), y(t)\big) = \frac{e^{-t}}{4\pi^2+1} \times$$

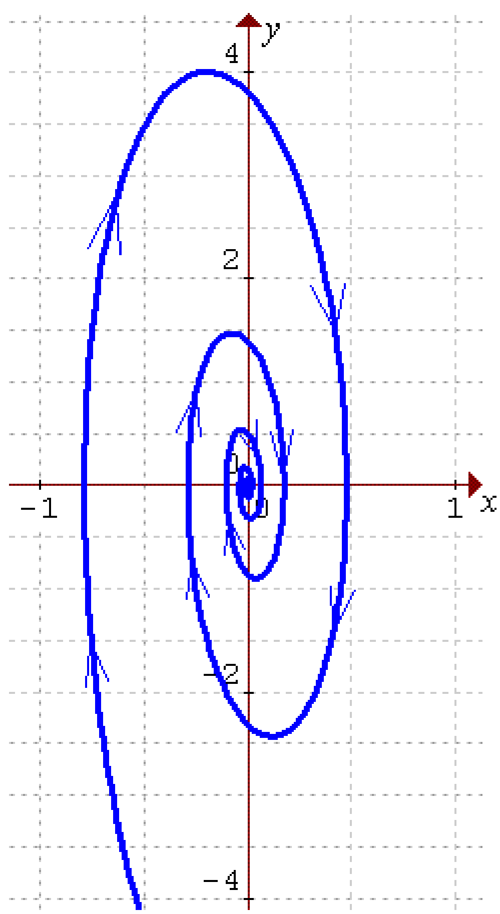
$$\Big(2\pi\big(2\pi \sin 2\pi t - \cos 2\pi t\big) + \big(2\pi \cos 2\pi t + \sin 2\pi t\big),\ \big(4\pi^2+1\big)\big(2\pi \cos 2\pi t\ -\ \sin 2\pi t\big)\Big)$$

$\Rightarrow$

$$\boxed{\big(x(t), y(t)\big) = e^{-t}\big(\sin 2\pi t,\ \big(2\pi \cos 2\pi t\ -\ \sin 2\pi t\big)\big)}$$

As $t \to \infty$, both functions $x(t)$ and $y(t)$ tend to zero.

The resulting phase space diagram is



Example 4.12.2

Examine the stability of the linear second order differential equation

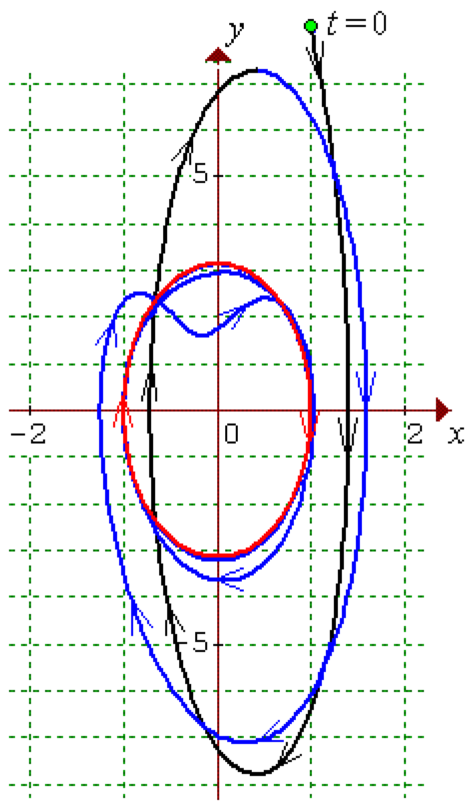
$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + \left(4\pi^2+1\right)x = \left(3\pi^2+1\right)\cos\pi t - 2\pi\sin\pi t$$

The complete solution for the initial conditions
$x(0) = 1, \quad y(0) = \dot{x}(0) = 2\pi$.
can be obtained by building upon the solution to Example 4.12.1 and is

$$\boxed{\left(x(t), y(t)\right) = \left(\cos\pi t + e^{-t}\sin 2\pi t,\ -\pi\sin\pi t + e^{-t}\left(2\pi\cos 2\pi t - \sin 2\pi t\right)\right)}$$

In this case, the steady state solution (after the transient terms have vanished) is
$\lim\limits_{t\to\infty}\left(x(t), y(t)\right) = \left(\cos\pi t,\ -\pi\sin\pi t\right)$
so that the orbit in the phase space approaches the ellipse

$$\pi^2x^2 + y^2 = \pi^2$$



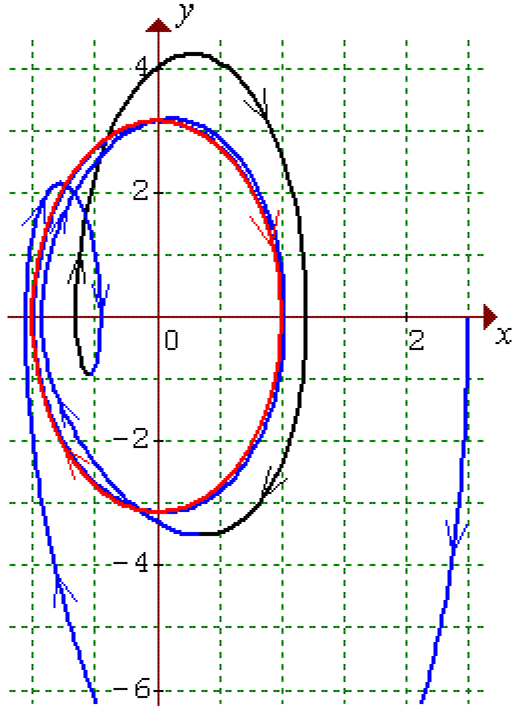
This ellipse must therefore be the limit cycle for the system.

A plot of the orbit in the phase space is shown here.

Note how the solution curve in the phase space can wander inside and outside the limit cycle more than once, before finally settling down to its asymptotic approach as the transient terms become negligible.

Example 4.12.2 (continued)

Different sets of initial conditions can generate orbits that look very different at first, before they settle down into their steady-state configuration near the limit cycle.



4.13 Liénard's Theorem

If $f(x)$ is an even function for all x
and $g(x)$ is an odd function for all x
and $g(x) > 0$ for all $x > 0$
and $F(x) = \int_0^x f(t)\,dt$ is such that $F(x) = 0$ has exactly one positive root, γ, and

$F(x) < 0$ for $0 < x < \gamma$ and $F(x) > 0$ and non-decreasing for $x > \gamma$,

then

the system

$$\dot{x} = y \ , \quad \dot{y} = -f(x)\,y - g(x)$$

or, equivalently,

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$$

has a unique limit cycle enclosing the origin and that limit cycle is asymptotically stable.

When all of the conditions of Liénard's theorem are satisfied, the system has exactly one periodic solution, towards which all other trajectories spiral as $t \to \infty$.

Example 4.13.1

Let $f(x) = -\mu(1 - x^2)$ and $g(x) = x$, (with $\mu > 0$),
then Liénard's ODE becomes

$$\frac{d^2x}{dt^2} - \mu\left(1-x^2\right)\frac{dx}{dt} + x = 0$$

which is Van der Pol's equation (section 4.08).

Checking the conditions of Liénard's theorem:
$f(x) = -\mu(1 - x^2)$ is an even function.
$g(x) = x$ is an odd function, positive for all $x > 0$.

$$F(x) = \int_0^x -\mu\left(1-t^2\right)dt = -\mu\left[\left(t-\frac{t^3}{3}\right)\right]_0^x = +\mu\left(\frac{x^3}{3}-x\right) = \mu x\left(\frac{x^2}{3}-1\right)$$

$F(x) = 0$ has only one positive root, $\gamma = \sqrt{3}$.
$F(x) < 0$ for $0 < x < \sqrt{3}$ and $F(x) > 0$ and increasing for $x > \sqrt{3}$.

Therefore Van der Pol's equation possesses a unique and asymptotically stable limit cycle.