6. Calculus of Variations

The method of calculus of variations involves finding the path between two points that provides the minimum (or maximum) value of integrals of the form

$$\int_a^b F(x, y, y')\, dx$$

Sections in this Chapter:

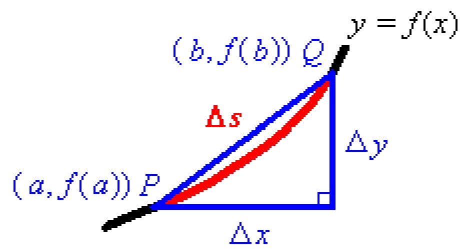
Sections for reference; **not** examinable:

6.01 Introduction

Example 6.01.1

To find the shortest path, (the **geodesic**), between two points, we need to find an expression for the arc length along a path between the two points.

Consider a pair of nearby points.
The element of arc length Δs is approximately the hypotenuse of the triangle.



$$(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2$$

$$\Rightarrow \quad \frac{(\Delta s)^2}{(\Delta x)^2} \approx \frac{(\Delta x)^2}{(\Delta x)^2} + \frac{(\Delta y)^2}{(\Delta x)^2}$$

In the limit as the two points approach each other and $\Delta x \to 0$, we obtain

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad \left(\frac{ds}{dx}\right) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

The arc length s between any two points $x = a$ and $x = b$ along any path C in $\mathbb{R}^2$ is the line integral

$$s = \int_C ds = \int_C \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_C \sqrt{1 + \left(f'(x)\right)^2}\, dx \quad \text{where } C \text{ is the path } y = f(x)$$

The geodesic will be the path C for which the line integral for s attains its minimum value. Of course, in a flat space such as $\mathbb{R}^2$, that geodesic is just the straight line between the two points.

6.02 Theory

We wish to find the curve $y(x)$ which passes through the points (x_0, y_0) and (x_1, y_1) and which minimizes the integral

$$I = \int_{x_0}^{x_1} F\left(x, y(x), y'(x)\right) dx$$

Consider the one parameter family of curves $y(x) = u(x) + \alpha\,\eta(x)$, where α is a real parameter, $\eta(x)$ is an arbitrary function except for the requirement $\eta(x_0) = \eta(x_1) = 0$ and $u(x)$ represents the (as yet unknown) solution.

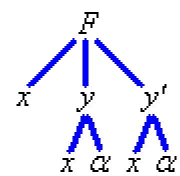
Every member of this family of curves passes through the points (x_0, y_0) and (x_1, y_1).
For any member of the family,

$$I(\alpha) = \int_{x_0}^{x_1} F\left(x, u(x) + \alpha\,\eta(x), u'(x) + \alpha\,\eta'(x)\right) dx$$

we know that $y(x) = u(x)$ minimizes I.

Therefore the minimum for I occurs when $\alpha = 0$, so that $\left.\dfrac{dI}{d\alpha}\right|_{\alpha=0} = 0$.

Carrying out a Leibnitz differentiation of the integral $I(\alpha)$,



$$\frac{dI}{d\alpha} = 0 - 0 + \int_{x_0}^{x_1} \frac{\partial}{\partial\alpha} F\left(x, u(x) + \alpha\,\eta(x), u'(x) + \alpha\,\eta'(x)\right) dx$$

$$= \int_{x_0}^{x_1} \left[0 + \frac{\partial F}{\partial y}\frac{\partial}{\partial\alpha}\left(u(x) + \alpha\,\eta(x)\right) + \frac{\partial F}{\partial y'}\frac{\partial}{\partial\alpha}\left(u'(x) + \alpha\,\eta'(x)\right) \right] dx$$

At the minimum $\alpha = 0$, so that $y(x) = u(x)$ and $y'(x) = u'(x)$. Therefore

$$0 = \int_{x_0}^{x_1} \left[\eta(x)\frac{\partial F}{\partial u} + \eta'(x)\frac{\partial F}{\partial u'} \right] dx$$

Also note, by the product rule of differentiation, that

$$\frac{d}{dx}\left(\eta(x)\frac{\partial F}{\partial u'}\right) = \eta'(x)\frac{\partial F}{\partial u'} + \eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial u}\right)$$

Therefore the integral can be written as

$$0 = \int_{x_0}^{x_1} \left[\eta(x)\frac{\partial F}{\partial u} + \frac{d}{dx}\left(\eta(x)\frac{\partial F}{\partial u'}\right) - \eta(x)\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) \right] dx$$

$$0 = \int_{x_0}^{x_1} \eta(x)\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) \right] dx + \int_{x_0}^{x_1} \frac{d}{dx}\left(\eta(x)\frac{\partial F}{\partial u'}\right) dx$$

$$0 = \int_{x_0}^{x_1} \eta(x)\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) \right] dx + \left[\eta(x)\frac{\partial F}{\partial u'} \right]_{x_0}^{x_1}$$

But $\eta(x_0) = \eta(x_1) = 0$

Therefore the minimizing curve $u(x)$ satisfies

$$\int_{x_0}^{x_1} \eta(x)\left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right]dx = 0$$

But $\eta(x)$ is an arbitrary function of x, which leads to

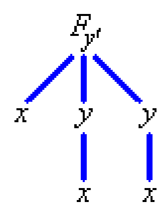
$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$$

Thus, if $y = f(x)$ is a path that minimizes the integral $\int_a^b F(x, y, y')\,dx$, then $y = f(x)$ and $F(x, y, y')$ must satisfy the **Euler equation for extremals**

$$\boxed{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0}$$

Euler's equation requires the assumption that $F(x, y, y')$ has continuous second derivatives in all three of its variables and that all members of the family $y(x) = u(x) + \alpha\,\eta(x)$ have continuous second derivatives.

Expansion of Euler's Equation:



$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}(x, y(x), y'(x))\right) - \frac{\partial F}{\partial y}(x, y(x), y'(x)) = 0$$

$$\Rightarrow \quad \frac{\partial^2 F}{\partial x\,\partial y'} + y'(x)\frac{\partial^2 F}{\partial y\,\partial y'} + y''(x)\frac{\partial^2 F}{\partial y'^2} - \frac{\partial F}{\partial y} = 0$$

or

$$\boxed{y'' F_{y'y'} + y' F_{yy'} + \left(F_{xy'} - F_y\right) = 0}$$

Note: Leibnitz differentiation of $I(z) = \int_{f(z)}^{g(z)} F(x, z)\,dx$ with respect to z is:

$$\frac{dI}{dz} = g'(z)F(g(z), z) - f'(z)F(f(z), z) + \int_{f(z)}^{g(z)} \frac{\partial}{\partial z}F(x, z)\,dx$$

A special case of this is $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$.

6.03 Examples

Example 6.03.1

(a) Find extremals $y(x)$ for $I = \displaystyle\int_{x_0}^{x_1} \frac{(y')^2}{x^3}\,dx$.

(b) Find the extremal that passes through the points (0, 1) and (1, 4).

(c) Prove that the extremal in part (b) minimizes the integral I.

(a) $F = \dfrac{(y')^2}{x^3} \quad \Rightarrow \quad \dfrac{\partial F}{\partial y} = 0$

Euler's equation simplifies to

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}\frac{(y')^2}{x^3}\right) = 0 \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 0$$

$$\Rightarrow \quad \frac{2y'}{x^3} = c_1 \quad \Rightarrow \quad y' = \tfrac{1}{2}c_1 x^3 \quad \Rightarrow \quad y = \tfrac{1}{8}c_1 x^4 + c_2$$

Redefining the arbitrary constants, this leads to the two-parameter family of extremals

$$y(x) = A\,x^4 + B$$

(b) The curve $y(x) = A\,x^4 + B$ must pass through both (0, 1) and (1, 4).

$1 = 0 + B \quad \Rightarrow \quad B = 1$

$4 = A(1)^4 + 1 \quad \Rightarrow \quad A = 3$

Therefore the extremal through (0, 1) and (1, 4) is $\Gamma_o : \quad y = 3x^4 + 1$.

Example 6.03.1 (continued)

(c) To prove that $y = 3x^4 + 1$ really is the path between (0, 1) and (1, 4) that minimizes the value of the integral $I = \int_{x_0}^{x_1} \frac{(y')^2}{x^3}\,dx$, consider the related family of functions

$\Gamma: \quad y = 3x^4 + 1 + g(x)$, where $g(0) = g(1) = 0$ and $g(x)$ is otherwise arbitrary.

$$I(\Gamma) = \int_0^1 \frac{(y')^2}{x^3}\,dx = \int_0^1 \frac{\left(12x^3 + g'(x)\right)^2}{x^3}\,dx$$

$$= \int_0^1 \frac{\left(12x^3\right)^2 + 24x^3 g'(x) + \left(g'(x)\right)^2}{x^3}\,dx$$

$$= \int_0^1 \frac{\left(12x^3\right)^2}{x^3}\,dx + 24\int_0^1 g'(x)\,dx + \int_0^1 \frac{\left(g'(x)\right)^2}{x^3}\,dx$$

$$\Rightarrow \quad I(\Gamma) = I(\Gamma_0) + 24\left(\cancel{g(1)} - \cancel{g(0)}\right) + \int_0^1 \frac{\left(g'(x)\right)^2}{x^3}\,dx > I(\Gamma_0) \quad \text{for } 0 < x < 1.$$

Note that the integral $\int_0^1 \frac{\left(g'(x)\right)^2}{x^3}\,dx$ is necessarily positive, because $g'(x)$ cannot be identically zero on [0, 1] and the integrand is non-negative on [0, 1].
Also $g(0) = g(1) = 0$. Therefore $I(\Gamma) > I(\Gamma_o)$ and Γ_o minimizes I.

If F is explicitly independent of x and y, so that the integral to be minimized is of the form $I = \int_{x_0}^{x_1} F(y')\,dx$, then Euler's equation simplifies to

$$\frac{d}{dx}\left(F_{y'}\right) - F_y = 0 \quad \Rightarrow \quad \underbrace{F_{y'x}}_{=0} + y'\underbrace{F_{y'y}}_{=0} + y''F_{y'y'} - \underbrace{F_y}_{=0} \equiv 0$$

$$\Rightarrow \quad \boxed{y''F_{y'y'} \equiv 0}$$

If $F_{y'y'} \not\equiv 0$ then $y'' \equiv 0 \quad \Rightarrow \quad \boxed{y(x) = Ax + B}$ (a line).

Example 6.03.2 (Example 6.01.1 revisited)

Show that the geodesic on $\mathbb{R}^2$ between any two points $x = a$ and $x = b$ is the straight line between the two points.

The arc length s between any two points $x = a$ and $x = b$ along any path C in $\mathbb{R}^2$ is the line integral

$$s = \int_C ds = \int_C \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

This integral is of the form $I = \int_{x_0}^{x_1} F(y')\,dx$, where $F(y') = \sqrt{1 + (y')^2}$.

Clearly

$$F_{y'y'} \not\equiv 0 \quad \Rightarrow \quad y''(x) = 0 \;\; \forall x \quad \Rightarrow \quad y(x) = Ax + B$$

which is a straight line.
But the extremal must pass through both points.
Only one straight line can pass through a pair of distinct points on $\mathbb{R}^2$.
Therefore the geodesic is the straight line between the two points.

If F is explicitly independent of y, so that the integral to be minimized is of the form $I = \int_{x_0}^{x_1} F(x, y')\, dx$, then Euler's equation simplifies to

$$\frac{d}{dx}\left(F_{y'}\right) - 0 = 0 \quad \Rightarrow \quad \boxed{F_{y'} = c_1}$$

If F is explicitly independent of x, so that the integral to be minimized is of the form $I = \int_{x_0}^{x_1} F(y, y')\, dx$, then multiply Euler's equation

$$\frac{d}{dx}\left(F_{y'}\right) - F_y = 0$$

by y' to obtain

$$y'\frac{d}{dx}\left(F_{y'}\right) - y'F_y = 0 \quad \Rightarrow \quad \left(\frac{d}{dx}\left(y'F_{y'}\right) - y''F_{y'}\right) - y'F_y = 0$$

But $\dfrac{dF}{dx} = \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial F}{\partial y'}\dfrac{dy'}{dx} = 0 + y'\dfrac{\partial F}{\partial y} + y''\dfrac{\partial F}{\partial y'}$

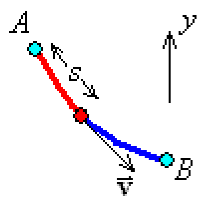
$$\Rightarrow \quad \frac{d}{dx}\left(y'F_{y'}\right) - \frac{dF}{dx} = 0 \quad \Rightarrow \quad \boxed{y'F_{y'} - F = c_1}$$

Example 6.03.3 The Brachistochrone Problem of Bernoulli (1696)

Find the curve $y = f(x)$ such that a particle sliding under gravity but without friction on the curve from the point $A(x_0, y_0)$ to the point $B(x_1, y_1)$ reaches B in the least time.

The sum of kinetic and potential energy of the particle is constant along the curve:

$$E = \tfrac{1}{2}mv^2 + mgy = \text{const.}$$



$$v = \frac{ds}{dt} \quad \Rightarrow \quad E = \tfrac{1}{2}m\left(\frac{ds}{dt}\right)^2 + mgy = \text{const.}$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{\frac{2E}{m} - 2gy} \quad \Rightarrow \quad dt = \frac{ds}{\sqrt{\frac{2E}{m} - 2gy}} = \frac{\sqrt{1 + (y')^2}\, dx}{\sqrt{\frac{2E}{m} - 2gy}}$$

Example 6.03.3 (continued)

Therefore the time taken to slide down the curve $y(x)$ is

$$t\left[y(x)\right] = \int_{x_A}^{x_B} \sqrt{\frac{1 + (y')^2}{\frac{2E}{m} - 2gy}}\,dx$$

If the point A is at the origin, y is measured *downwards* and the particle is released from rest, then the total energy is $E = \frac{1}{2}m(0)^2 + mg(0) = 0$ and the integral for travel time simplifies to

$$t\left[y(x)\right] = \int_0^{x_B} \sqrt{\frac{1 + (y')^2}{2gy}}\,dx$$

The integrand is an explicit function of y and y' only, not x.

When F is explicitly independent of x, $y'F_{y'} - F = c_1$

$$\Rightarrow \quad y'\frac{y'}{\sqrt{2gy}\sqrt{1 + (y')^2}} - \sqrt{\frac{1 + (y')^2}{2gy}} = c_1$$

$$\Rightarrow \quad (y')^2 - \left(1 + (y')^2\right) = c_1\sqrt{2gy\left(1 + (y')^2\right)}$$

$$\Rightarrow \quad 1 = c_1^{\,2}\,2gy\left(1 + (y')^2\right) \qquad \Rightarrow \quad y\left(1 + (y')^2\right) = c_2$$

Use the substitution $y' = \dfrac{dy}{dx} = \tan\phi$. Then

$$y\left(1 + \tan^2\phi\right) = c_2 \quad \Rightarrow \quad y\sec^2\phi = c_2 \quad \Rightarrow \quad y = c_2\cos^2\phi = c_2\frac{(1+\cos 2\phi)}{2}$$

$$dx = \frac{dy}{\tan\phi} = c_2\frac{(2\cos\phi)(-\sin\phi)}{\tan\phi}\,d\phi = -c_2\frac{2\sin\phi\cos\phi}{\left(\frac{\sin\phi}{\cos\phi}\right)}\,d\phi$$

$$\Rightarrow \quad dx = -c_2\left(2\cos^2\phi\right)d\phi = -c_2\left(1+\cos 2\phi\right)d\phi$$

$$\Rightarrow \quad x = -c_2\int(1+\cos 2\phi)\,d\phi = -c_2\left(\phi + \frac{\sin 2\phi}{2}\right) + c_3$$

Therefore the solution can be expressed in parametric form by

$$\left(x(\phi), y(\phi)\right) = \left(c_3 + r(2\phi + \sin 2\phi), -r(1 + \cos 2\phi)\right)$$

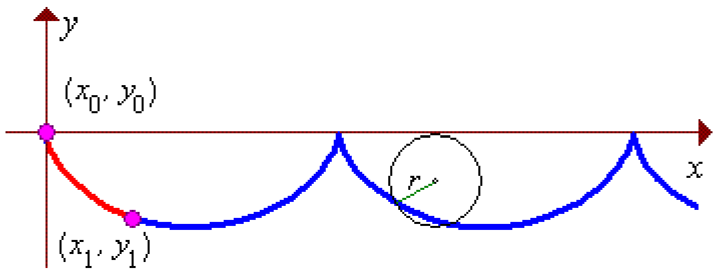
where $r = -\dfrac{c_2}{2}$.

Example 6.03.3 (continued)

Replacing 2ϕ by $\theta + \pi$ and defining $a = c_3 + r\pi$,

$$\boxed{\left(x(\theta), y(\theta)\right) = \left(a + r(\theta - \sin\theta), -r(1 - \cos\theta)\right)}$$

which is the parametric equation of a two-parameter family of **cycloids**.

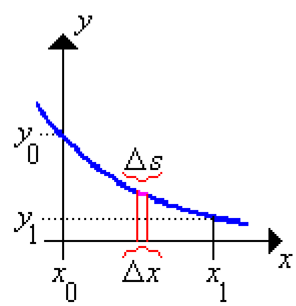


Parameter a shifts the curve horizontally, while r changes the magnitude of the radius of the generating circle. [A cycloid is the path generated by a point on the circumference of a circle that rolls without slipping along an axis. θ is the angle through which the rolling circle of radius r has rotated.]

Example 6.03.4 (The Catenary)

Find the equation $y = f(x)$ of the curve between points $A(x_0, y_0)$ and $B(x_1, y_1)$ which is such that the curved surface of the surface of revolution swept out by the curve around the x-axis has the least possible area.

The element of curved surface area is $2\pi y\,\Delta s$, where Δs is the element of arc length.



The total curved surface area is therefore

$$A = 2\pi \int_{x=x_0}^{x=x_1} y\,ds = 2\pi \int_{x_0}^{x_1} y \frac{ds}{dx}\,dx$$

$$= 2\pi \int_{x_0}^{x_1} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

The integrand is of the form $F(y, y')$, with no explicit dependence on x.

Therefore the extremal is the solution of $y' \frac{\partial F}{\partial y'} - F = c_1$, where

Example 6.03.4 (continued)

$$F(y, y') = y\sqrt{1+(y')^2}$$

$$\Rightarrow \quad y'y\frac{y'}{\sqrt{1+(y')^2}} - y\sqrt{1+(y')^2} = c_1$$

$$\Rightarrow \quad y\left((y')^2 - 1 - (y')^2\right) = c_1\sqrt{1+(y')^2} \qquad \Rightarrow \quad y = -c_1\sqrt{1+(y')^2}$$

$$\Rightarrow \quad y^2 = c_1^{\,2}\left(1+(y')^2\right) \qquad \Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{\frac{y^2}{c_1^{\,2}} - 1}$$

Let $y = c_1 \cosh t$

then $\dfrac{dy}{dx} = \pm\sqrt{\cosh^2 t - 1} = \pm\sqrt{\sinh^2 t} = \pm\sinh t$

But $\dfrac{dy}{dx}\cdot\dfrac{dx}{dt} = \dfrac{dy}{dt} = c_1 \sinh t = \pm c_1 \dfrac{dy}{dx} \quad \Rightarrow \quad \dfrac{dx}{dt} = \pm c_1 \quad \Rightarrow \quad x = \pm c_1 t + c_2$

$$\Rightarrow \quad t = \pm\frac{x}{c_1} - \frac{c_2}{c_1}$$

Let $A = c_1$ and $B = -\dfrac{c_2}{c_1}$ and note that $\cosh(t)$ is an even function.

Then the two-parameter family of extremals is

$$y(x) = A\cosh\left(\frac{x}{A} + B\right)$$

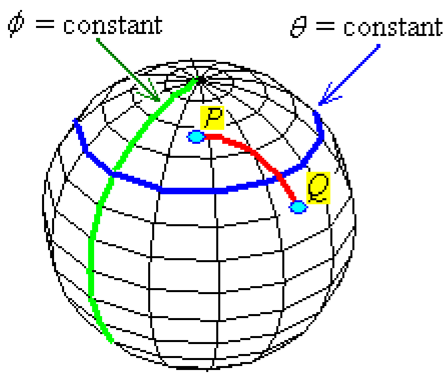
which is the catenary curve.

Example 6.03.5

Find the geodesic (shortest path) between two points P and Q on the surface of a sphere.

Let the radius of the sphere be a and choose the coordinate system such that the origin is at the centre of the sphere. The relationship between the Cartesian coordinates (x, y, z) of any point on the sphere and its spherical polar coordinates (θ, ϕ) is

$$x = a \sin\theta \cos\phi$$
$$y = a \sin\theta \sin\phi$$
$$z = a \cos\theta$$



Note that the radial coordinate r is constant ($r = a$) everywhere on the sphere.
The element of arc length in the spherical polar coordinate system is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

But, on the sphere, $dr = 0$

$$\Rightarrow \quad ds^2 = a^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$\Rightarrow \quad \left(\frac{ds}{d\theta}\right)^2 = a^2\left(1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

The distance along a path on the sphere between points P and Q is therefore

$$s = \int_P^Q ds = \int_P^Q \frac{ds}{d\theta}\, d\theta = \int_P^Q a\sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta$$

The geodesic between P and Q on the surface of the sphere is the function $\phi(\theta)$ that minimizes the integral for s.

For x read θ, for y read ϕ, for y' read $\dfrac{d\phi}{d\theta}$.

The integrand is $F(\theta, \phi, \phi') = a\sqrt{1 + \sin^2\theta\left(\dfrac{d\phi}{d\theta}\right)^2}$.

The integrand is an explicit function of θ and $\dfrac{d\phi}{d\theta}$, but not of ϕ.

Example 6.03.5 (continued)

Therefore the Euler equation for the extremal, $\frac{d}{d\theta}\left(F_{\phi'}\right) - F_{\phi} = 0$, simplifies to

$$\frac{\partial F}{\partial \phi'} = c \qquad \Rightarrow \quad \frac{a \sin^2\theta \left(\frac{d\phi}{d\theta}\right)}{\sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}} = c$$

$$\Rightarrow \quad \left(a \sin^2\theta \left(\frac{d\phi}{d\theta}\right)\right)^2 = c^2\left(1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

$$\Rightarrow \quad \sin^2\theta\left(a^2\sin^2\theta - c^2\right)\left(\frac{d\phi}{d\theta}\right)^2 = c^2 \qquad \Rightarrow \quad \left(\frac{d\phi}{d\theta}\right)^2 = \frac{c^2}{\sin^2\theta\left(a^2\sin^2\theta - c^2\right)}$$

$$\Rightarrow \quad \frac{d\phi}{d\theta} = \frac{c}{\sin\theta\sqrt{a^2\sin^2\theta - c^2}}$$

After substitutions, this can be integrated to a function $\phi(\theta)$, which, upon conversion back into Cartesian coordinates, can be found to lie entirely on a plane through the origin. But the intersection of any plane through the origin with the sphere is just a great circle on the sphere.

Alternatively, reorient the coordinate system (or rotate the sphere) so that one of the two points is at the north pole ($\theta = 0$). Then $\frac{\partial F}{\partial \phi'} = c$ becomes $\frac{\partial F}{\partial \phi'} = 0$ (because $\sin\theta = 0$ at the pole and c must have the same value everywhere on the path).

$$\Rightarrow \quad \frac{a \sin^2\theta \left(\frac{d\phi}{d\theta}\right)}{\sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}} = 0 \qquad \Rightarrow \quad \sin^2\theta\left(\frac{d\phi}{d\theta}\right) = 0$$

$\sin\theta \neq 0$ along the path between the points, so

$$\frac{d\phi}{d\theta} = 0 \qquad \Rightarrow \quad \phi = \text{constant}$$

which, again, is an arc of a great circle (a line of longitude from the north pole to the other point).

Therefore the geodesic between any two points on the sphere is the shorter arc of the great circle that passes through both points.

Example 6.03.6

Find the path $y = f(x)$ between the points (0, 0) and ($\pi/2$, 0) that provides an extremum for the value of the integral

$$I = \int_0^{\pi/2} \left((y')^2 - y^2 - 4y \sin x \right) dx$$

$$F = (y')^2 - y^2 - 4y \sin x$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = \frac{d}{dx}(2y') + 2y + 4 \sin x = 0$$

$$\Rightarrow \quad y'' + y = -2 \sin x$$

This is a second order linear ODE with constant coefficients and a pair of boundary conditions (solution curve passes through (0, 0) and ($\pi/2$, 0)).

A.E.: $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm j$

C.F.: $y_C = A \cos x + B \sin x$

P.S.: Method of undetermined coefficients:
$R(x) = -2 \sin x$, but $\sin x$ is part of the complementary function.

Therefore try $y_P = cx \cos x + dx \sin x \quad \Rightarrow \quad y_P'' = -2c \sin x + 2d \cos x - y_P$

Substitute y_P into the ODE:

$$y_P'' + y_P = -2c \sin x + 2d \cos x = -2 \sin x$$

$$\Rightarrow \quad c = 1, \quad d = 0$$

$$\therefore \quad y_P = x \cos x$$

G.S.: $y = (x + A) \cos x + B \sin x$

Impose the boundary conditions:

$(0,0)$: $0 = A + 0$

$\left(\frac{\pi}{2}, 0\right)$: $0 = 0 + B$

Therefore the complete solution is

$$\boxed{y = f(x) = x \cos x}$$

It can be shown that this sole extremal solution leads to $I = -\frac{\pi}{4} < 0$

The trivial path $y \equiv 0$ leads to $I = 0$

Therefore the extremum must be an absolute minimum.

6.04 Integrals with more than One Dependent Variable
[for reference; **not** examinable]

Let the required curve $y = f(x)$ be defined parametrically, as $x = x(t)$, $y = y(t)$.
Then the integral to be minimized (or maximized) is of the form

$$I = \int_{t_0}^{t_1} F\left(t, x(t), y(t), \dot{x}(t), \dot{y}(t)\right) dt \quad , \quad \text{where} \quad x_0 = x(t_0), x_1 = x(t_1)$$

To derive the necessary conditions on $x(t)$ and $y(t)$, we proceed as before.
Let $\xi(t)$ and $\eta(t)$ be functions that are arbitrary except for the requirements
$\xi(t_0) = \xi(t_1) = \eta(t_0) = \eta(t_1) = 0$
Let $u(t)$ and $v(t)$ be the optimum solutions for $x(t)$ and $y(t)$ respectively.
Then an arbitrary curve can be written as
$x(t) = u(t) + \alpha\,\xi(t)$, $y(t) = v(t) + \alpha\,\eta(t)$.

The integral for which the extremum is required becomes

$$I(\alpha) = \int_{t_0}^{t_1} F\left(t, u(t)+\alpha\,\xi(t), v(t)+\alpha\,\eta(t), \dot{u}(t)+\alpha\,\dot{\xi}(t), \dot{v}(t)+\alpha\,\dot{\eta}(t)\right) dt$$

Performing a Leibnitz differentiation of this integral,

$$\frac{dI}{d\alpha} = \int_{t_0}^{t_1} \frac{\partial F}{\partial \alpha}\, dt = \int_{t_0}^{t_1} \left(0 + \frac{\partial F}{\partial x}\xi(t) + \frac{\partial F}{\partial y}\eta(t) + \frac{\partial F}{\partial \dot{x}}\dot{\xi}(t) + \frac{\partial F}{\partial \dot{y}}\dot{\eta}(t) \right) dt$$

The extremum of I occurs at $\alpha = 0$, where $\dfrac{dI}{d\alpha} = 0$.
Set $\alpha = 0$, $x(t) = u(t)$ and $y(t) = v(t)$, so that

$$0 = \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial u}\xi(t) + \frac{\partial F}{\partial v}\eta(t) + \frac{\partial F}{\partial \dot{u}}\dot{\xi}(t) + \frac{\partial F}{\partial \dot{v}}\dot{\eta}(t) \right) dt$$

However, from the product rule of differentiation,

$$\frac{\partial F}{\partial \dot{u}}\frac{d\xi}{dt} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\xi\right) - \xi\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right) \quad \text{and} \quad \frac{\partial F}{\partial \dot{v}}\frac{d\eta}{dt} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}}\eta\right) - \eta\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}}\right)$$

so that

$$0 = \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial u}\xi + \frac{\partial F}{\partial v}\eta + \frac{d}{dt}\left(\xi\frac{\partial F}{\partial \dot{u}}\right) - \xi\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right) + \frac{d}{dt}\left(\eta\frac{\partial F}{\partial \dot{v}}\right) - \eta\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}}\right) \right) dt$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial u}\xi + \frac{\partial F}{\partial v}\eta - \xi\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right) - \eta\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}}\right) \right) dt + \left[\xi\frac{\partial F}{\partial \dot{u}} + \eta\frac{\partial F}{\partial \dot{v}} \right]_{t_0}^{t_1}$$

But $\xi(t_0) = \xi(t_1) = \eta(t_0) = \eta(t_1) = 0$.

$$\Rightarrow \quad 0 = \int_{t_0}^{t_1} \left(\xi\left(\frac{\partial F}{\partial u} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right) \right) + \eta\left(\frac{\partial F}{\partial v} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}}\right) \right) \right) dt$$

The functions ξ and η are arbitrary and independent of each other, yet the integral must equal zero. The optimal path must therefore satisfy the Euler equations

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right) - \frac{\partial F}{\partial u} = 0 \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{v}}\right) - \frac{\partial F}{\partial v} = 0$$

Any path $x = x(t)$, $y = y(t)$ that minimizes (or maximizes) the integral

$$I = \int_{t_0}^{t_1} F\left(t, x(t), y(t), \dot{x}(t), \dot{y}(t)\right) dt \quad , \quad \text{where} \quad x_0 = x(t_0),\ x_1 = x(t_1)$$

must therefore satisfy

$$\boxed{\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) - \frac{\partial F}{\partial x} = 0 \qquad \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{y}}\right) - \frac{\partial F}{\partial y} = 0}$$

This concept can be extended to problems involving a set of n dependent variables $\{y_i(x)\}$:

With $y_i(x_0)$ and $y_i(x_1)$ $(i = 1, 2, \ldots, n)$ all prescribed, the integral

$$I = \int_{x_0}^{x_1} F\left(x, y_1(x), y_2(x), \ldots, y_n(x), y_1'(x), y_2'(x), \ldots, y_n'(x)\right) dx$$

is minimized (or maximized) only if all members of the set $\{y_i(x)\}$ satisfy the Euler equations

$$\boxed{\frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) - \frac{\partial F}{\partial y_i} = 0}$$

<u>Example 6.04.1</u>

Find the functions $y(x)$ and $z(x)$ that, between the fixed points $(x_0, y(x_0))$ and $(x_1, y(x_1))$ and between the fixed points $(x_0, z(x_0))$ and $(x_1, z(x_1))$ respectively, provide an extreme value for the integral

$$I = \int_{x_0}^{x_1} \left(2yz - 2y^2 + (y')^2 - (z')^2 \right) dx$$

$$F = 2yz - 2y^2 + (y')^2 - (z')^2$$

$$\Rightarrow \quad \frac{\partial F}{\partial y} = 2z - 4y + 0 - 0\,, \qquad \frac{\partial F}{\partial y'} = 0 - 0 + 2y' - 0$$

$$\frac{\partial F}{\partial z} = 2y - 0 + 0 - 0\,, \qquad \frac{\partial F}{\partial z'} = 0 - 0 + 0 - 2z'$$

The Euler equations become

$$\frac{d}{dx}(2y') - (2z - 4y) = 0 \qquad \Rightarrow \quad y'' + 2y = z \qquad \textbf{(1)}$$

and

$$\frac{d}{dx}(-2z') - (2y) = 0 \qquad \Rightarrow \quad y = -z'' \qquad \textbf{(2)}$$

Substituting **(2)** into **(1)**:

$$(-z'')'' + 2(-z'') = z \qquad \Rightarrow \quad z^{(4)} + 2z'' + z = 0$$

The auxiliary equation for this ODE is

$$\lambda^4 + 2\lambda^2 + 1 = 0 \qquad \Rightarrow \quad (\lambda^2 + 1)^2 = 0 \quad \Rightarrow \quad \lambda = \pm j, \pm j$$

The complementary function (which is also the general solution) for z is

$$\boxed{z = (Ax + B)\cos x + (Cx + D)\sin x}$$

Substituting back into **(2)**:

$$y = -z'' = +2A\sin x - 2C\cos x + (Ax + B)\cos x + (Cx + D)\sin x$$

$$\boxed{\Rightarrow \quad y = (Ax + B - 2C)\cos x + (Cx + D + 2A)\sin x}$$

The values of the four arbitrary constants can be determined from the values of the four constants $y(x_0)$, $y(x_1)$, $z(x_0)$ and $z(x_1)$ (although it is likely that that determination will need to be by numerical methods).

6.05 Integrals with Higher Derivatives [for reference; **not** examinable]

The path $y(x)$ which passes through the point (x_0, y_0) with prescribed slope $y'(x_0)$, passes through the point (x_1, y_1) with prescribed slope $y'(x_1)$ and which minimizes the integral

$$I = \int_{x_0}^{x_1} F\left(x, y(x), y'(x), y''(x)\right) dx$$

must be a solution of the Euler equation

$$\boxed{\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} = 0}$$

For an integrand of the form $F = F\left(x, y', y'', \ldots, y^{(n)}\right)$,

the Euler equation is

$$\boxed{(-1)^n \frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right) + \ldots + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} = 0}$$

<u>Example 6.05.1</u>

Find the path $y(x)$ which minimizes the integral

$$I = \int_{x_0}^{x_1} \left(16y^2 - (y'')^2 + \phi(x)\right) dx$$

where $\phi(x)$ is any twice differentiable function of x only and y and y' are prescribed at both endpoints x_0 and x_1.

$$F = 16y^2 - (y'')^2 + \phi(x)$$

$$\Rightarrow \quad \frac{\partial F}{\partial y''} = -2y'', \qquad \frac{\partial F}{\partial y'} = 0, \qquad \frac{\partial F}{\partial y} = 32y$$

The Euler equation becomes

$$\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{\partial F}{\partial y} = 0 \quad \Rightarrow \quad -2\frac{d^4 y}{dx^4} - 0 + 32y = 0$$

$$\Rightarrow \quad \frac{d^4 y}{dx^4} - 16y = 0$$

The auxiliary equation is $\lambda^4 - 16 = 0 \quad \Rightarrow \quad (\lambda^2 + 4)(\lambda^2 - 4) = 0$

$\Rightarrow \quad \lambda = \pm 2j, \pm 2.$

Therefore

$$y(x) = A e^{2x} + B e^{-2x} + C \cos 2x + D \sin 2x$$

The values of y and y' at both endpoints can be used to find the values of A, B, C, D.

6.06 Integrals with Several Independent Variables
[for reference; **not** examinable]

Consider the problem of finding $w(x, y)$ that minimizes the integral

$$I = \iint_D F\left(x, y, w(x,y), w_x(x,y), w_y(x,y)\right) dx\, dy$$

where D is some region in the x-y plane bounded by a simply connected curve C and $w(x, y)$ (or its normal derivative) is prescribed on C. The Euler equation in this case is

$$\boxed{\frac{\partial}{\partial x}\left(\frac{\partial F}{\partial w_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial w_y}\right) - \frac{\partial F}{\partial w} = 0}$$

<u>Example 6.06.1</u>

Find a constraint on the function $w(x, y)$ that minimizes the integral

$$I = \frac{1}{2}\iint_D \left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + 2pw\right) dx\, dy$$

where p is a constant and w is prescribed on the closed boundary C of the domain D.

$$F = \tfrac{1}{2} w_x^{\,2} + \tfrac{1}{2} w_y^{\,2} + pw$$

$$\Rightarrow \quad \frac{\partial F}{\partial w_x} = w_x\,, \quad \frac{\partial F}{\partial w_y} = w_y\,, \quad \frac{\partial F}{\partial w} = p$$

The Euler equation therefore becomes

$$\frac{\partial}{\partial x}(w_x) + \frac{\partial}{\partial y}(w_y) - (p) = 0 \qquad \Rightarrow \quad w_{xx} + w_{yy} = p$$

Therefore $w(x, y)$ must be such that

$$\nabla^2 w = p$$

Note (from section 9.05) that if $p \geq 0$, then w is subharmonic and, everywhere in D, w is bounded above by the maximum value of w on the boundary C.

If $p \leq 0$, then w is superharmonic and, everywhere in D, w is bounded below by the minimum value of w on the boundary C.

If $p = 0$, then w is harmonic and, everywhere in D, w is bounded between the minimum and maximum values of w on the boundary C.

6.07 Integrals subject to a Constraint [for reference; **not** examinable]

<u>Example 6.07.1</u>

Find the curve $y(x)$ between (0,0) and (1,0) of fixed length L (> 1) that maximizes the area under the curve, $A = \int_0^1 y\,dx$.

We are required to maximize the integral $\int_0^1 y\,dx$ subject to the constraint $\int_0^1 \sqrt{1+(y')^2}\,dx = L$.

Introduce the Lagrange multiplier λ.
Then this optimization problem becomes one of finding the minimum value of

$$I(y, y'; \lambda) = \int_0^1 \left(y + \lambda\sqrt{1+(y')^2} \right) dx.$$

The Euler equation for extremals, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$, becomes

$$\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+(y')^2}}\right) - 1 = 0 \quad \Rightarrow \quad \frac{\lambda y'}{\sqrt{1+(y')^2}} = x + c_1$$

$$\Rightarrow \quad (\lambda y')^2 = (x+c_1)^2\left(1+(y')^2\right) \quad \Rightarrow \quad \left(\lambda^2 - (x+c_1)^2\right)(y')^2 = (x+c_1)^2$$

$$\Rightarrow \quad (y')^2 = \frac{(x+c_1)^2}{\lambda^2 - (x+c_1)^2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{x+c_1}{\sqrt{\lambda^2 - (x+c_1)^2}}$$

$$\Rightarrow \quad \int 1\,dy = \int \frac{x+c_1}{\sqrt{\lambda^2 - (x+c_1)^2}}\,dx$$

Let $x + c_1 = \lambda \sin\theta \quad \Rightarrow \quad dx = \lambda\cos\theta\,d\theta$
Then

$$y = \int \frac{\lambda\sin\theta \cdot \lambda\cos\theta}{\sqrt{\lambda^2\left(1-\sin^2\theta\right)}}\,d\theta = \int \lambda\sin\theta\,d\theta = -\lambda\cos\theta - c_2$$

$$\Rightarrow \quad y + c_2 = -\sqrt{\lambda^2 - \lambda^2\sin^2\theta} = -\sqrt{\lambda^2 - (x+c_1)^2}$$

$$\Rightarrow \quad (x+c_1)^2 + (y+c_2)^2 = \lambda^2$$

Example 6.07.1 (continued)

But (0,0) and (1,0) are both on the curve.

$\Rightarrow \quad c_1^{\,2} + c_2^{\,2} = \left(1+c_1\right)^2 + c_2^{\,2} = \lambda^2 \qquad \Rightarrow \quad c_1 = \pm\sqrt{1+c_1}$

The positive root yields a contradiction (0 = 1).

The negative root yields $2c_1 = -1 \quad \Rightarrow \quad c_1 = -\frac{1}{2} \quad \Rightarrow \quad c_2 = \pm\sqrt{\lambda^2 - \frac{1}{4}}$.

We therefore obtain the **upper arc of one of the two circles**

$$\boxed{\left(x-\tfrac{1}{2}\right)^2 + \left(y \pm \sqrt{\lambda^2 - \tfrac{1}{4}}\right)^2 = \lambda^2}$$

of radius λ and centre $\left(\frac{1}{2}, \mp\sqrt{\lambda^2 - \frac{1}{4}}\right)$, passing through both (0,0) and (1,0).

The value of the parameter λ depends on the length L of the curve.

For a circle $\left(x+c_1\right)^2 + \left(y+c_2\right)^2 = \lambda^2$,

$$2\left(x+c_1\right) + 2\left(y+c_2\right)\frac{dy}{dx} = 0 \quad \Rightarrow \quad y' = -\frac{x+c_1}{y+c_2}$$

$$\Rightarrow \quad 1+\left(y'\right)^2 = \frac{\left(y+c_2\right)^2 + \left(x+c_1\right)^2}{\left(y+c_2\right)^2} = \frac{\lambda^2}{\lambda^2 - \left(x+c_1\right)^2}$$

$$\Rightarrow \quad L = \int_0^1 \sqrt{1+\left(y'\right)^2}\, dx = \int_0^1 \frac{\lambda}{\sqrt{\lambda^2 - \left(x-\frac{1}{2}\right)^2}}\, dx .$$

Using the substitution $x - \frac{1}{2} = \lambda \sin\phi$, we have

$\lambda^2 - \left(x-\frac{1}{2}\right)^2 = \lambda^2 \cos^2\phi \quad$ and $\quad dx = \lambda\cos\phi\, d\phi$

$$\Rightarrow \quad L = \int_{x=0}^{x=1} \frac{\lambda \cdot \lambda\cos\phi}{\lambda\cos\phi}\, d\phi = \left[\lambda\,\phi\right]_{x=0}^{x=1} = \left[\lambda \operatorname{Arcsin}\left(\frac{x-\frac{1}{2}}{\lambda}\right)\right]_0^1 = 2\lambda \operatorname{Arcsin}\left(\frac{1}{2\lambda}\right).$$

Therefore λ is related to the given arc length L by

$$2\lambda \sin\left(\frac{L}{2\lambda}\right) = 1 .$$

As examples, if $L = 2$, then $\lambda \approx 0.528$.

If $L = \frac{\pi}{2}$ then $\lambda = \frac{1}{2}$, for which the optimal curve is the entire upper semicircle of $\left(x-\frac{1}{2}\right)^2 + y^2 = \left(\frac{1}{2}\right)^2$.

END OF CHAPTER 6