First order linear ODE:

The general solution of $\frac{dy}{dx} + P(x)y = R(x)$ is

$$y(x) = e^{-h(x)}\left(\int R(x)e^{h(x)}dx + C\right), \quad \text{where} \quad h(x) = \int P(x)\,dx$$

Integrating Factor to convert a non-exact first order ODE into an exact form:

For the ODE $P(x,y)\,dx + Q(x,y)\,dy = 0$

If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ then the ODE is exact

else if $R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ is a function of x only, then $I(x) = e^{\int R(x)\,dx}$

else if $S = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$ is a function of y only, then $I(y) = e^{\int S(y)\,dy}$

and $I \cdot P\,dx + I \cdot Q\,dy = 0$ is exact.

Bernoulli ODEs

$$\frac{dy}{dx} + P(x)y = R(x)y^n$$

If $n = 0$ then the ODE is linear
If $n = 1$ then the ODE is separable (and linear)
For all other values of n

$$\frac{y^{1-n}}{1-n} = u(x) = e^{-h(x)}\left(\int e^{h(x)}R(x)\,dx + C\right), \quad \text{where} \quad h(x) = (1-n)\int P(x)\,dx$$

Variation of parameters for second order linear ODEs $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = r(x)$

If the complementary function is $y_C(x) = A\,y_1(x) + B\,y_2(x)$ then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix} = -y_2 r, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix} = +y_1 r$$

$$u' = \frac{W_1}{W}, \quad v' = \frac{W_2}{W} \quad \rightarrow \quad y_P(x) = u(x)y_1(x) + v(x)y_2(x)$$

Some properties of **Laplace transforms** are listed here
(from pages 1.31 and 1.32 of the lecture notes)

Linearity:

$$\mathcal{L}\left\{ a\, f(t) + b\, g(t) \right\} = a\, \mathcal{L}\left\{ f(t) \right\} + b\, \mathcal{L}\left\{ g(t) \right\} \qquad (a, b = \text{constants})$$

Polynomial functions:

$$\mathcal{L}\left\{ t^n \right\} = \frac{n!}{s^{n+1}} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{ \frac{1}{s^n} \right\} = \frac{t^{n-1}}{(n-1)!}$$

First Shift Theorem:

$$\mathcal{L}\left\{ f(t) \right\} = F(s) \;\Rightarrow\; \mathcal{L}\left\{ e^{at} f(t) \right\} = F(s-a)$$

$$\Rightarrow \mathcal{L}^{-1}\left\{ \frac{1}{s+a} \right\} = e^{-at} \quad \text{and} \quad \mathcal{L}^{-1}\left\{ \frac{1}{(s+a)^n} \right\} = \frac{t^{n-1} e^{-at}}{(n-1)!}$$

Trigonometric Functions:

$$\mathcal{L}\left\{ e^{at} \sin \omega t \right\} = \frac{\omega}{(s-a)^2 + \omega^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{ \frac{1}{(s+a)^2 + \omega^2} \right\} = \frac{e^{-at} \sin \omega t}{\omega}$$

$$\mathcal{L}\left\{ e^{at} \cos \omega t \right\} = \frac{s-a}{(s-a)^2 + \omega^2} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{ \frac{s+a}{(s+a)^2 + \omega^2} \right\} = e^{-at} \cos \omega t$$

Derivatives:

$$\mathcal{L}\left\{ f'(t) \right\} = s\, \mathcal{L}\left\{ f(t) \right\} - f(0)$$

$$\mathcal{L}\left\{ f''(t) \right\} = s^2\, \mathcal{L}\left\{ f(t) \right\} - s\, f(0) - f'(0)$$

Integration:

$$\mathcal{L}^{-1}\left\{ \frac{1}{s} G(s) \right\} = \int_0^t \mathcal{L}^{-1}\left\{ G(s) \right\} d\tau$$

$$\frac{d}{ds} \mathcal{L}\left\{ f(t) \right\} = -\mathcal{L}\left\{ t\, f(t) \right\} \quad \Rightarrow \quad \mathcal{L}^{-1}\left\{ F'(s) \right\} = -t \cdot \mathcal{L}^{-1}\left\{ F(s) \right\}$$

Properties of **Laplace transforms** (continued)

Second shift theorem:

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \Rightarrow \quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = H(t-a)\,f(t-a)$$

where $H(t-a) = \begin{cases} 0 & (t<a) \\ 1 & (t \ge a) \end{cases}$ is the Heaviside (unit step) function.

Dirac delta function

$$\mathscr{L}\{\delta(t-a)\} = e^{-as}$$

where $\int_c^d f(t)\,\delta(t-a)\,dt = \begin{cases} f(a) & (\text{if} \quad c<a<d) \\ 0 & (a<c \ \text{ or } \ a>d) \end{cases}$.

For a periodic function $f(t)$ with fundamental period p,

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)\,dt$$

Convolution:

$$\mathscr{L}^{-1}\{F(s)\,G(s)\} = \mathscr{L}^{-1}\{F(s)\} * \mathscr{L}^{-1}\{G(s)\}$$

where $(f * g)(t)$ denotes the convolution of $f(t)$ and $g(t)$ and is defined by

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

The identity function for convolution is the Dirac delta function:

$$\delta(t-a) * f(t) = f(t-a)\,H(t-a) \quad \Rightarrow \quad \delta(t) * f(t) = f(t)$$

Frobenius series solution of an ODE $\quad P(x)\,y'' + Q(x)\,y' + R(x)\,y = F(x)$

If $P(x_o) \ne 0$ and $P(x),\ Q(x),\ R(x),\ F(x)$ are all analytic at $x = x_o$, then try a Taylor series solution $y(x) = \sum_{n=0}^{\infty} c_n (x-x_o)^n$ [Usually $x_o = 0 \ \to$ Maclaurin series]

If $P(x_o) = 0$, but $(x-x_o)\dfrac{Q(x)}{P(x)}$, $(x-x_o)^2\dfrac{R(x)}{P(x)}$ and $\dfrac{F(x)}{P(x)}$ are all analytic at x_o

then try $y(x) = \sum_{n=0}^{\infty} c_n (x-x_o)^{n+r}$. $c_0 \ne 0$ leads to the values of r .

Here is a summary of **inverse Laplace transforms** (page 1.33 of the lecture notes).

$F(s)$	$f(t)$
$\int_0^\infty e^{-st} f(t)\, dt$	$f(t)$
$\dfrac{1}{s^n} \quad (n \in \mathbb{N})$	$\dfrac{t^{n-1}}{(n-1)!}$
$\dfrac{1}{\sqrt{s}}$	$\dfrac{1}{\sqrt{\pi t}}$
$\dfrac{1}{s-a}$	e^{at}
$\dfrac{1}{(s-a)^n} \quad (n \in \mathbb{N})$	$\dfrac{t^{n-1} e^{at}}{(n-1)!}$
e^{-as}	$\delta(t-a)$
$\dfrac{e^{-as}}{s}$	$H(t-a)$
$\dfrac{1}{s^2+\omega^2}$	$\dfrac{\sin \omega t}{\omega}$
$\dfrac{1}{(s+a)^2+\omega^2}$	$\dfrac{e^{-at} \sin \omega t}{\omega}$
$\dfrac{1}{(s-a)^2-b^2}$	$\dfrac{e^{at} \sinh bt}{b}$
$\dfrac{(s+a)}{(s+a)^2+\omega^2}$	$e^{-at} \cos \omega t$
$\dfrac{(s-a)}{(s-a)^2-b^2}$	$e^{at} \cosh bt$

$F(s)$	$f(t)$
$\dfrac{1}{s(s^2+\omega^2)}$	$\dfrac{1-\cos \omega t}{\omega^2}$
$\dfrac{1}{s^2(s^2+\omega^2)}$	$\dfrac{\omega t - \sin \omega t}{\omega^3}$
$\dfrac{1}{(s^2+\omega^2)^2}$	$\dfrac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t \sin \omega t}{2\omega}$
$\dfrac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$t \cos \omega t$
$\dfrac{1}{s} \tanh\left(\dfrac{as}{2}\right)$	Square wave, period $2a$, amplitude 1
$\dfrac{1}{s^2} \tanh\left(\dfrac{as}{2}\right)$	Triangular wave, period $2a$, amplitude a
$\dfrac{b}{as^2} - \dfrac{b}{s(e^{as}-1)}$	Sawtooth wave, period a, amplitude b
$\{ s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0) \}$	$\dfrac{d^n f}{dt^n}$
$\dfrac{1}{s} F(s)$	$\int_0^t f(\tau)\, d\tau$
$\dfrac{dF}{ds}$	$-t\, f(t)$

Newton's method to estimate the solution x to $f(x) = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

To estimate the value of $y(x_0 + nh)$ where $y(x)$ is the solution of

$$y' = f(x, y), \qquad y(x_0) = y_0$$

Euler's method

$$y_{n+1} = y_n + h\,f(x_n, y_n)$$

RK4 algorithm

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h\,k_1\right) \\ k_3 &= f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h\,k_2\right) \\ k_4 &= f(x_n + h,\ y_n + h\,k_3) \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}$$

Stability Analysis

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy, \qquad (a, b, c, d = \text{constants})$$

Characteristic equation:

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Discriminant

$$D = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc$$

Roots of characteristic equation (= eigenvalues of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$):

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2}$$

Cases:

$a + d$	D	other condition	λ	Type of point
$a + d < 0$	$D > 0$	$ad - bc > 0$	real, distinct negative	**Stable** node
$a + d < 0$	$D = 0$	$b = c = 0$	real, equal negative	**Stable** star shape
$a + d < 0$	$D = 0$	b, c not both 0	real, equal negative	**Stable** node
$a + d < 0$	$D < 0$		complex conjugate pair	**Stable** focus [spiral]
$a + d = 0$	$D < 0$		Pure imaginary pair	**Stable** centre
$a + d > 0$	$D > 0$	$ad - bc > 0$	real, distinct positive	**Unstable** node
(*any*)	$D > 0$	$ad - bc < 0$	real, distinct opposite signs	**Unstable** saddle point
$a + d > 0$	$D = 0$	$b = c = 0$	real, equal positive	**Unstable** star shape
$a + d > 0$	$D = 0$	b, c not both 0	real, equal positive	**Unstable** node
$a + d > 0$	$D < 0$		complex conjugate pair	**Unstable** focus [spiral]

Note that $ad - bc = \det A$ and that $a + d$ = the trace of the matrix A.

In brief, if the real parts of both eigenvalues are negative (or both zero), then the origin is stable. Otherwise it is unstable.

Eigenvectors are the non-trivial solutions of $(\mathrm{A} - \lambda \mathrm{I})\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

Stability Analysis (continued)

If $D > 0$ then the general solution is

$$\boxed{\left(x(t), y(t)\right) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\; c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)},$$

where

$$\lambda_1 = \frac{(a+d) - \sqrt{D}}{2}, \quad \lambda_2 = \frac{(a+d) + \sqrt{D}}{2},$$

$$\begin{pmatrix}\alpha_1 \\ \beta_1\end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix}\dfrac{(a-d) - \sqrt{D}}{2} \\ c\end{pmatrix},$$

$$\begin{pmatrix}\alpha_2 \\ \beta_2\end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix}\dfrac{(a-d) + \sqrt{D}}{2} \\ c\end{pmatrix}$$ and c_1, c_2 are arbitrary constants.

[An exception may occur if $c = 0$: use $\begin{pmatrix} b \\ \dfrac{(d-a) \pm \sqrt{D}}{2}\end{pmatrix}$ instead.]

If $D < 0$ then the general solution is

$$\boxed{\begin{gathered}\left(x(t), y(t)\right) = \\ e^{ut}\left(c_3\left((u-d)\cos vt - v\sin vt\right) + c_4\left(v\cos vt + (u-d)\sin vt\right),\; c\left(c_3\cos vt + c_4\sin vt\right)\right)\end{gathered}}$$

where $u = \dfrac{a+d}{2}$ $\left(\Rightarrow \; u - d = \dfrac{a-d}{2}\right)$ and $v = \dfrac{\sqrt{-(a-d)^2 - 4bc}}{2} = \dfrac{\sqrt{-D}}{2}$

and c_3, c_4 are [real] arbitrary constants.

If $D = 0$ then the general solution is

$$\boxed{\left(x(t), y(t)\right) = \left(\left(c_1\left(\frac{a-d}{2}\right) + c_2\left(1 + \left(\frac{a-d}{2}\right)(1+t)\right)\right)e^{\lambda t},\; c\left(c_1 + c_2(1+t)\right)e^{\lambda t}\right)},$$

unless $a = d$ and $c = 0$ but $b \neq 0$, in which case

$$\boxed{\left(x(t), y(t)\right) = \left((c_1 + c_2 t)e^{at},\; \frac{c_2}{b}e^{at}\right)}$$

or the decoupled system $a = d$ and $b = c = 0$, in which case

$$\boxed{\left(x(t), y(t)\right) = \left(c_1 e^{at},\; c_2 e^{at}\right)}$$

Stability Analysis (continued)

where the sole distinct eigenvalue and eigenvector are

$$\lambda = \frac{(a+d)}{2},$$

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \text{ any non-zero multiple of } \begin{pmatrix} \frac{a-d}{2} \\ c \end{pmatrix} \quad \left(\text{or } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ if } a = d \text{ and } c = 0\right).$$

Linear approximation of a non-linear system:

$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y)$$

Near each critical point (a, b), the coefficient matrix A for the linear system is

$$A = \left. \begin{bmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{bmatrix} \right|_{(a,b)}$$

Curvilinear coordinates

Scale factors are $h_i = \left| \frac{\partial \vec{\mathbf{r}}}{\partial u_i} \right|$

For cylindrical polar coordinates (ρ, ϕ, z), $h_\rho = h_z = 1$, $h_\phi = \rho$.
For spherical polar coordinates (r, θ, ϕ), $h_r = 1$, $h_\theta = r$, $h_\phi = r \sin\theta$.

The **gradient operator** is $\vec{\nabla} = \frac{\hat{\mathbf{e}}_1}{h_1} \frac{\partial}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2} \frac{\partial}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3} \frac{\partial}{\partial u_3}$

The **divergence** of a vector field $\vec{\mathbf{F}}(u_1, u_2, u_3)$ is

$$\operatorname{div} \vec{\mathbf{F}} = \vec{\nabla} \bullet \vec{\mathbf{F}} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial(h_2 h_3 F_1)}{\partial u_1} + \frac{\partial(h_3 h_1 F_2)}{\partial u_2} + \frac{\partial(h_1 h_2 F_3)}{\partial u_3} \right)$$

The **curl** of a vector field $\vec{\mathbf{F}}(u_1, u_2, u_3)$ is

$$\operatorname{curl}\vec{\mathbf{F}} = \vec{\nabla}\times\vec{\mathbf{F}} = \frac{1}{h_1\,h_2\,h_3}\begin{vmatrix} h_1\,\hat{\mathbf{e}}_1 & \dfrac{\partial}{\partial u_1} & h_1\,F_1 \\ h_2\,\hat{\mathbf{e}}_2 & \dfrac{\partial}{\partial u_2} & h_2\,F_2 \\ h_3\,\hat{\mathbf{e}}_3 & \dfrac{\partial}{\partial u_3} & h_3\,F_3 \end{vmatrix}$$

The **Laplacian** of a scalar field $V(u_1, u_2, u_3)$ is

$$\nabla^2 V = \frac{1}{h_1\,h_2\,h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2\,h_3}{h_1}\frac{\partial V}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3\,h_1}{h_2}\frac{\partial V}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1\,h_2}{h_3}\frac{\partial V}{\partial u_3}\right)\right)$$

Basis vectors

Cylindrical Polar Coordinates:

$x = \rho\cos\phi\,, \quad y = \rho\sin\phi\,, \quad z = z$

$$\frac{d}{dt}\hat{\boldsymbol{\rho}} = \frac{d\phi}{dt}\hat{\boldsymbol{\phi}}$$
$$\frac{d}{dt}\hat{\boldsymbol{\phi}} = -\frac{d\phi}{dt}\hat{\boldsymbol{\rho}}$$
$$\frac{d}{dt}\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

$$\mathbf{r} = \rho\,\hat{\boldsymbol{\rho}} + z\,\hat{\mathbf{k}}$$
$$\Rightarrow \quad \vec{\mathbf{v}} = \dot{\rho}\,\hat{\boldsymbol{\rho}} + \rho\dot{\phi}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\mathbf{k}}$$

[radial and transverse components of $\vec{\mathbf{v}}$]

Spherical Polar Coordinates:

$x = r\sin\theta\cos\phi\,, \quad y = r\sin\theta\sin\phi\,, \quad z = r\cos\theta\,.$

$$\frac{d}{dt}\hat{\mathbf{r}} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{d\phi}{dt}\sin\theta\,\hat{\boldsymbol{\phi}}$$
$$\frac{d}{dt}\hat{\boldsymbol{\theta}} = -\frac{d\theta}{dt}\hat{\mathbf{r}} + \frac{d\phi}{dt}\cos\theta\,\hat{\boldsymbol{\phi}}$$
$$\frac{d}{dt}\hat{\boldsymbol{\phi}} = -\frac{d\phi}{dt}\left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right)$$

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$$
$$\Rightarrow \quad \vec{\mathbf{v}} = \dot{r}\,\hat{\mathbf{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\dot{\phi}\sin\theta\,\hat{\boldsymbol{\phi}}$$

The **Euler equation for the extremals** of $\int_a^b F(x, y, y')\,dx$ is

$$\boxed{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0}$$

If F is explicitly independent of y, $I = \int_{x_0}^{x_1} F(x, y')\,dx$, then $\boxed{F_{y'} = c_1}$

If F is explicitly independent of x, $I = \int_{x_0}^{x_1} F(y, y')\,dx$, then $\boxed{y' F_{y'} - F = c_1}$

If F is explicitly independent of x and y, $I = \int_{x_0}^{x_1} F(y')\,dx$, then $\boxed{y'' F_{y'y'} \equiv 0}$

The **Fourier series** of $f(x)$ on the interval $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

The half-range Fourier cosine series of $f(x)$ on the interval $(0, L)$ has coefficients

$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$$

The half-range Fourier sine series of $f(x)$ on the interval $(0, L)$ has coefficients

$$b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

Partial differential equations

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$$

Let $D = B^2 - 4AC$ then the PDE is
Hyperbolic, wherever (x, y) is such that $D > 0$;
Parabolic, wherever (x, y) is such that $D = 0$;
Elliptic, wherever (x, y) is such that $D < 0$.

Wave equation on a finite string $(0 \le x \le L)$

The general solution of $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ subject to $y(0,t) = y(L,t) = 0 \quad (t \ge 0)$,

$y(x,0) = f(x) \quad (0 \le x \le L)$, $\left.\dfrac{\partial y}{\partial t}\right|_{(x,0)} = g(x) \quad (0 \le x \le L)$ is

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right) + \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

Wave equation on an infinite string

The general solution of $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ subject to $y(x,0) = f(x)$, $\left.\dfrac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$

is

$$y(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$$

Heat equation

The general solution of $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2}$ subject to $u(0,t) = T_1$, $u(L,t) = T_2$

and $u(x, 0) = f(x)$ is $u(x,t) = v(x,t) + \left(\dfrac{T_2 - T_1}{L}\right)x + T_1$, where

$$v(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L \left(f(z) - \frac{T_2 - T_1}{L}z - T_1\right)\sin\left(\frac{n\pi z}{L}\right)dz\right)\sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2 k t}{L^2}\right)$$

d'Alembert solution

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

$$\boxed{u(x, y) = f_1(y+\lambda_1 x) + f_2(y+\lambda_2 x),}$$

where

$$\boxed{\lambda_1 = \frac{-B-\sqrt{D}}{2A} \quad \text{and} \quad \lambda_2 = \frac{-B+\sqrt{D}}{2A}}$$

and $D = B^2 - 4AC$

Equal roots case:

$$u(x,y) = f_1(y+\lambda x) + h(x,y)\,f_2(y+\lambda x)$$

where $h(x, y)$ is any non-trivial linear function of x and/or y (except $y + \lambda x$).

If $\nabla^2 u \ge 0$ in Ω, then u is **subharmonic** and

$$u(\vec{\mathbf{r}}) < M \quad \text{or} \quad u(\vec{\mathbf{r}}) \equiv M \qquad \forall \vec{\mathbf{r}} \text{ in } \Omega$$

If $\nabla^2 u \le 0$ in Ω, then u is **superharmonic** and

$$u(\vec{\mathbf{r}}) > m \quad \text{or} \quad u(\vec{\mathbf{r}}) \equiv m \qquad \forall \vec{\mathbf{r}} \text{ in } \Omega$$

If $\nabla^2 u = 0$ in Ω, then u is **harmonic** (both subharmonic and superharmonic) and u is either constant on $\overline{\Omega}$ or $m < u < M$ everywhere on Ω.
