1. Ordinary Differential Equations

An equation involving a function of one independent variable and the derivative(s) of that function is an ordinary differential equation (ODE).

The highest order derivative present determines the order of the ODE and the power to which that highest order derivative appears is the degree of the ODE. A general n^{th} order ODE is

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

Example 1.00.1

$$\frac{d^2 y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 = x^2 y \quad \text{is a}$$

Example 1.00.2

$$x\left(\frac{dy}{dx}\right)^2 = x^2 y \quad \text{is a}$$

In this course we will usually consider first degree ODEs of first or second order only. The topics in this chapter are treated briefly, because it is assumed that graduate students will have seen this material during their undergraduate years.

Sections in this Chapter:

1.01 **First Order ODEs - Separation of Variables**

Example 1.01.1

A particle falls under gravity from rest through a viscous medium such that the drag force is proportional to the square of the speed. Find the speed $v(t)$ at any time $t > 0$ and find the terminal speed v_∞.

Example 1.01.1 (continued)

Example 1.01.1 (continued)

General solution:

$$v(t) = \frac{k\left(1 + A e^{-pt}\right)}{1 - A e^{-pt}}$$

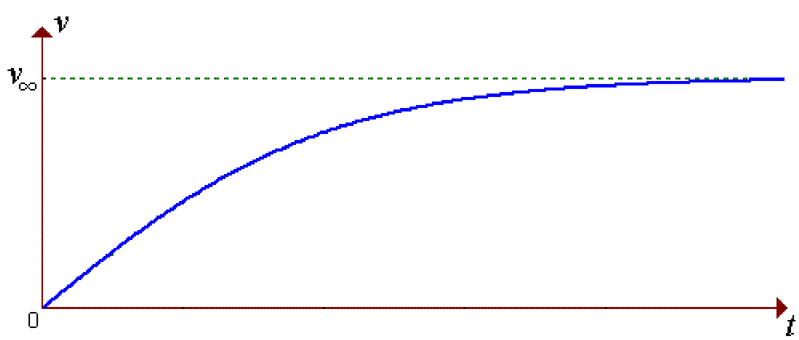
Initial condition:

Complete solution:

Terminal speed v_∞:

The terminal speed can also be found directly from the ODE.
At terminal speed, the acceleration is zero, so that the ODE simplifies to

Graph of speed against time:



[For a 90 kg person in air, $b \approx 1 \text{ kg m}^{-1} \rightarrow k \approx 30 \text{ ms}^{-1} \approx 100$ km/h.
$v(t)$ is approximately linear at first, but air resistance builds quickly.
One accelerates to within 10 km/h of terminal velocity very fast, in just a few seconds.]

1.02 <u>Exact First Order ODEs</u>

If x and y are related implicitly by the equation $u(x, y) = c$ (constant), then the chain rule for differentiation leads to the ODE

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

Therefore, for the functions $M(x, y)$ and $N(x, y)$ in the first order ODE

$$M\,dx + N\,dy = 0 \,,$$

if a **potential function** $u(x, y)$ exists such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y} \,,$$

then $u(x, y) = c$ is the general solution to the ODE and the ODE is said to be **exact**.

Note that, for nearly all functions of interest, Clairault's theorem results in the identity

$$\frac{\partial^2 u}{\partial y\,\partial x} \equiv \frac{\partial^2 u}{\partial x\,\partial y}$$

This leads to a simple test to determine whether or not an ODE is exact:

$$\boxed{\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x} \quad \Rightarrow \quad M\,dx + N\,dy = 0 \quad \text{is exact}}$$

A separable first order ODE is also exact (after suitable rearrangement).

However, the converse is false. One counter-example will suffice.

Example 1.02.1

The ODE

$$\left(y\, e^{x} - x\right)dx \;+\; e^{x}dy \;=\; 0$$

is exact,

Example 1.02.2

Is the ODE $2y\,dx \;+\; x\,dy \;=\; 0$ exact?

Example 1.02.3

Is the ODE

$$A\left(2x^{2n+1}y^{n+1}\,dx \;+\; x^{2n+2}y^{n}\,dy\right) \;=\; 0$$

(where n is any constant and A is any non-zero constant) exact?
Find the general solution.

Note that the exact ODE in example 1.02.3 is just the non-exact ODE of example 1.02.2 multiplied by the factor $I(x,y) = Ax^{2n+1}y^{n}$. The ODEs are therefore equivalent and share the same general solution. The function $I(x,y) = Ax^{2n+1}y^{n}$ is an **integrating factor** for the ODE of example 1.02.2.

Also note that the integrating factor is not unique. In this case, *any* two distinct values of n generate two distinct integrating factors that both convert the non-exact ODE into an exact form. However, we need to guard against introducing a spurious singular solution $y \equiv 0$.

1.03 Integrating Factor

Occasionally it is possible to transform a non-exact first order ODE into exact form, using an integrating factor $I(x, y)$.
Suppose that

$$P\,dx + Q\,dy = 0$$

is not exact, but that

$$IP\,dx + IQ\,dy = 0$$

is exact.

Then, using the product rule,

Example 1.03.1 (Example 1.02.2 again)

Find the general solution of the ODE

$$2y\,dx + x\,dy = 0$$

Example 1.03.2

Find the general solution of the ODE

$$2xy\,dx + (2x^2 + 3y)\,dy = 0$$

1.04 First Order Linear ODEs [+ Integration by Parts]

A special case of a first order ODE is the linear ODE:

$$\boxed{\frac{dy}{dx} + P(x)\,y = R(x)}$$

[or, in some cases,

$$\frac{dx}{dy} + Q(y)\,x = S(y) \text{]}$$

Rearranging the first ODE into standard form,

$$(P(x)\,y - R(x))\,dx + 1\,dy = 0$$

Therefore the general solution of $\frac{dy}{dx} + P(x)\,y = R(x)$ is

$$\boxed{y(x) = e^{-h(x)}\left(\int e^{h(x)}R(x)\,dx + C\right), \quad \text{where} \quad h(x) = \int P(x)\,dx}$$

Example 1.04.1

Solve the ordinary differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = 1$$

Example 1.04.1 (continued)

Examples of Integration by Parts

The method of integration by parts will be required in the next example of a first order linear ODE (Example 1.04.4). There are three main cases for integration by parts:

Example 1.04.2

Integrate $x^3 e^x$ with respect to *x*.

Example 1.04.3

Integrate $\ln x$ with respect to *x*.

Example 1.04.4

An electrical circuit that contains a resistor, $R = 8\ \Omega$ (ohm), an inductor, $L = 0.02$ millihenry, and an applied emf, $E(t) = 2\cos(5t)$, is governed by the differential equation

$$L\frac{di}{dt} + Ri = \frac{dE}{dt}$$

Determine the current at any time $t \geq 0$, if initially there is a current of 1 ampere in the circuit.

First note that the inductance $L = 2\times10^{-5}$ H is very small. The ODE is therefore not very different from

$$0 + R\,i = dE/dt$$

which has the immediate solution

$$i = (1/R)\,dE/dt = (1/8)\times(-10\sin 5t)$$

We therefore anticipate that $i = -(5/4)\sin 5t$ will be a good approximation to the exact solution.

Substituting all values ($R = 8$, $L = 2\times10^{-5}$, $E = 2\cos 5t \Rightarrow E' = -10\sin 5t$) into the ODE yields

$$\frac{di}{dt} + 4\times10^5\, i = -5\times10^5 \sin 5t$$

which is a linear first order ODE.

$P(t) = 400\,000$ and $R(t) = -500\,000\sin 5t \Rightarrow h = \int P\,dt = 400\,000\,t$

$\Rightarrow$ integrating factor $= e^h = e^{400\,000t}$

$\Rightarrow \int e^h R\,dt = -500\,000\int e^{400\,000t}\sin 5t\,dt$

Integration by parts of the general case $\int e^{ax}\sin bx\,dx$:

Example 1.04.4 (continued)

$$\Rightarrow \quad \int e^{ax} \sin bx \, dx = \frac{1}{a^2+b^2}\left[e^{ax}\left(a \sin bx - b \cos bx \right)\right] + C$$

Set $a = 400\,000$, $b = 5$ and $x = t$:

$$\Rightarrow \quad \int e^{h} R \, dt = -500\,000 \frac{1}{400\,000^2 + 5^2} e^{400000t} \left(400\,000 \sin 5t - 5 \cos 5t \right)$$

The general solution is

$$i(t) = e^{-h}\left(\int e^{h} R \, dt + C \right)$$

$$\Rightarrow \quad i(t) = A e^{-400000t} - \frac{500\,000}{400\,000^2 + 25}\left(400\,000 \sin 5t - 5 \cos 5t \right)$$

But $i(0) = 1$

$$\Rightarrow \quad 1 = A - \frac{500\,000}{400\,000^2 + 25}(0 - 5)$$

$$\Rightarrow \quad A = (400\,000^2 + 25 - 2\,500\,000) / (400\,000^2 + 25)$$

Therefore the complete solution is [exactly]

$$i(t) = \frac{159\,997\,500\,025\, e^{-400000t} - 500\,000\left(400\,000 \sin 5t - 5 \cos 5t \right)}{160\,000\,000\,025}$$

To an excellent approximation, this complete solution is

$$\Rightarrow \quad i(t) \approx e^{-400000t} - \frac{5}{4} \sin 5t$$

After only a few microseconds, the transient term is negligible.
The complete solution is then, to an excellent approximation,

$$i(t) \approx -\frac{5}{4} \sin 5t$$

as before.

1.05 Bernoulli ODEs

The first order linear ODE is a special case of the Bernoulli ODE

$$\frac{dy}{dx} + P(x)\,y = R(x)\,y^n$$

If $n = 0$ then the ODE is linear.
If $n = 1$ then the ODE is separable.

For any other value of n, the change of variables $u = \dfrac{y^{1-n}}{1-n}$ will convert the Bernoulli ODE for y into a linear ODE for u.

$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = \frac{1-n}{1-n}\,y^{-n}\frac{dy}{dx} \quad\Rightarrow\quad \frac{dy}{dx} = y^n\frac{du}{dx}$$

The ODE transforms to

$$y^n\frac{du}{dx} + P(x)\,y = R(x)\,y^n \quad\Rightarrow\quad \frac{du}{dx} + P(x)\,y^{1-n} = R(x)$$

We therefore obtain the linear ODE for u:

$$\frac{du}{dx} + \big((1-n)P(x)\big)u = R(x)$$

whose solution is

$$\boxed{\frac{y^{1-n}}{1-n} = u(x) = e^{-h(x)}\left(\int e^{h(x)}R(x)\,dx + C\right), \quad \text{where} \quad h(x) = (1-n)\int P(x)\,dx}$$

together with the singular solution $y \equiv 0$ in the cases where $n > 0$.

Example 1.05.1

Find the general solution of the logistic population model

$$\frac{dy}{dx} = ay - by^2$$

where a, b are positive constants.

The Bernoulli equation is

$$\frac{dy}{dx} + (-a)\,y = (-b)\,y^2$$

with $P = -a, \quad R = -b, \quad n = 2.$

1.06 Second Order Homogeneous Linear ODEs

The general second order linear ordinary differential equation with constant real coefficients may be written in the form

$$\frac{d^2 y}{dx^2} + p\frac{dy}{dx} + q\,y = r(x)$$

If, in addition, the right-side function $r(x)$ is identically zero, then the ODE is said to be **homogeneous**. Otherwise it is inhomogeneous.

The most general possible solution y_C to the homogeneous ODE $y'' + p\,y' + q\,y = 0$ is called the **complementary function**.
A solution y_P to the inhomogeneous ODE $y'' + p\,y' + q\,y = r(x)$ is called the **particular solution**.
The linearity of the ODE leads to the following two properties:
Any linear combination of two solutions to the homogeneous ODE is another solution to the homogeneous ODE; and
The sum of any solution to the homogeneous ODE and a particular solution is another solution to the inhomogeneous ODE.

It can be shown that the following is a valid method for obtaining the complementary function:

From the ODE $y'' + p\,y' + q\,y = r(x)$ form the **auxiliary equation** (or “characteristic equation”)

$$\lambda^2 + p\lambda + q = 0$$

If the roots λ_1, λ_2 of this quadratic equation are distinct, then a basis for the entire set of possible complementary functions is $\{y_1, y_2\} = \left\{e^{\lambda_1 x}, e^{\lambda_2 x}\right\}$.

If the roots are not real (and therefore form a complex conjugate pair $a \pm bj$), then the basis can be expressed instead as the equivalent real set $\left\{e^{ax}\cos bx,\ e^{ax}\sin bx\right\}$.

If the roots are equal (and therefore real), then a basis for the entire set of possible complementary functions is $\{y_1, y_2\} = \left\{e^{\lambda x}, x e^{\lambda x}\right\}$.
The complementary function, in the form that captures all possibilities, is then

$$y_C = A y_1 + B y_2$$

where A and B are arbitrary constants.

Example 1.06.1

A simple unforced mass-spring system (with damping coefficient per unit mass = 6 s^{-1} and restoring coefficient per unit mass = 9 s^{-2}) is released from rest at an extension 1 m beyond its equilibrium position ($s = 0$). Find the position $s(t)$ at all subsequent times t.

The simple mass-spring system may be modelled by a second order linear ODE.

The $\frac{d^2 s}{dt^2}$ term represents the acceleration of the mass, due to the net force.

The $\frac{ds}{dt}$ term represents the friction (damping) term.

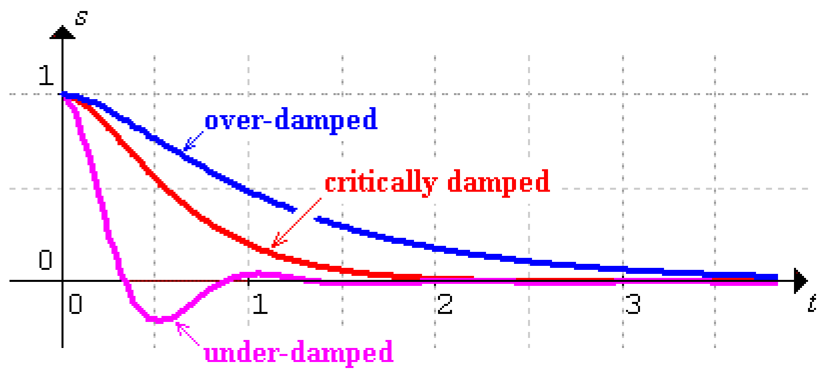
The s term represents the restoring force.

The model is

$$\frac{d^2 s}{dt^2} + 6\frac{ds}{dt} + 9s = 0$$

Example 1.06.1 (continued)

This is an example of **critical damping**.
Real distinct roots for λ correspond to **over-damping**.
Complex conjugate roots for λ correspond to **under-damping** (damped oscillations).



Illustrated here are a critically damped case $s(t) = (1+3t)e^{-3t}$ (the solution to Example 1.06.1), an over-damped case $s(t) = \frac{1}{3}\left(4e^{-t} - e^{-4t}\right)$ and an under-damped case $s(t) = e^{-3t}\left(\cos 6t + \frac{1}{2}\sin 6t\right)$, all of which share the same initial conditions $s(0) = 1$ and $s'(0) = 0$.

1.07 Variation of Parameters

A particular solution y_P to the inhomogeneous ODE $y'' + p\,y' + q\,y = r(x)$ may be constructed from the set of basis functions $\{y_1, y_2\}$ for the complementary function by varying the parameters:

Try $y_P(x) = u(x)\,y_1(x) + v(x)\,y_2(x)$, where the functions $u(x)$ and $v(x)$ are such that

(i) y_P is a solution of $y'' + p\,y' + q\,y = r(x)$ and

(ii) one free constraint is imposed, to ease the search for $u(x)$ and $v(x)$.

Substituting $y_P = u\,y_1 + v\,y_2$ into the ODE,

[space to continue the derivation of the method of variation of parameters]

Example 1.07.1

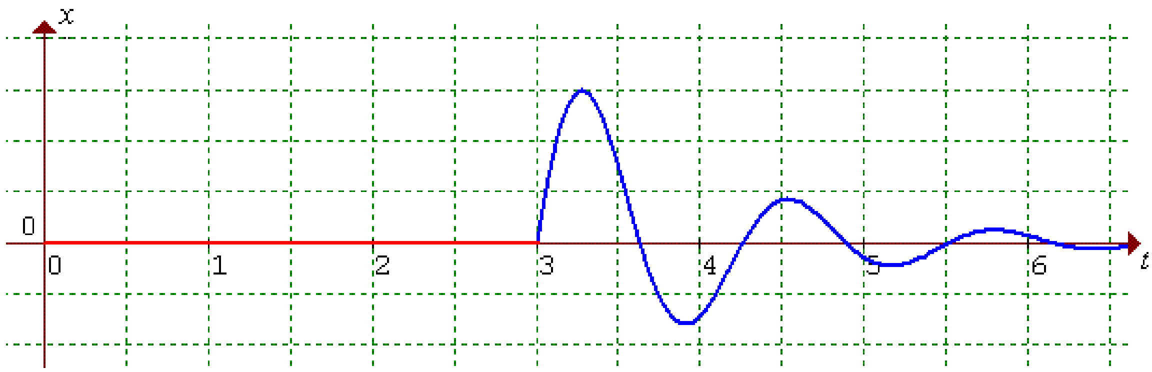
A mass spring system is at rest until the instant $t = 3$, when a sudden hammer blow, of impulse 10 Ns, sets the system into motion. No further external force is applied to the system, which has a mass of 1 kg, a restoring force coefficient of 26 kg s^{-2} and a friction coefficient of 2 kg s^{-1}. The response $x(t)$ at any time $t > 0$ is governed by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 26x = 10\,\delta(t-3)$$

(where $\delta(t-a)$ is the Dirac delta function),
together with the initial conditions $x(0) = x'(0) = 0$.
Find the complete solution to this initial value problem.

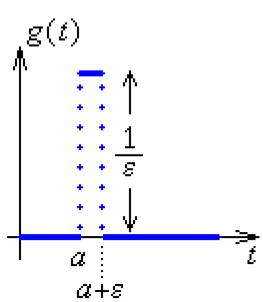
Example 1.07.1 (continued)

This complete solution is continuous at $t = 3$.
It is not differentiable at $t = 3$, because of the infinite discontinuity of the Dirac delta function inside $r(t)$ at $t = 3$.



Note: $\delta(t-a) = \lim_{\varepsilon \to 0} g(t; a, \varepsilon)$

$$H(t-a) = \begin{cases} 0 & (t < a) \\ 1 & (t \geq a) \end{cases}$$



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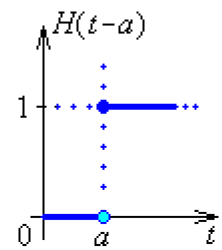


Example 1.07.2

Find the general solution of the ODE $y'' + 2y' - 3y = x^2 + e^{2x}$.

Example 1.07.2 (continued)

1.08 Method of Undetermined Coefficients

When trying to find the particular solution of the inhomogeneous ODE

$$\frac{d^2 y}{dx^2} + p\frac{dy}{dx} + q\,y = r(x)$$

an alternative method to variation of parameters is available only when $r(x)$ is one of the following special types:

e^{kx}, $\cos kx$, $\sin kx$, $\sum_{k=1}^{n} a_k x^k$ and any linear combinations of these types and any products of these types. When it is available, this method is often faster than the method of variation of parameters.

The method involves the substitution of a form for y_P that resembles $r(x)$, with coefficients yet to be determined, into the ODE.

If $r(x) = c\,e^{kx}$, then try $y_P = d\,e^{kx}$, with the coefficient *d* to be determined.

If $r(x) = a\cos kx$ or $b\sin kx$, then try $y_P = c\cos kx + d\sin kx$, with the coefficients *c* and *d* to be determined.

If $r(x)$ is an n^{th} order polynomial function of *x*, then set y_P equal to an n^{th} order polynomial function of *x*, with all (*n* + 1) coefficients to be determined.

However, if $r(x)$ contains a constant multiple of either part of the complementary function $(y_1 \text{ or } y_2)$, then that part must be multiplied by *x* in the trial function for y_P.

Example 1.08.1 (Example 1.07.2 again)

Find the general solution of the ODE $y'' + 2y' - 3y = x^2 + e^{2x}$.

A.E.: $\lambda^2 + 2\lambda - 3 = 0$

$\Rightarrow \quad (\lambda + 3)(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda = -3, 1$

C.F.: $y_C = A e^{-3x} + B e^{x}$

Particular Solution by Undetermined Coefficients:

Example 1.08.2

Find the general solution of the ODE

$$\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

1.09 Laplace Transforms

Laplace transforms can convert some initial value problems into algebra problems. It is assumed here that students have met Laplace transforms before. Only the key results are displayed here, before they are employed to solve some initial value problems.

The Laplace transform of a function $f(t)$ is the integral

$$\boxed{F(s) = \mathscr{L}\{ f(t)\} = \int_0^\infty e^{-st} f(t)\, dt}$$

where the integral exists.

Some standard transforms and properties are:

Linearity:

$$\mathscr{L}\{ a\, f(t) + b\, g(t)\} = a\,\mathscr{L}\{ f(t)\} + b\,\mathscr{L}\{ g(t)\} \qquad (a, b = \text{constants})$$

Polynomial functions:

$$\mathscr{L}\{ t^n \} = \frac{n!}{s^{n+1}} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{ \frac{1}{s^n} \right\} = \frac{t^{n-1}}{(n-1)!}$$

First Shift Theorem:

$$\mathscr{L}\{ f(t)\} = F(s) \;\Rightarrow\; \mathscr{L}\{ e^{at} f(t)\} = F(s-a)$$

$$\Rightarrow \quad \mathscr{L}^{-1}\left\{ \frac{1}{s-a} \right\} = e^{at} \quad \text{and} \quad \mathscr{L}^{-1}\left\{ \frac{1}{(s-a)^n} \right\} = \frac{t^{n-1} e^{at}}{(n-1)!}$$

Trigonometric Functions:

$$\mathscr{L}\{ e^{at} \sin \omega t \} = \frac{\omega}{(s-a)^2 + \omega^2} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{ \frac{1}{(s-a)^2 + \omega^2} \right\} = \frac{e^{at} \sin \omega t}{\omega}$$

$$\mathscr{L}\{ e^{at} \cos \omega t \} = \frac{s-a}{(s-a)^2 + \omega^2} \quad \Rightarrow \quad \mathscr{L}^{-1}\left\{ \frac{s-a}{(s-a)^2 + \omega^2} \right\} = e^{at} \cos \omega t$$

Derivatives:

$$\mathscr{L}\{ f'(t)\} = s\,\mathscr{L}\{ f(t)\} - f(0)$$

$$\mathscr{L}\{ f''(t)\} = s^2\,\mathscr{L}\{ f(t)\} - s\, f(0) - f'(0)$$

Integration:

$$\mathscr{L}^{-1}\left\{ \frac{1}{s} G(s) \right\} = \int_0^t \mathscr{L}^{-1}\{ G(s) \}\, d\tau$$

$$\frac{d}{ds}\mathscr{L}\{ f(t) \} = -\mathscr{L}\{ t\, f(t) \} \quad \Rightarrow \quad \mathscr{L}^{-1}\{ F'(s) \} = -t \cdot \mathscr{L}^{-1}\{ F(s) \}$$

Second shift theorem:

$$\mathscr{L}^{-1}\{ F(s) \} = f(t) \quad \Rightarrow \quad \mathscr{L}^{-1}\{ e^{-as} F(s) \} = H(t-a)\, f(t-a)$$

where $H(t-a) = \begin{cases} 0 & (t < a) \\ 1 & (t \ge a) \end{cases}$ is the Heaviside (unit step) function.

Dirac delta function

$$\mathscr{L}\{ \delta(t-a) \} = e^{-as}$$

where $\int_c^d f(t)\,\delta(t-a)\,dt = \begin{cases} f(a) & (\text{if} \quad c < a < d) \\ 0 & (a < c \ \text{ or } \ a > d) \end{cases}$.

For a periodic function $f(t)$ with fundamental period p,

$$\mathscr{L}\{ f(t) \} = \frac{1}{1 - e^{-ps}} \int_0^p e^{-st} f(t)\, dt$$

Convolution:

$$\mathscr{L}^{-1}\{ F(s)\, G(s) \} = \mathscr{L}^{-1}\{ F(s) \} * \mathscr{L}^{-1}\{ G(s) \}$$

where $(f * g)(t)$ denotes the convolution of $f(t)$ and $g(t)$ and is defined by

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

The identity function for convolution is the Dirac delta function:

$$\delta(t-a) * f(t) = f(t-a)\, H(t-a) \quad \Rightarrow \quad \delta(t) * f(t) = f(t)$$

Here is a summary of inverse Laplace transforms.

$F(s)$	$f(t)$
$\int_0^\infty e^{-st} f(t)\, dt$	$f(t)$
$\frac{1}{s^n} \quad (n \in \mathbb{N})$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
$\frac{1}{s-a}$	e^{at}
$\frac{1}{(s-a)^n} \quad (n \in \mathbb{N})$	$\frac{t^{n-1} e^{at}}{(n-1)!}$
e^{-as}	$\delta(t-a)$
$\frac{e^{-as}}{s}$	$H(t-a)$
$\frac{1}{s^2+\omega^2}$	$\frac{\sin \omega t}{\omega}$
$\frac{1}{(s-a)^2+\omega^2}$	$\frac{e^{at} \sin \omega t}{\omega}$
$\frac{1}{(s-a)^2-b^2}$	$\frac{e^{at} \sinh bt}{b}$
$\frac{(s-a)}{(s-a)^2+\omega^2}$	$e^{at} \cos \omega t$
$\frac{(s-a)}{(s-a)^2-b^2}$	$e^{at} \cosh bt$

$F(s)$	$f(t)$
$\frac{1}{s(s^2+\omega^2)}$	$\frac{1-\cos \omega t}{\omega^2}$
$\frac{1}{s^2(s^2+\omega^2)}$	$\frac{\omega t - \sin \omega t}{\omega^3}$
$\frac{1}{(s^2+\omega^2)^2}$	$\frac{\sin \omega t - \omega t \cos \omega t}{2\omega^3}$
$\frac{s}{(s^2+\omega^2)^2}$	$\frac{t \sin \omega t}{2\omega}$
$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$t \cos \omega t$
$\frac{1}{s} \tanh\left(\frac{as}{2}\right)$	Square wave, period $2a$, amplitude 1
$\frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$	Triangular wave, period $2a$, amplitude a
$\frac{b}{as^2} - \frac{b}{s(e^{as}-1)}$	Sawtooth wave, period a, amplitude b
$\{ s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0) \}$	$\frac{d^n f}{dt^n}$
$\frac{1}{s} F(s)$	$\int_0^t f(\tau)\, d\tau$
$\frac{dF}{ds}$	$-t\, f(t)$

<u>Example 1.09.1</u> (Example 1.08.2 again)

Find the general solution of the ODE

$$\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

The initial conditions are unknown, so let $a = y(0)$ and $b = y'(0)$.
Taking the Laplace transform of the initial value problem,

Example 1.09.2 (Example 1.07.1 again)

Find the complete solution to the initial value problem

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 26x = 10\,\delta(t-3)$$

(where $\delta(t-a)$ is the Dirac delta function),
together with the initial conditions $x(0) = x'(0) = 0$.

Let $X(s) = \mathcal{L}\{x(t)\}$ be the Laplace transform of the solution $x(t)$.

Taking the Laplace transform of the initial value problem,

1.10 Series Solutions of ODEs

If the functions $p(x)$, $q(x)$ and $r(x)$ in the ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = r(x)$$

are all analytic in some interval $x_o - h < x < x_o + h$ (and therefore possess Taylor series expansions around x_o with radii of convergence of at least h), then a series solution to the ODE around x_o with a radius of convergence of at least h exists:

$$y(x) = \sum_{n=0}^{\infty} a_n \left(x - x_o\right)^n , \qquad a_n = \frac{y^{(n)}(x_o)}{n!}$$

Example 1.10.1

Find a series solution as far as the term in x^3, to the initial value problem

$$\frac{d^2y}{dx^2} - x\frac{dy}{dx} + e^x y = 4 ; \qquad y(0) = 1, \quad y'(0) = 4$$

None of our previous methods apply to this problem.
The functions $-x$, e^x and 4 are all analytic everywhere.

The solution of this ODE, expressed as a power series, is

$$y(x) = y(0) + y'(0)\,x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \ldots$$

But $y(0) = 1$ and $y'(0) = 4$.
From the ODE,

Example 1.10.2

Find the general solution (as a power series about $x = 0$) to the ordinary differential equation

$$\frac{d^2 y}{dx^2} + x^2 y = 0$$

Let the general solution be $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then $y'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

Substitute into the ODE:

1.11 The Gamma Function

The gamma function $\Gamma(x)$ is a special function that will be needed in the solution of Bessel's ODE. $\Gamma(x)$ is a generalisation of the factorial function $n\,!$ from positive integers to most real numbers. For any positive integer *n*,
$n! = n\times(n-1)\times(n-2)\times\ldots\times 3\times 2\times 1$ (with 0! defined to be 1)

When x is a positive integer *n*, $\boxed{\Gamma(n) = (n-1)!}$

We know that $n! = n\times(n-1)!$
The gamma function has a similar recurrence relationship: $\Gamma(x+1) = x\cdot\Gamma(x)$

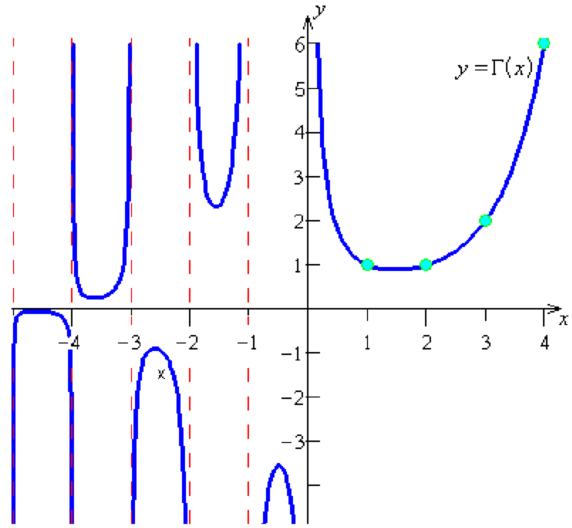
This allows $\Gamma(x)$ to be defined for non-integer negative x, using $\boxed{\Gamma(x) = \dfrac{\Gamma(x+1)}{x}}$

For example,
it can be shown that $\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$

$$\Rightarrow \quad \Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(+\tfrac{1}{2}\right)}{-\tfrac{1}{2}} = -2\sqrt{\pi} \qquad \Rightarrow \quad \Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-\tfrac{1}{2}\right)}{-\tfrac{3}{2}} = +\frac{4\sqrt{\pi}}{3}\text{, etc.}$$

$\Gamma(x)$ is infinite when x is a negative integer or zero. It is well defined for all other real numbers *x*.

In this graph of $y = \Gamma(x)$, values of the factorial function (at positive integer values of *x*) are highlighted.



There are several ways to define the gamma function, such as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt \qquad (x>0)$$

and

$$\Gamma(x) = \lim_{n\to\infty} \frac{n!\,n^x}{x(x+1)\ldots(x+n)}$$

A related special function is the **beta function**:

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Among the many results involving the gamma function are:

For the closed region V in the first octant, bounded by the coordinate planes and the surface $\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta + \left(\frac{z}{c}\right)^\gamma = 1$, with all constants positive,

$$I = \iiint_V x^{p-1} y^{q-1} z^{r-1} dx\,dy\,dz = \frac{a^p b^q c^r}{\alpha\beta\gamma} \cdot \frac{\Gamma\left(\frac{p}{\alpha}\right)\Gamma\left(\frac{q}{\beta}\right)\Gamma\left(\frac{r}{\gamma}\right)}{\Gamma\left(1+\frac{p}{\alpha}+\frac{q}{\beta}+\frac{r}{\gamma}\right)}$$

For the closed area A in the first quadrant, bounded by the coordinate axes and the curve $\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta = 1$, with all constants positive,

$$I = \iint_A x^{p-1} y^{q-1} dx\,dy = \frac{a^p b^q}{\alpha\beta} \cdot \frac{\Gamma\left(\frac{p}{\alpha}\right)\Gamma\left(\frac{q}{\beta}\right)}{\Gamma\left(1+\frac{p}{\alpha}+\frac{q}{\beta}\right)}$$

<u>Example 1.11.1</u>

Establish the formula for the area enclosed by an ellipse.

The Cartesian equation of a standard ellipse is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

1.12 Bessel and Legendre ODEs

Frobenius Series Solution of an ODE

If the ODE

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = F(x)$$

is such that $P(x_o) = 0$, but $(x - x_o)\dfrac{Q(x)}{P(x)}$, $(x - x_o)^2 \dfrac{R(x)}{P(x)}$ and $\dfrac{F(x)}{P(x)}$ are all analytic at x_o, then $x = x_o$ is a **regular singular point** of the ODE.

A Frobenius series solution of the ODE about $x = x_o$ exists:

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_o)^{n+r}$$

for some real number(s) r and for some set of values $\{ c_n \}$.

Example 1.12.1

Find a solution of Bessel's ordinary differential equation of order v, ($v \geq 0$),

$$x^2 y'' + x\,y' + \left(x^2 - v^2\right) y = 0$$

Example 1.12.1 (continued)

Example 1.12.1 (continued)

One Frobenius solution of Bessel's equation of order v is therefore

$$y(x) = c_0 \Gamma(\nu+1) \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k! \Gamma(\nu+k+1)} x^{2k+\nu} = c_0 \Gamma(\nu+1) J_\nu(x)$$

where $J_\nu(x)$ is the **Bessel function of the first kind of order ν**.

It turns out that the Frobenius series found by setting $r = -v$ generates a second linearly independent solution $J_{-v}(x)$ of the Bessel equation only if v is not an integer.

The Bessel ODE in standard form,

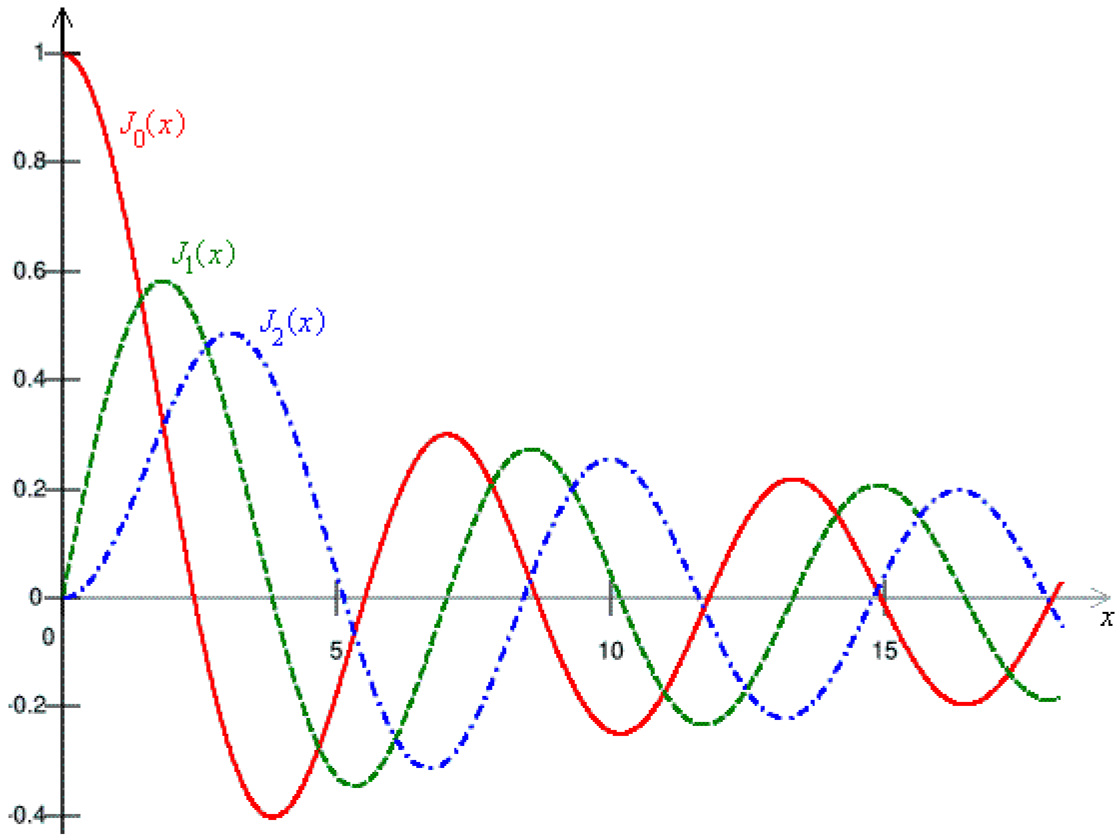
$$x^2 y'' + x y' + \left(x^2 - \nu^2\right) y = 0$$

has the general solution

$$y(x) = A J_\nu(x) + B Y_\nu(x)$$

unless ν is not an integer, in which case $Y_\nu(x)$ can be replaced by $J_{-\nu}(x)$.
$Y_\nu(x)$ is the Bessel function of the second kind.
When ν is an integer, $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$.

Graphs of Bessel functions of the first kind, for $\nu = 0, 1, 2$:



The series expression for the Bessel function of the first kind is

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

This function has a simpler form when ν is an odd half-integer. For example,

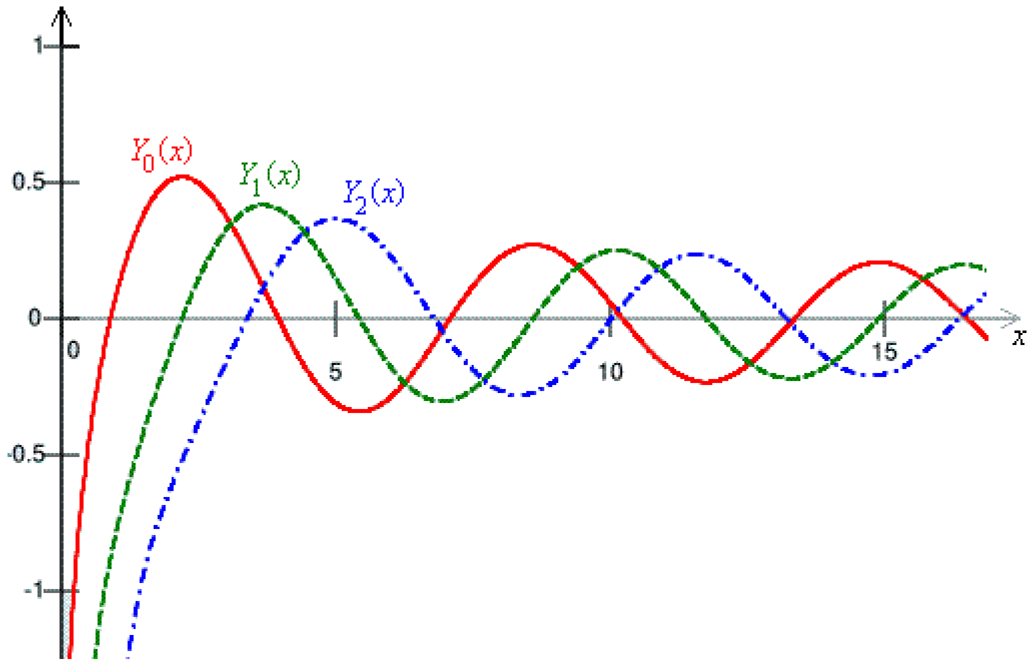
$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x , \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

The Bessel function of the second kind is

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

$Y_\nu(x)$ is unbounded as $x \to 0$: $\lim_{x\to 0^+} Y_\nu(x) = -\infty$

Bessel functions of the second kind (all of which have a singularity at $x = 0$):



Bessel functions arise frequently in situations where cylindrical or spherical polar coordinates are used.

A generalised Bessel ODE is

$$x^2 \frac{d^2y}{dx^2} + (1-2a)x\frac{dy}{dx} + \left(b^2c^2x^{2c} + \left(a^2 - c^2\nu^2\right)\right)y = 0$$

whose general solution is

$$y(x) = x^a\left(A\,J_\nu\left(bx^c\right) + B\,Y_\nu\left(bx^c\right)\right)$$

For a generalised Bessel ODE with $a \geq 0$, whenever the solution must remain bounded as $x \to 0$, the general solution simplifies to $y(x) = A\,x^a J_\nu\left(bx^c\right)$.

Example 1.12.2

Find a Maclaurin series solution to Legendre's ODE

$$\left(1-x^2\right)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + p\left(p+1\right)y = 0$$

in the case when p is a non-negative integer.

Example 1.12.2 (continued)

If we set $a_1 = 0$ when p is even, then the series solution terminates as a p^{th} order polynomial (and therefore converges for all x).

If we set $a_0 = 0$ when p is odd, then the series solution terminates as a p^{th} order polynomial (and therefore converges for all x).

With suitable choices of a_0 and a_1, so that $P_n(1) = 1$,
we have the set of **Legendre polynomials**:
$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$,
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$,
$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$, etc.
Each $P_n(x)$ is a solution of Legendre's ODE with $p = n$.

Rodrigues' formula generates all of the Legendre polynomials:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(\left(x^2 - 1 \right)^n \right)$$

Among the properties of Legendre polynomials is their orthogonality on [–1, 1]:

$$\int_{-1}^{+1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & (m \neq n) \\ \dfrac{2}{2n+1} & (m = n) \end{cases}$$

[Space for any additional notes]

END OF CHAPTER 1