7. Fourier Series and Fourier Transforms

Fourier series have multiple purposes, including the provision of series solutions to some linear partial differential equations with boundary conditions (as will be reviewed in Chapter 8). Fourier transforms are often used to extract frequency information from time series data. For lack of time in this course, only a brief introduction is provided here.

Sections in this Chapter:

Sections for reference only, ***not*** examinable in this course:

7.01 Orthogonal Functions

The inner product (or scalar product or dot product) of two vectors **u** and **v** is defined in Cartesian coordinates in $\mathbb{R}^3$ by

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \sum_{k=1}^{3} u_k v_k = u_1 v_1 + u_2 v_2 + u_3 v_3$$

The inner product possesses the four properties:

Commutative: $\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \vec{\mathbf{v}} \bullet \vec{\mathbf{u}}$

Scalar multiplication: $(k\vec{\mathbf{u}}) \bullet \vec{\mathbf{v}} = k(\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}), \quad k \in \mathbb{R}$

Positive definite: $\vec{\mathbf{u}} \bullet \vec{\mathbf{u}} \begin{cases} = 0 & (\text{if } \vec{\mathbf{u}} = \vec{\mathbf{0}}) \\ > 0 & (\text{if } \vec{\mathbf{u}} \neq \vec{\mathbf{0}}) \end{cases}$

Associative: $\vec{\mathbf{u}} \bullet (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}} + \vec{\mathbf{u}} \bullet \vec{\mathbf{w}}$

Vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are orthogonal if and only iff $\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = 0$.
A pair of non-zero orthogonal vectors intersects at right angles.

The inner product of two real-valued functions f_1 and f_2 on an interval $[a, b]$ may be defined in a way that also possesses these four properties:

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) \, dx$$

Two functions f_1 and f_2 are said to be **orthogonal** on an interval $[a, b]$ if their inner product is zero:

$$(f_1, f_2) = \int_a^b f_1(x) f_2(x) \, dx = 0$$

A set of real-valued functions $\{ \phi_0(x), \phi_1(x), \phi_2(x), \cdots, \phi_n(x) \}$ is orthogonal on the interval $[a, b]$ if the inner product of any two of them is zero:

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x) \phi_n(x) \, dx = 0 \quad (m \neq n)$$

If, in addition, the inner product of any function in the set with itself is unity, then the set is **orthonormal**:

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x) \phi_n(x) \, dx = \delta_{mn} = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases}$$

where δ_{mn} is the "Kronecker delta" symbol.

Just as any vector in $\mathbb{R}^3$ may be represented by a linear combination of the three Cartesian basis vectors, (which form the orthonormal set {**i**, **j**, **k**}), so a real valued function $f(x)$ defined on $[a, b]$ may be written as a linear combination of the elements of an infinite orthonormal set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ on $[a, b]$:

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots$$

To find the coefficients c_n, multiply $f(x)$ by $\phi_n(x)$ and integrate over $[a, b]$:

$$\int_a^b f(x)\phi_n(x)\,dx = c_0\int_a^b \phi_0(x)\phi_n(x)\,dx + c_1\int_a^b \phi_1(x)\phi_n(x)\,dx + \ldots$$

$$= \sum_{m=0}^{\infty} c_m \int_a^b \phi_m(x)\phi_n(x)\,dx$$

But the $\{\phi_n(x)\}$ are an orthonormal set. Therefore all but one of the terms in the infinite series are zero. The exception is the term for which $m = n$, where the integral is unity. Therefore

$$c_n = \int_a^b f(x)\phi_n(x)\,dx$$

and

$$f(x) = \sum_{n=0}^{\infty}\left(\left(\int_a^b f(x)\phi_n(x)\,dx\right)\phi_n(x)\right)$$

If the set is orthogonal but not orthonormal, then the form for $f(x)$ changes to

$$f(x) = \sum_{n=0}^{\infty}\left(\left(\frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}\right)\phi_n(x)\right)$$

The orthogonal set $\{\phi_n(x)\}$ is **complete** if the only function that is orthogonal to all members of the set is the zero function $f(x) \equiv 0$. An expansion of every function $f(x)$ in terms of an orthogonal or orthonormal set $\{\phi_n(x)\}$ is not possible if $\{\phi_n(x)\}$ is not complete.

Also note that a generalised form of an inner product can be defined using a weighting function $w(x)$, so that, in terms of a complete orthogonal set $\{\phi_n(x)\}$,

$$f(x) = \sum_{n=0}^{\infty}\left(\left(\frac{\int_a^b w(x) f(x)\phi_n(x)\,dx}{\int_a^b w(x)\phi_n^2(x)\,dx}\right)\phi_n(x)\right)$$

We shall usually be concerned with the case $w(x) \equiv 1$ only.

Example 7.01.1

Show that the set { sin nx } $(n \in \mathbb{N})$ is orthogonal but not orthonormal and not complete on $[-\pi, +\pi]$.

7.02 Definitions of Fourier Series

Example 7.02.1

Show that the set $\left\{1, \left\{\cos\left(\frac{n\pi x}{L}\right)\right\}, \left\{\sin\left(\frac{n\pi x}{L}\right)\right\}\right\}, \quad (n \in \mathbb{N})$ is orthogonal but not orthonormal on $[-L, L]$.

Example 7.02.1 (continued)

Using the results from Example 7.02.1, we can express most real-valued functions $f(x)$ defined on $(-L, L)$, in terms of an infinite series of trigonometric functions:

The **Fourier series** of $f(x)$ on the interval $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

The $\{a_n, b_n\}$ are the Fourier coefficients of $f(x)$.

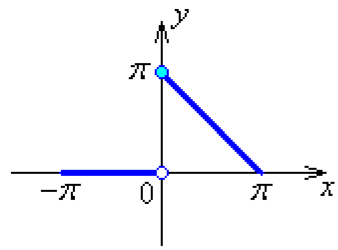
Note that the cosine functions (and the function 1) are even, while the sine functions are odd.

If $f(x)$ is even ($f(-x) = +f(x)$ for all x), then $b_n = 0$ for all n, leaving a Fourier cosine series (and perhaps a constant term) only for $f(x)$.

If $f(x)$ is odd ($f(-x) = -f(x)$ for all x), then $a_n = 0$ for all n, leaving a Fourier sine series only for $f(x)$.

Example 7.02.2

Expand $f(x) = \begin{cases} 0 & (-\pi < x < 0) \\ \pi - x & (0 \le x < +\pi) \end{cases}$ in a Fourier series.



Example 7.02.2 (Additional Notes – also see "www.engr.mun.ca/~ggeorge/9420/demos/")

The first few partial sums in the Fourier series

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left(\frac{1-(-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right) \qquad (-\pi < x < +\pi)$$

are

$$S_0 = \frac{\pi}{4}$$

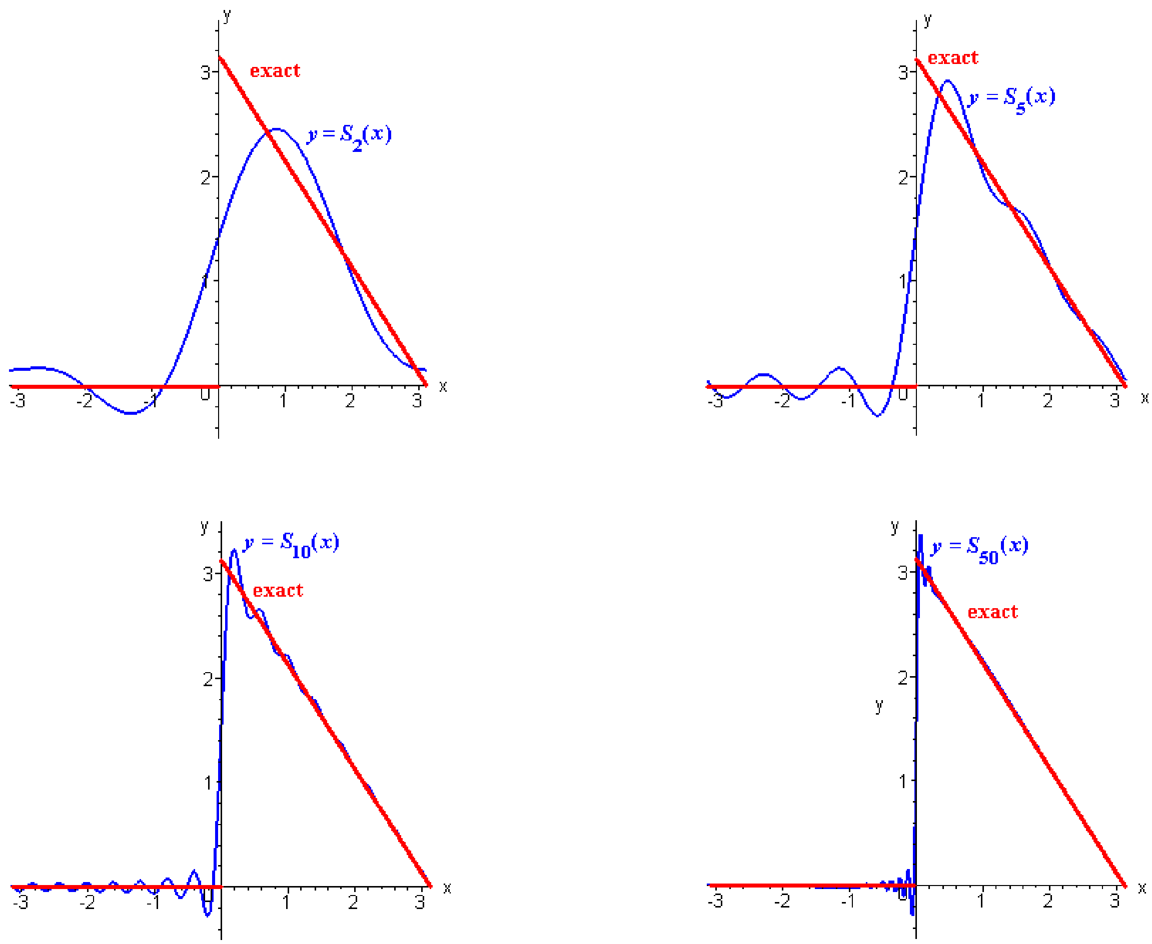
$$S_1 = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x$$

$$S_2 = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x + \frac{1}{2} \sin 2x$$

$$S_3 = \frac{\pi}{4} + \frac{2}{\pi} \cos x + \sin x + \frac{1}{2} \sin 2x + \frac{2}{9\pi} \cos 3x + \frac{1}{3} \sin 3x$$

and so on.

The graphs of successive partial sums approach $f(x)$ more closely, except in the vicinity of any discontinuities, (where a systematic overshoot occurs, the **Gibbs phenomenon**).

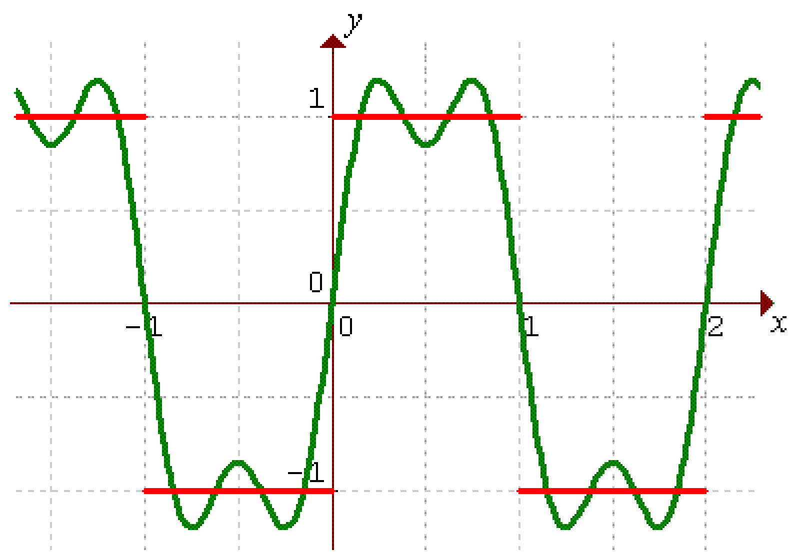


Example 7.02.3

Find the Fourier series expansion for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

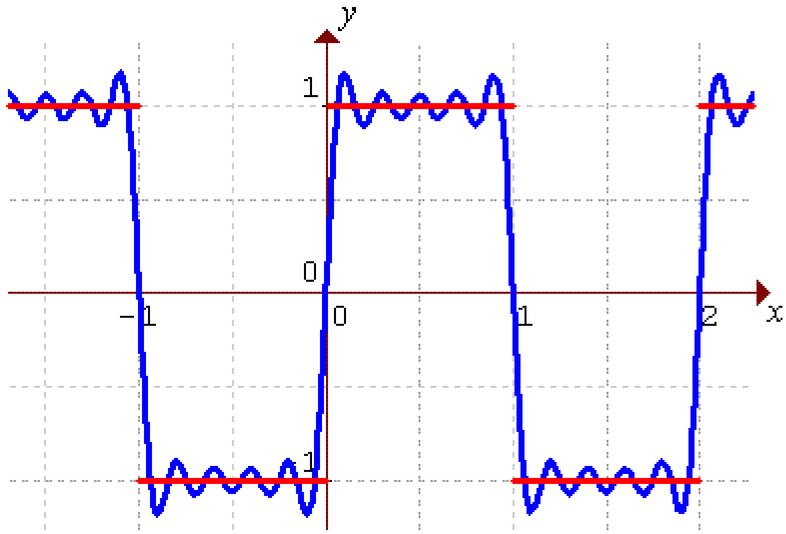
The graphs of the third and ninth partial sums (containing two and five non-zero terms respectively) are displayed here, together with the exact form for $f(x)$, with a **periodic extension** beyond the interval (–1, +1) that is appropriate for the square wave.



$y = S_3(x)$

Example 7.02.3 (continued)

$y = S_9(x)$

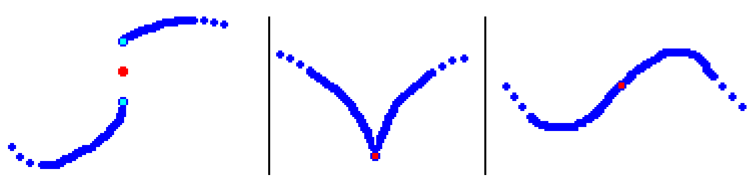


Convergence

At all points $x = x_o$ in $(-L, L)$ where $f(x)$ is continuous and is either differentiable or the limits $\lim_{x \to x_o^-} f'(x)$ and $\lim_{x \to x_o^+} f'(x)$ both exist, the Fourier series converges to $f(x)$.

At finite discontinuities, (where the limits $\lim_{x \to x_o^-} f'(x)$ and $\lim_{x \to x_o^+} f'(x)$ both exist), the Fourier series converges to $\dfrac{f(x_o -) + f(x_o +)}{2}$,

(using the abbreviations $f(x_o -) = \lim_{x \to x_o^-} f(x)$ and $f(x_o +) = \lim_{x \to x_o^+} f(x)$).



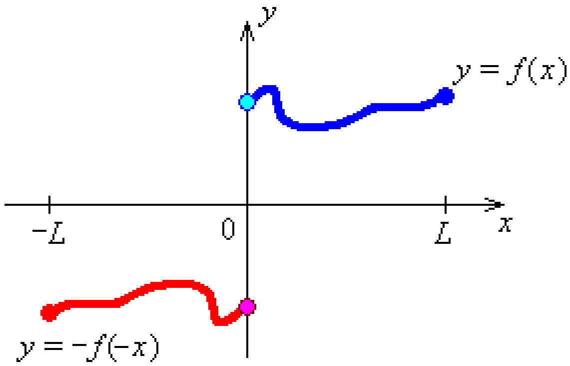
$f(x)$ not continuous at $x = x_o$ | continuous but not differentiable | continuous and differentiable

In all cases, the Fourier series at $x = x_o$ converges to $\dfrac{f(x_o -) + f(x_o +)}{2}$ (the red dot).

7.03 Half-Range Fourier Series

A Fourier series for $f(x)$, valid on $[0, L]$, may be constructed by extension of the domain to $[-L, L]$.

An odd extension leads to a **Fourier sine series**:

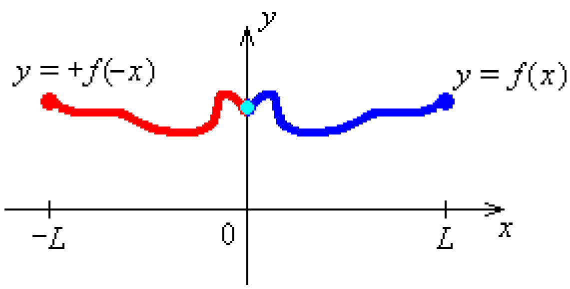


$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

An even extension leads to a **Fourier cosine series**:



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

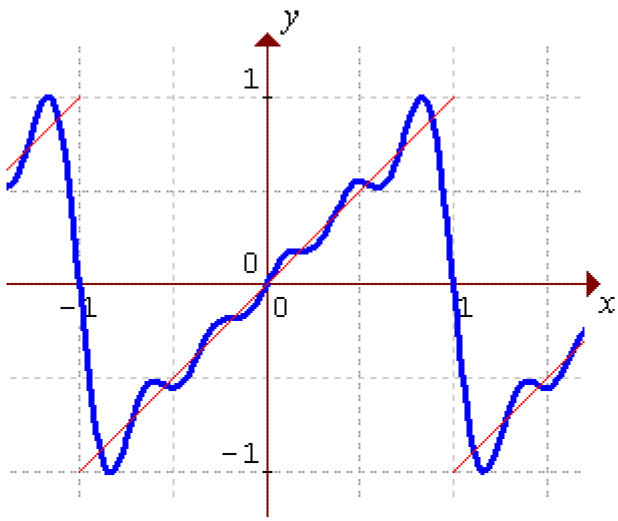
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$$

and there is automatic continuity of the Fourier cosine series at $x = 0$ and at $x = \pm L$.

Example 7.03.1

Find the Fourier sine series and the Fourier cosine series for $f(x) = x$ on $[0, 1]$.

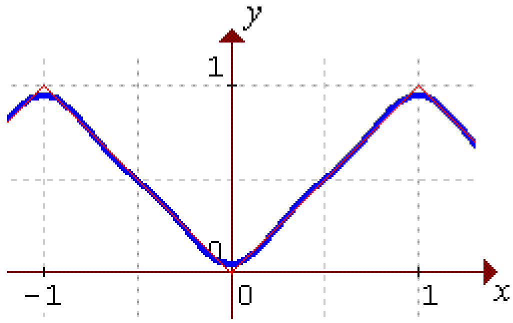
Fifth order partial sum of the Fourier sine series for $f(x) = x$ on $[0, 1]$



Example 7.03.1 (continued)

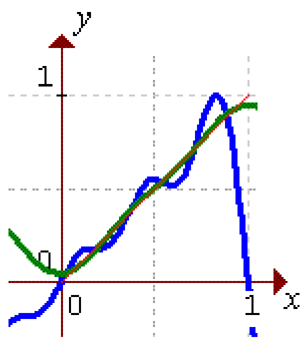
Example 7.03.1 (continued)

Third order partial sum of the Fourier cosine series for $f(x) = x$ on [0, 1]



Note how rapid the convergence is for the cosine series compared to the sine series.

$y = S_3(x)$ for cosine series and $y = S_5(x)$ for sine series for $f(x) = x$ on [0, 1]



7.04 Frequency Spectrum

The Fourier series may be combined into a single cosine series.
Let p be the fundamental period. If the function $f(x)$ is not periodic at all on [–L, L], then the fundamental period of the extension of $f(x)$ to the entire real line is $p = 2L$.

Define the fundamental frequency $\omega = \frac{2\pi}{p} = \frac{\pi}{L}$.
The Fourier series for $f(x)$ on [–L, L] is, from page 7.07,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega x) + b_n \sin(n\omega x)\right)$$

where

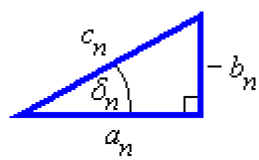
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\omega x)\, dx, \quad (n = 0, 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\omega x)\, dx, \quad (n = 1, 2, 3, \ldots)$$

Let the **phase** angle δ_n be such that $\tan\delta_n = -\frac{b_n}{a_n}$,

so that $\sin\delta_n = -\frac{b_n}{c_n}$ and $\cos\delta_n = +\frac{a_n}{c_n}$



where the **amplitude** is $c_n = \sqrt{a_n^{\,2} + b_n^{\,2}}$.
Also, in the trigonometric identity $\cos A \cos B - \sin A \sin B \equiv \cos(A+B)$,
replace A by $n\omega x$ and B by δ_n. Then

$$a_n \cos(n\omega x) + b_n \sin(n\omega x) = (c_n \cos\delta_n)\cos(n\omega x) - (c_n \sin\delta_n)\sin(n\omega x)$$

$$= c_n \cos(n\omega x + \delta_n), \quad \text{where} \quad \boxed{\omega = \frac{2\pi}{p} = \frac{\pi}{L}}, \quad \boxed{c_n = \sqrt{a_n^{\,2} + b_n^{\,2}}} \quad \text{and} \quad \boxed{\tan\delta_n = -\frac{b_n}{a_n}}$$

Therefore the phase angle or **harmonic** form of the Fourier series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega x + \delta_n)}$$

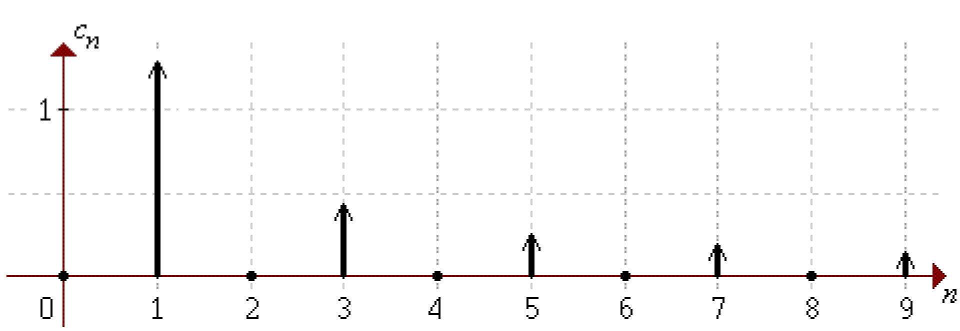
Example 7.04.1

Plot the frequency spectrum for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

From Example 7.02.3, the Fourier series for the standard square wave is

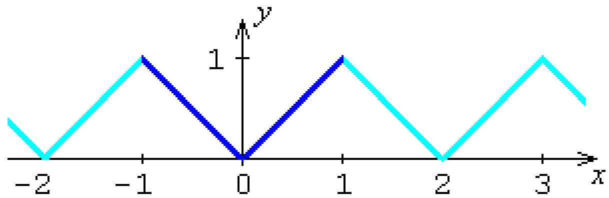
$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\sin n\pi x\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\sin(2k-1)\pi x\right)$$



Example 7.04.2

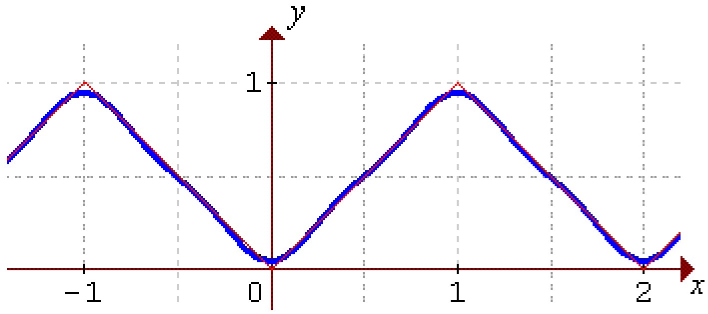
Plot the frequency spectrum for the periodic extension of

$$f(x) = |x|, \quad -1 < x < 1$$



Example 7.04.2 (continued)

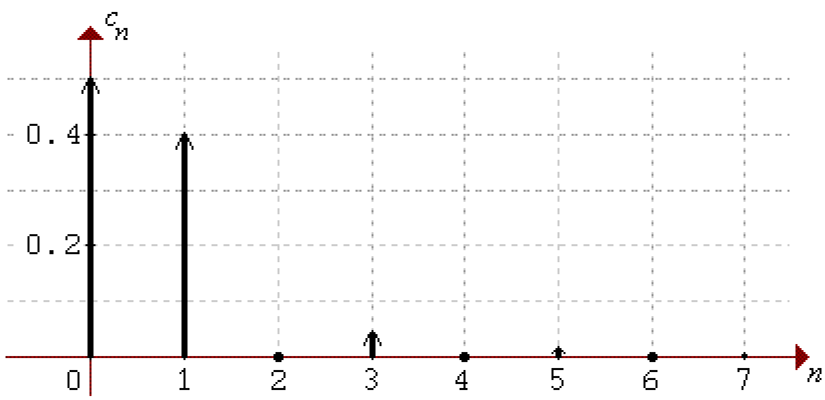
(which converges very rapidly, as this third partial sum demonstrates)



The harmonic amplitudes are

$$c_n = \begin{cases} \dfrac{1}{2} & (n=0) \\ \dfrac{2\left(1-(-1)^n\right)}{(n\pi)^2} & (n \in \mathbb{N}) \end{cases} = \begin{cases} \dfrac{1}{2} & (n=0) \\ 0 & (n \text{ even}, n \geq 2) \\ \dfrac{4}{(n\pi)^2} & (n \text{ odd}) \end{cases}$$

The frequencies therefore diminish rapidly:



7.05 Complex Fourier Series [for reference only, ***not*** examinable]

Note that the Euler identity $e^{j\theta} \equiv \cos\theta + j\sin\theta$ leads to

$$\cos n\omega x \equiv \frac{e^{jn\omega x} + e^{-jn\omega x}}{2} \quad \text{and} \quad \sin n\omega x \equiv \frac{e^{jn\omega x} - e^{-jn\omega x}}{2j}$$

The Fourier series for the periodic extension of $f(x)$ from the original interval $[a, a+p)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(n\omega x) + b_n \sin(n\omega x) \right),$$

where $\omega = \dfrac{2\pi}{p}$ and

$$a_n = \frac{2}{p}\int_a^{a+p} f(x)\cos(n\omega x)\,dx, \quad b_n = \frac{2}{p}\int_a^{a+p} f(x)\sin(n\omega x)\,dx$$

The Fourier series becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \frac{e^{jn\omega x} + e^{-jn\omega x}}{2} + b_n \frac{e^{jn\omega x} - e^{-jn\omega x}}{2j} \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\left(\frac{a_n - jb_n}{2}\right) e^{jn\omega x} + \left(\frac{a_n + jb_n}{2}\right) e^{-jn\omega x} \right]$$

$$= d_0 + \sum_{n=1}^{\infty}\left[d_n e^{jn\omega x} + d_n^* e^{-jn\omega x} \right]$$

where $d_n = \dfrac{a_n - jb_n}{2}$ for all non-negative integers n.

$$d_n = \frac{a_n - jb_n}{2} = \frac{1}{2}\left[\frac{2}{p}\int_a^{a+p} f(x)\cos(n\omega x)\,dx - j\frac{2}{p}\int_a^{a+p} f(x)\sin(n\omega x)\,dx \right]$$

$$= \frac{1}{p}\int_a^{a+p} f(x)\left[\cos(n\omega x) - j\sin(n\omega x)\right]dx = \frac{1}{p}\int_a^{a+p} f(x)\, e^{-jn\omega x}\,dx$$

$$\Rightarrow \quad d_n^* = \frac{1}{p}\int_a^{a+p} f(x)\, e^{+jn\omega x}\,dx = \frac{1}{p}\int_a^{a+p} f(x)\, e^{-j(-n)\omega x}\,dx = d_{-n}$$

Therefore the entire Fourier series may be re-written more concisely as

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} d_n e^{jn\omega x}, \quad \text{where} \quad d_n = \frac{1}{p}\int_a^{a+p} f(x) e^{-jn\omega x}\,dx}$$

The numbers $\{ \ldots, d_{-2}, d_{-1}, d_0, d_1, d_2, \ldots \}$ are the complex Fourier coefficients of f.
The harmonic amplitudes of f are just the magnitudes $\{ |d_n| \}$ for $n = 0, \pm 1, \pm 2, \ldots$.

Example 7.05.1

Find the complex Fourier series expansion for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

$$p = 2 \quad \Rightarrow \quad \omega = \frac{2\pi}{2} = \pi$$

$$d_0 = \frac{1}{2}\int_{-1}^{1} f(x)\,dx = \frac{1}{2}\left(\int_{-1}^{0} -1\,dx + \int_{0}^{1} +1\,dx\right) = \frac{-1+1}{2} = 0$$

For $n \neq 0$,

$$d_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-jn\pi x}\,dx = \frac{1}{2}\left(\int_{-1}^{0} -e^{-jn\pi x}\,dx + \int_{0}^{1} +e^{-jn\pi x}\,dx\right)$$

$$= \frac{1}{2}\left(\left[\frac{-e^{-jn\pi x}}{-jn\pi}\right]_{-1}^{0} + \left[\frac{+e^{-jn\pi x}}{-jn\pi}\right]_{0}^{1}\right) = \frac{j}{2n\pi}\left(\left(-1+e^{jn\pi}\right) + \left(e^{-jn\pi} - 1\right)\right)$$

$$= \frac{j}{n\pi}\left(\left(\frac{e^{jn\pi} + e^{-jn\pi}}{2}\right) - 1\right) = \frac{j}{n\pi}\left(\cos(n\pi) - 1\right) = \frac{j}{n\pi}\left((-1)^n - 1\right)$$

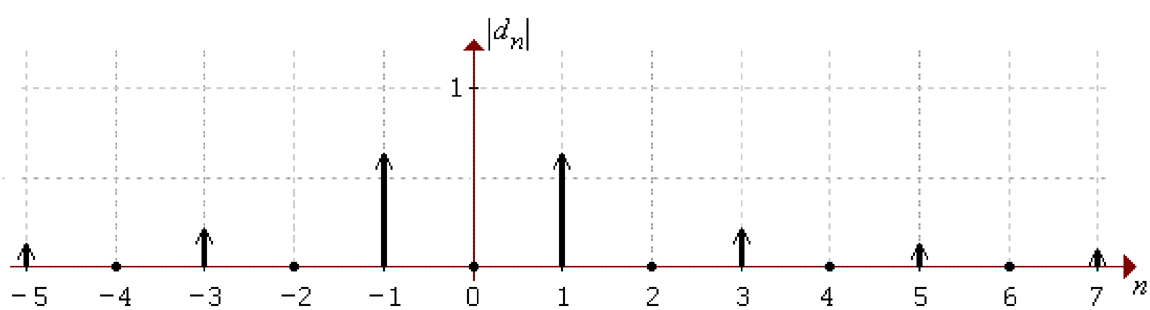
or

$$d_n = \begin{cases} 0 & (n \text{ even}) \\ -\dfrac{2j}{n\pi} & (n \text{ odd}) \end{cases}$$

In its most compact form, the complex Fourier series for the square wave is

$$f(x) = +\frac{2}{j\pi}\sum_{k=-\infty}^{\infty} \frac{1}{2k-1} e^{j(2k-1)\pi x}$$

The amplitude spectrum is the set $\left(n\omega, |d_n|\right) = \left(n\pi, \dfrac{2}{|n|\pi}\right)$, for odd n only.



7.06 Fourier Integrals [for reference only, ***not*** examinable]

The Fourier series may be extended from (–*L*, *L*) to the entire real line.

Let $\omega_n = \frac{n\pi}{L} \quad \Rightarrow \quad \omega_n - \omega_{n-1} = \frac{\pi}{L} = \Delta\omega \quad \Rightarrow \quad \frac{1}{L} = \frac{\Delta\omega}{\pi}$

The Fourier series for $f(x)$ on (–*L*, *L*) is

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt \; +$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{L}\left(\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt\right)\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{L}\left(\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt\right)\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$\Rightarrow \quad f(x) = \frac{\Delta\omega}{2\pi}\int_{-L}^{L} f(t)\,dt \; +$$

$$\sum_{n=1}^{\infty}\left(\frac{\Delta\omega}{\pi}\left(\int_{-L}^{L} f(t)\cos(\omega_n t)\,dt\right)\cos(\omega_n x) + \frac{\Delta\omega}{\pi}\left(\int_{-L}^{L} f(t)\sin(\omega_n t)\,dt\right)\sin(\omega_n x)\right)$$

Now take the limit as $L \to \infty \quad \Rightarrow \quad \Delta\omega \to 0$:

The first integral converges to some finite number, so the first term vanishes in the limit.
The summation becomes an integral over all frequencies in the limit:

$$f(x) \to 0 \; +$$

$$\int_{0}^{\infty}\left(\frac{1}{\pi}\left(\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt\right)\cos(\omega x)\,d\omega + \frac{1}{\pi}\left(\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt\right)\sin(\omega x)\,d\omega\right)$$

Therefore the Fourier integral of $f(x)$ is

$$\boxed{f(x) = \int_{0}^{\infty}\left(A_\omega\cos(\omega x) + B_\omega\sin(\omega x)\right)d\omega}$$

where the Fourier integral coefficients are

$$\boxed{A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\,dt \quad \text{and} \quad B_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin(\omega t)\,dt}$$

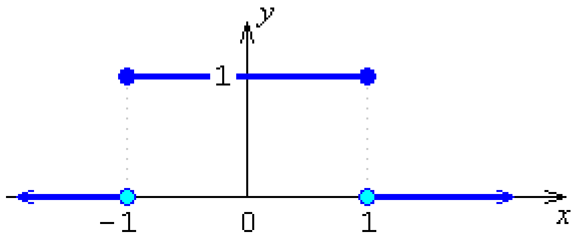
provided $\int_{-\infty}^{\infty}\left|f(x)\right|dx$ converges.

Example 7.06.1

Find the Fourier integral of

$$f(x) = \begin{cases} 1 & (-1 \le x \le +1) \\ 0 & (\text{otherwise}) \end{cases}$$

From the functional form and from the graph of $f(x)$, it is obvious that $f(x)$ is piecewise smooth and that $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges to the value



$$A_\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos(\omega t)\, dt = \frac{1}{\pi}\int_{-1}^{1} \cos(\omega t)\, dt = \frac{1}{\pi}\left[\frac{\sin(\omega t)}{\omega}\right]_{-1}^{1} = \frac{2\sin\omega}{\pi\omega}$$

The function $f(x)$ is even $\Rightarrow$ $B_\omega = 0$ for all ω.

Therefore the Fourier integral of $f(x)$ is

$$\boxed{f(x) = \int_0^\infty \frac{2\sin\omega}{\pi\omega}\cos(\omega x)\, d\omega}$$

It also follows that

$$\int_0^\infty \frac{2\sin\omega}{\pi\omega}\cos(\omega x)\, d\omega = \begin{cases} 1 & (-1 < x < 1) \\ \frac{1}{2} & (x = \pm 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Fourier series and Fourier integrals can be used to evaluate summations and definite integrals that would otherwise be difficult or impossible to evaluate. For example, setting $x = 0$ in Example 7.06.1, we find that

$$\int_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$$

7.07 Complex Fourier Integrals [for reference only, ***not*** examinable]

$$f(x) = \int_0^\infty \left(A_\omega \cos(\omega x) + B_\omega \sin(\omega x)\right) d\omega$$

$$= \int_0^\infty \left(A_\omega \frac{e^{j\omega x} + e^{-j\omega x}}{2} + B_\omega \frac{e^{j\omega x} - e^{-j\omega x}}{2j}\right) d\omega$$

$$= \int_0^\infty \left(\left(\frac{A_\omega - jB_\omega}{2}\right) e^{j\omega x} + \left(\frac{A_\omega + jB_\omega}{2}\right) e^{-j\omega x}\right) d\omega$$

$$= \int_0^\infty \left(C_\omega e^{j\omega x} + C_\omega^* e^{-j\omega x}\right) d\omega, \quad \text{where} \quad C_\omega = \frac{A_\omega - jB_\omega}{2}$$

But $$C_\omega^* = \frac{1}{\pi}\int_{-\infty}^\infty f(t) \frac{\cos(\omega t) + j\sin(\omega t)}{2}\, dt$$

$$= \frac{1}{\pi}\int_{-\infty}^\infty f(t) \frac{\cos(-\omega t) - j\sin(-\omega t)}{2}\, dt = C_{-\omega}$$

and $$\int_0^\infty \left(C_{-\omega} e^{-j\omega x}\right) d\omega = \int_{-\infty}^0 \left(C_{+\omega} e^{+j\omega x}\right) d\omega$$

By convention, the factor of $\frac{1}{2\pi}$ is extracted from the coefficients.
Therefore the complex Fourier integral of $f(t)$ is

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^\infty C_\omega e^{j\omega t}\, dt$$

where the complex Fourier integral coefficients are

$$C_\omega = \int_{-\infty}^\infty f(t) e^{-j\omega t}\, dt$$

(which is also the **Fourier transform** of f, $f(\omega) = \mathcal{F}[f(t)](\omega)$).

ω is the **frequency** of the signal $f(t)$.

7.08 Some Fourier Transforms [for reference only, ***not*** examinable]

If $\hat{f}(\omega)$ is the Fourier transform of $f(t)$, then

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt$$

and the inverse Fourier transform is

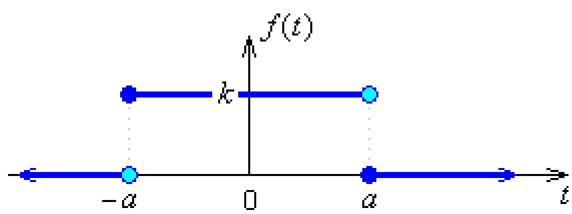
$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{j\omega t} \, d\omega$$

<u>Example 7.08.1</u>

Find the Fourier transform of the pulse function

$$f(t) = k\left(H(t+a) - H(t-a)\right) = \begin{cases} k & (-a \le t < a) \\ 0 & (\text{otherwise}) \end{cases}$$

From the functional form and from the graph of $f(t)$, it is obvious that $f(t)$ is piecewise smooth and that $\int_{-\infty}^{\infty} \left| f(t) \right| dt$ converges to the value $2ak$.



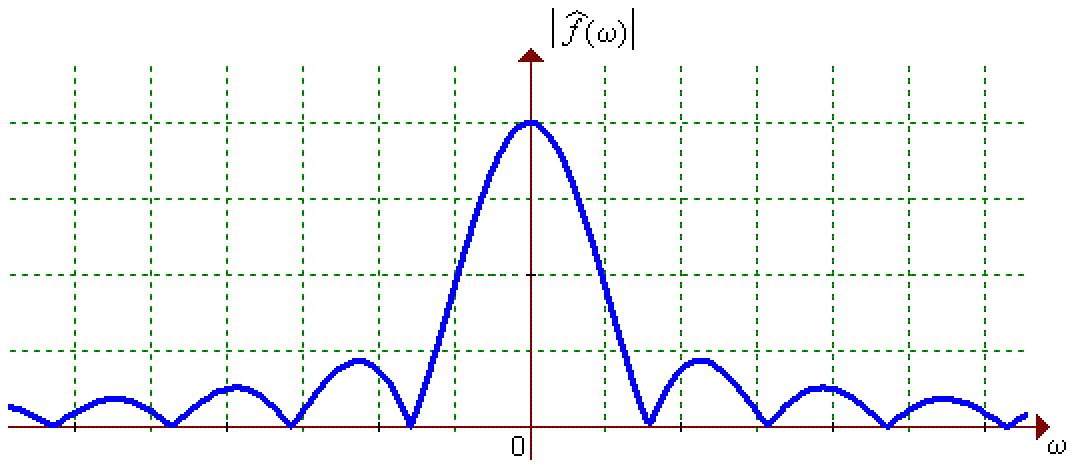
$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt = \int_{-a}^{a} k e^{-j\omega t} \, dt = \left[\frac{k e^{-j\omega t}}{-j\omega} \right]_{-a}^{a} = \frac{2k}{\omega} \cdot \frac{\left(-e^{-j\omega a} + e^{+j\omega a}\right)}{2j}$$

Therefore

$$\hat{f}(\omega) = 2k \frac{\sin(a\omega)}{\omega}$$

Example 7.08.1 (continued)

The transform is real. Therefore the frequency spectrum follows quickly:

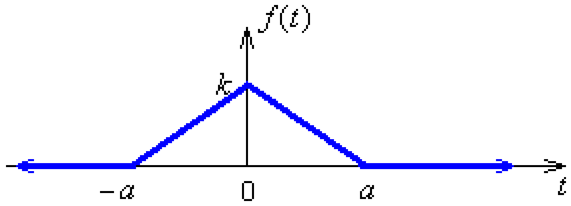


Example 7.08.2

Find the Fourier transform of the triangle function

$$f(t) = \begin{cases} \frac{k}{a}\left(a - |t|\right) & (-a \le t \le a) \\ 0 & (\text{otherwise}) \end{cases}$$

From the functional form and from the graph of $f(t)$, it is obvious that $f(t)$ is piecewise smooth and that $\int_{-\infty}^{\infty} |f(t)|\, dt$ converges to the value ak.



$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t}\, dt = \frac{k}{a}\int_{-a}^{0} (a+t) e^{-j\omega t}\, dt + \frac{k}{a}\int_{0}^{a} (a-t) e^{-j\omega t}\, dt$$

$$= \frac{k}{a}\left[\left(\frac{a+t}{-j\omega} - \frac{1}{(-j\omega)^2}\right) e^{-j\omega t}\right]_{-a}^{0} + \frac{k}{a}\left[\left(\frac{a-t}{-j\omega} - \frac{-1}{(-j\omega)^2}\right) e^{-j\omega t}\right]_{0}^{a}$$

$$= \frac{k}{a}\left[\left(\frac{ja}{\omega} + \frac{1}{\omega^2}\right) - \left(0 + \frac{1}{\omega^2}\right) e^{j\omega a} + \left(0 - \frac{1}{\omega^2}\right) e^{-j\omega a} - \left(\frac{ja}{\omega} - \frac{1}{\omega^2}\right)\right]$$

$$= \frac{k}{a\omega^2}\left(2 - e^{j\omega a} - e^{-j\omega a}\right) = \frac{2k}{a\omega^2}\left(1 - \cos(a\omega)\right)$$

Therefore

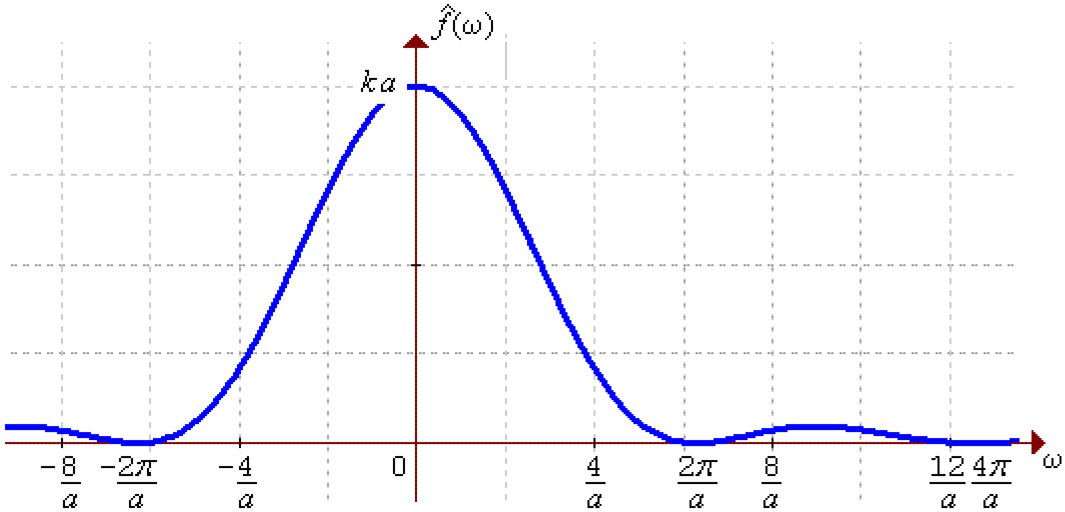
$$\boxed{\hat{f}(\omega) = \frac{2k}{a} \cdot \frac{1 - \cos(a\omega)}{\omega^2}}$$

An equivalent form is

$$\boxed{\hat{f}(\omega) = ak \cdot \left(\frac{\sin\left(\frac{a\omega}{2}\right)}{\left(\frac{a\omega}{2}\right)}\right)^2}$$

Example 7.08.2 (continued)

The Fourier transform of the triangle function happens to be real and non-negative, so that it is its own frequency spectrum $\left|\hat{f}(\omega)\right| = \hat{f}(\omega)$.



Example 7.08.3

Time Shift Property:

Let $\hat{f}(\omega) = \mathscr{F}\{f(t)\}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt$.

Then

$$\mathscr{F}\{f(t-t_o)\}(\omega) = \int_{-\infty}^{\infty} f(t-t_o)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} f(u)e^{-j\omega(u+t_o)}\,du$$

$$= e^{-j\omega t_o}\int_{-\infty}^{\infty} f(u)e^{-j\omega u}\,du = e^{-j\omega t_o}\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\,dt = e^{-j\omega t_o}\hat{f}(\omega)$$

Therefore the time shift property of Fourier transforms is

$$\boxed{\mathscr{F}\{f(t-t_o)\}(\omega) = e^{-j\omega t_o}\mathscr{F}\{f(t)\}(\omega)}$$

There are many other properties of Fourier transforms to explore, (such as sampling, windowing, filtering, Fourier [Co]sine Transforms, discrete Fourier transforms and Fast Fourier transforms) and their applications to signal analysis. However, there is insufficient time in this course to proceed beyond this introduction.

7.09 Summary of Fourier Transforms [for reference only, ***not*** examinable]

In this table, $a > 0$.

$f(t)$	$\hat{f}(\omega) = \mathscr{F}\{f(t)\}(\omega)$
$e^{-a\lvert t\rvert}$	$\dfrac{2a}{a^2+\omega^2}$
$t\,e^{-a\lvert t\rvert}$	$\dfrac{-4a\,j\omega}{\left(a^2+\omega^2\right)^2}$
$\lvert t\rvert\,e^{-a\lvert t\rvert}$	$\dfrac{2\left(a^2-\omega^2\right)}{\left(a^2+\omega^2\right)^2}$
$e^{-at}H(t)$	$\dfrac{1}{a+j\omega}$
$e^{-(at)^2}$	$\dfrac{\sqrt{\pi}}{a}\,e^{-\omega^2/4a^2}$
Pulse (or gate) $k\left(H(t+a)-H(t-a)\right)$	$2k\,\dfrac{\sin(a\omega)}{\omega}$
Triangle $\begin{cases} \dfrac{k}{a}\left(a-\lvert t\rvert\right) & (-a\le t\le a) \\ 0 & (\text{otherwise}) \end{cases}$	$\dfrac{2k}{a}\cdot\dfrac{1-\cos(a\omega)}{\omega^2}$
Time shift $f(t-t_o)$	$e^{-j\omega t_o}\,\hat{f}(\omega)$
Scaling $f(at)$	$\dfrac{1}{a}\,\hat{f}\left(\dfrac{\omega}{a}\right)$

Table of Fourier Transforms (continued)

$f(t)$	$\hat{f}(\omega) = \mathscr{F}\{f(t)\}(\omega)$
Time Differentiation $f^{(n)}(t)$	$(j\omega)^n \hat{f}(\omega)$ [provided f continuous]
Frequency Differentiation $t^n f(t)$	$j^n \frac{d^n}{d\omega^n} \hat{f}(\omega)$
Time Integration $\int_{-\infty}^{t} f(x)\,dx$	$\frac{\hat{f}(\omega)}{j\omega}$ [provided $\hat{f}(0)=0$]
Time Convolution $f * g$	$\hat{f} \cdot \hat{g}$
Frequency Convolution $f \,.\, g$	$\frac{\hat{f} * \hat{g}}{2\pi}$
Dirac delta $\delta(t-a)$	$e^{-ja\omega}$
$\frac{1}{t^2+a^2}$	$\frac{\pi}{a} e^{-a\lvert\omega\rvert}$
$\frac{t}{t^2+a^2}$	$\frac{-j\pi}{2a} \omega e^{-a\lvert\omega\rvert}$
$\frac{1}{t}$	$j \operatorname{sgn}(\omega)$

Shannon Sampling Theorem

The entire signal $f(t)$ may be reconstructed from the discrete sample at times $t = \left\{\ldots, -\frac{2\pi}{L}, -\frac{\pi}{L}, 0, +\frac{\pi}{L}, +\frac{2\pi}{L}, \ldots\right\}$:

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n\pi}{L}\right) \frac{\sin(Lt - n\pi)}{Lt - n\pi}$$

END OF CHAPTER 7

[Space for additional notes]