1. Ordinary Differential Equations

An equation involving a function of one independent variable and the derivative(s) of that function is an ordinary differential equation (ODE).

The highest order derivative present determines the order of the ODE and the power to which that highest order derivative appears is the degree of the ODE. A general n^{th} order ODE is

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

Example 1.00.1

$$\frac{d^2 y}{d x^2} + x\left(\frac{dy}{dx}\right)^2 = x^2 y$$ is a second order first degree ODE.

Example 1.00.2

$$x\left(\frac{dy}{dx}\right)^2 = x^2 y$$ is a first order second degree ODE.

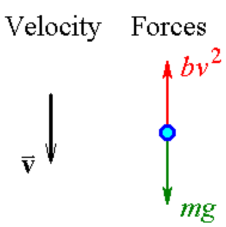
In this course we will usually consider first degree ODEs of first or second order only. The topics in this chapter are treated briefly, because it is assumed that graduate students will have seen this material during their undergraduate years.

Sections in this Chapter:

1.01 First Order ODEs - Separation of Variables

Example 1.01.1

A particle falls under gravity from rest through a viscous medium such that the drag force is proportional to the square of the speed. Find the speed $v(t)$ at any time $t > 0$ and find the terminal speed v_∞.



The forces acting on the ball bearing are its weight downwards and friction upwards. Let m be the mass of the object, $g \approx 9.81 \text{ m s}^{-2}$ be the gravitational acceleration due to gravity.

Newton's second law of motion states

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt}$$

The ODE governing the motion follows:

$$m\frac{dv}{dt} = mg - bv^2 \quad \text{(Net force = weight − drag force)}$$

In standard form,

$$\underbrace{\left(bv^2 - mg\right)}_{M(t,v)} dt + \underbrace{(m)}_{N(t,v)} dv = 0$$

$M(t,v)$ ↑ $f(v)$ only; $N(t,v)$ ↑ const. $\therefore$ type **separable**.

Whenever a first order ODE can be rewritten in the form

$$f(x)\,dx = g(y)\,dy$$

the method of separation of variables may be used.

The ODE in this problem may be separated into the form

$$\frac{m}{mg - bv^2}\,dv = dt \quad \Rightarrow \quad \frac{m}{-b\left(-\frac{mg}{b} + v^2\right)}\,dv = dt$$

Example 1.01.1 (continued)

$$\Rightarrow \quad \int \frac{dv}{v^2 - \frac{mg}{b}} = -\frac{b}{m}\int dt$$

$$\Rightarrow \quad \int \frac{dv}{v^2 - k^2} = -\frac{b}{m}\int dt \qquad \text{where} \quad k^2 = \frac{mg}{b}$$

Partial fractions:

$$\frac{1}{(v-k)(v+k)} = \frac{A}{v-k} + \frac{B}{v+k}$$

Using the "**cover-up rule**":

$$A = \frac{1}{\boxed{\cancel{(k-k)}}\,(k+k)} = \frac{1}{2k}$$

$$B = \frac{1}{(-k-k)\,\boxed{\cancel{(k+k)}}} = \frac{-1}{2k}$$

$$\Rightarrow \quad \frac{1}{v^2 - k^2} = \frac{1}{2k}\left(\frac{1}{v-k} - \frac{1}{v+k}\right)$$

$$\Rightarrow \quad \frac{1}{2k}\big(\ln(v-k) - \ln(v+k)\big) = -\frac{bt}{m} + C_1$$

$$\Rightarrow \quad \ln\left(\frac{v-k}{v+k}\right) = -\frac{2kbt}{m} + C_2 = -pt + C_2 \,,$$

where $p = \dfrac{2kb}{m} = \dfrac{2b}{m}\sqrt{\dfrac{mg}{b}} = 2\sqrt{\dfrac{bg}{m}}$

$$\Rightarrow \quad \frac{v-k}{v+k} = e^{-pt + C_2} = A\,e^{-pt}$$

$$\Rightarrow \quad v - k = v\,A\,e^{-pt} + k\,A\,e^{-pt}$$

$$\Rightarrow \quad v\,(1 - A\,e^{-pt}) = k\,(1 + A\,e^{-pt})$$

Example 1.01.1 (continued)

General solution:

$$v(t) = \frac{k\left(1 + A e^{-pt}\right)}{1 - A e^{-pt}}$$

Initial condition: $v(0) = 0$

$$\Rightarrow \quad 0 = \frac{k(1+A)}{1-A} \quad \Rightarrow \quad A = -1$$

Complete solution:

$$\boxed{v(t) = k \cdot \frac{1 - e^{-pt}}{1 + e^{-pt}}, \quad \text{where} \quad k = \sqrt{\frac{mg}{b}} \quad \text{and} \quad p = 2\sqrt{\frac{bg}{m}}}$$

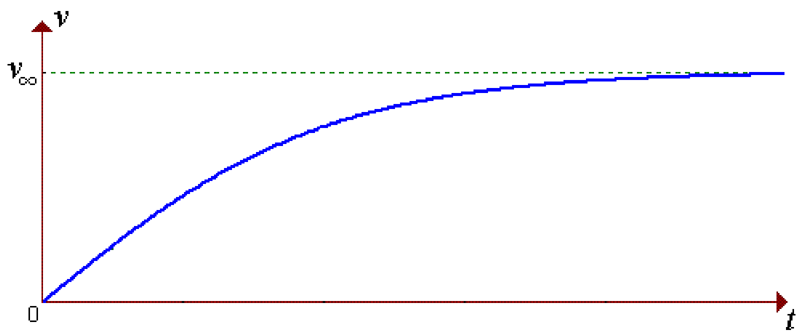
Terminal speed v_∞:

$$v_\infty = \lim_{t \to \infty} v(t) = k \frac{1-0}{1+0} = k = \sqrt{\frac{mg}{b}}$$

The terminal speed can also be found directly from the ODE.
At terminal speed, the acceleration is zero, so that the ODE simplifies to

$$m \frac{dv}{dt} = 0 = mg - b v_\infty^{\,2} \quad \Rightarrow \quad v_\infty^{\,2} = \frac{mg}{b}$$

Graph of speed against time:



[For a 90 kg person in air, $b \approx 1 \text{ kg m}^{-1} \rightarrow k \approx 30 \text{ ms}^{-1} \approx 100$ km/h.
$v(t)$ is approximately linear at first, but air resistance builds quickly.
One accelerates to within 10 km/h of terminal velocity very fast, in just a few seconds.]

1.02 Exact First Order ODEs

If x and y are related implicitly by the equation $u(x, y) = c$ (constant), then the chain rule for differentiation leads to the ODE

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

Therefore, for the functions $M(x, y)$ and $N(x, y)$ in the first order ODE

$$M\,dx + N\,dy = 0 ,$$

if a **potential function** $u(x, y)$ exists such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y} ,$$

then $u(x, y) = c$ is the general solution to the ODE and the ODE is said to be **exact**.

Note that, for nearly all functions of interest, Clairault's theorem results in the identity

$$\frac{\partial^2 u}{\partial y\,\partial x} \equiv \frac{\partial^2 u}{\partial x\,\partial y}$$

This leads to a simple test to determine whether or not an ODE is exact:

$$\boxed{\frac{\partial M}{\partial y} \equiv \frac{\partial N}{\partial x} \quad \Rightarrow \quad M\,dx + N\,dy = 0 \quad \text{is exact}}$$

A separable first order ODE is also exact (after suitable rearrangement).

$$f(x)\,g(y)\,dx + h(x)\,k(y)\,dy = 0 \quad \text{[separable]}$$

$$\Rightarrow \quad \underbrace{\left(\frac{f(x)}{h(x)}\right)}_{M} dx + \underbrace{\left(\frac{k(y)}{g(y)}\right)}_{N} dy = 0$$

$$\Rightarrow \quad \frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

However, the converse is false. One counter-example will suffice.

<u>Example 1.02.1</u>

The ODE

$$\left(y\,e^x - x\right)dx + e^x dy = 0$$

is exact,

$$\left[M = y e^x - x\,, \quad N = e^x \qquad \Rightarrow \quad \frac{\partial M}{\partial y} = e^x = \frac{\partial N}{\partial x}\right]$$

but not separable.

To find the general solution, we seek a potential function $u(x, y)$ such that

$\dfrac{\partial u}{\partial x} = y e^x - x$ and $\dfrac{\partial u}{\partial y} = e^x$

It does not take long to discover that

$$u = y e^x - \tfrac{1}{2}x^2 = c$$

possesses the correct partial derivatives and is therefore the general solution of the ODE.

The solution may be expressed in explicit form as

$$y = \left(c + \tfrac{1}{2}x^2\right)e^{-x}$$

<u>Example 1.02.2</u>

Is the ODE $2y\,dx + x\,dy = 0$ exact?

$$M = 2y \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 2\,, \qquad N = x \quad \Rightarrow \quad \frac{\partial N}{\partial x} = 1 \neq \frac{\partial M}{\partial y}$$

Therefore **NO**, the ODE is not exact.

Example 1.02.3

Is the ODE

$$A\left(2x^{2n+1}y^{n+1}\,dx \;+\; x^{2n+2}y^{n}\,dy\right) \;=\; 0$$

(where n is any constant and A is any non-zero constant) exact?
Find the general solution.

$$M \;=\; 2Ax^{2n+1}y^{n+1} \quad\Rightarrow\quad \frac{\partial M}{\partial y} \;=\; 2Ax^{2n+1}(n+1)\,y^{n}$$

$$N \;=\; Ax^{2n+2}y^{n} \quad\Rightarrow\quad \frac{\partial N}{\partial x} \;=\; A(2n+2)\,x^{2n+1}y^{n} \;=\; \frac{\partial M}{\partial y}$$

Therefore **YES**, the ODE is exact (for any n and for any non-zero A).

To find the general solution, we seek a potential function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} \;=\; 2Ax^{2n+1}y^{n+1} \quad\text{and}\quad \frac{\partial u}{\partial y} \;=\; Ax^{2n+2}y^{n}$$

If $n \neq -1$ then $u \;=\; \dfrac{Ax^{2n+2}y^{n+1}}{n+1} \;=\; c_1$

If $n = -1$ then $u \;=\; A\ln\left(x^2 y\right) \;=\; c_1$

In either case, the general solution simplifies to $x^2 y = c$ or, in explicit form,

$$\boxed{y \;=\; \frac{c}{x^2}}$$

Note that the exact ODE in example 1.02.3 is just the non-exact ODE of example 1.02.2 multiplied by the factor $I(x, y) \;=\; Ax^{2n+1}y^{n}$. The ODEs are therefore equivalent and share the same general solution. The function $I(x, y) \;=\; Ax^{2n+1}y^{n}$ is an **integrating factor** for the ODE of example 1.02.2.

Also note that the integrating factor is not unique. In this case, *any* two distinct values of n generate two distinct integrating factors that both convert the non-exact ODE into an exact form. However, we need to guard against introducing a spurious singular solution $y \equiv 0$.

1.03 Integrating Factor

Occasionally it is possible to transform a non-exact first order ODE into exact form, using an integrating factor $I(x, y)$.
Suppose that

$$P\,dx + Q\,dy = 0$$

is not exact, but that

$$IP\,dx + IQ\,dy = 0$$

is exact.

Then, using the product rule,

$$M = I \cdot P \quad \Rightarrow \quad \frac{\partial M}{\partial y} = \frac{\partial I}{\partial y} P + I \frac{\partial P}{\partial y}$$

and

$$N = I \cdot Q \quad \Rightarrow \quad \frac{\partial N}{\partial x} = \frac{\partial I}{\partial x} Q + I \frac{\partial Q}{\partial x}$$

From the exactness condition

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow \quad \frac{\partial I}{\partial x} Q - \frac{\partial I}{\partial y} P = I \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

This is an awkward partial differential equation. Where it is valid, we may use the assumption that the integrating factor is a function of x alone, to simplify its derivation.

$$\frac{dI}{dx} Q - 0 = I \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

$$\Rightarrow \quad \frac{1}{I} \frac{dI}{dx} = \frac{1}{Q} \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

This assumption is valid only if $\boxed{\frac{1}{Q} \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = R(x)}$ is a function of x only.

If so, then the integrating factor is $\boxed{I(x) = e^{\int R(x)\,dx}}$

[Note that the arbitrary constant of integration can be omitted safely.] Then

$$u = \int M\,dx = \int e^{\int R(x)\,dx} \cdot P(x, y)\,dx, \quad \text{etc.}$$

Returning to

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \Rightarrow \quad \frac{\partial I}{\partial x} Q - \frac{\partial I}{\partial y} P = I \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right)$$

If we assume that the integrating factor is a function of y alone, then

$$0 - \frac{dI}{dy} \cdot P = I \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \quad \Rightarrow \quad \frac{1}{I} \cdot \frac{dI}{dy} = \frac{1}{P} \cdot \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

This assumption is valid only if $\boxed{\frac{1}{P} \cdot \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = S(y)}$ a function of y only.

If so, then the integrating factor is $\boxed{I(y) = e^{\int S(y)\, dy}}$ and

$$u = \int N \, dy = \int e^{\int S(y)\, dy} \cdot Q(x, y) \, dy , \quad \text{etc.}$$

Example 1.03.1 (Example 1.02.2 again)

Find the general solution of the ODE

$$2y \, dx + x \, dy = 0$$

$$P = 2y, \quad Q = x \quad \Rightarrow \quad \frac{1}{Q} \cdot \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = \frac{2 - 1}{x} = \frac{1}{x} = R(x)$$

Therefore an integrating factor that is a function of x only does exist.

$$\int R(x) \, dx = \int \frac{1}{x} dx = \ln x \qquad \Rightarrow \quad I(x) = e^{\int R(x)\, dx} = e^{\ln x} = x$$

Multiplying the original ODE by $I(x)$, we obtain the exact ODE

$$2xy \, dx + x^2 \, dy = 0$$

To find the general solution, we seek a potential function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial u}{\partial y} = x^2$$

This leads quickly to the general solution, $u = x^2 y = c$ or, in explicit form,

$$\boxed{y = \frac{c}{x^2}}$$

Example 1.03.2

Find the general solution of the ODE

$$2xy\,dx + (2x^2 + 3y)\,dy = 0$$

Test for exactness:

$$P = 2xy \;,\quad Q = 2x^2 + 3y \quad\Rightarrow\quad \frac{\partial P}{\partial y} = 2x \;,\quad \frac{\partial Q}{\partial x} = 4x \neq \frac{\partial P}{\partial y}$$

Therefore the ODE is not exact.

Assume an integrating factor of the form $I(x)$:

$$\frac{1}{Q}\cdot\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = \frac{2x - 4x}{2x^2 + 3y} \neq \boldsymbol{R(x)}$$

Therefore an integrating factor that is a function of x only does *not* exist.

Assume an integrating factor of the form $I(y)$:

$$\frac{1}{P}\cdot\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \frac{4x - 2x}{2xy} = \frac{1}{y} = R(y)$$

Therefore an integrating factor that is a function of y only *does* exist.

$$\int R(y)\,dy = \int \frac{1}{y}\,dy = \ln y \quad\Rightarrow\quad I(y) = e^{\int R(y)\,dy} = e^{\ln y} = y$$

Multiplying the original ODE by $I(y)$, we obtain the exact ODE

$$2xy^2\,dx + (2x^2y + 3y^2)\,dy = 0$$

To find the general solution, we seek a potential function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = 2xy^2 \quad\text{and}\quad \frac{\partial u}{\partial y} = 2x^2y + 3y^2$$

This leads quickly to the general solution, $u = \boxed{x^2y^2 + y^3 = c}$ or, in explicit form,

$$\boxed{x = \pm\frac{\sqrt{c - y^3}}{y}}$$

1.04 <u>First Order Linear ODEs</u> [+ Integration by Parts]

A special case of a first order ODE is the linear ODE:

$$\boxed{\frac{dy}{dx} + P(x)\,y = R(x)}$$

[or, in some cases,

$$\frac{dx}{dy} + Q(y)x = S(y)\ ]$$

Rearranging the first ODE into standard form,

$$(P(x)\,y - R(x))\,dx + 1\,dy = 0$$

Written in the standard exact form with a simple integrating factor in place, the ODE becomes

$$I(x)\,(P(x)\,y - R(x))\,dx + I(x)\,dy = 0$$

Compare this with the exact ODE

$$du = M(x, y)\,dx + N(x, y)\,dy = 0$$

The exactness condition $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \Rightarrow I(x)\cdot P(x) = \dfrac{dI}{dx} \Rightarrow \displaystyle\int \frac{dI}{I} = \int P\,dx$

Let $h(x) = \displaystyle\int P\,dx$, then $\ln I(x) = h(x)$

and the integrating factor is

$$I(x) = e^{h(x)}, \quad \text{where} \quad h(x) = \int P(x)\,dx \quad \left(\Rightarrow \frac{dh}{dx} = P(x)\right).$$

The ODE becomes the exact form

$$e^h\,(Py - R)\,dx + e^h\,dy = 0$$

Seek a potential function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = e^h\left(h'y - R\right) \quad \text{and} \quad \frac{\partial u}{\partial y} = e^h$$

$$\Rightarrow \quad u = y\,e^h - \int e^h R\,dx = C \qquad \Rightarrow \quad y\,e^h = \int e^h R\,dx + C$$

Therefore the general solution of $\dfrac{dy}{dx} + P(x)y = R(x)$ is

$$\boxed{y(x) = e^{-h(x)}\left(\int e^{h(x)} R(x)\,dx + C\right), \quad \text{where} \quad h(x) = \int P(x)\,dx}$$

Example 1.04.1

Solve the ordinary differential equation

$$\frac{dy}{dx} + \frac{2}{x}y = 1$$

This ODE is linear, with $P(x) = \frac{2}{x}$ and $R(x) = 1$.

$$h = \int P\,dx = \int \frac{2}{x}dx = 2\ln x = \ln\left(x^2\right)$$

The integrating factor is therefore $e^h = x^2$.

$$\int e^h R\,dx = \int x^2 1\,dx = \frac{x^3}{3}$$

$$\Rightarrow \quad y = e^{-h}\left(\int e^h R\,dx + C\right) = \frac{1}{x^2}\left(\frac{x^3}{3} + C\right)$$

The general solution is therefore

$$\boxed{y = \frac{x}{3} + \frac{C}{x^2}}$$

Alternative methods:

The ODE is not separable.

Re-arrange the ODE into the form

$(2y - x)\,dx + x\,dy = 0$

$P = 2y - x$ and $Q = x$ $\Rightarrow$ $P_y = 2$, $Q_x = 1 \neq P_y$

This ODE is not exact.

$$\frac{P_y - Q_x}{Q} = \frac{2-1}{x} = \frac{1}{x} = R(x)$$

$$\Rightarrow \quad \int R\,dx = \int \frac{1}{x}dx = \ln x \quad \Rightarrow \quad I(x) = x$$

Example 1.04.1 (continued)

The exact ODE is therefore

$$(2xy - x^2)\, dx + x^2\, dy = 0$$

$$\frac{\partial u}{\partial x} = 2xy - x^2 \quad \text{and} \quad \frac{\partial u}{\partial y} = x^2 \qquad \Rightarrow \quad u = x^2 y - \frac{x^3}{3} = c$$

The same explicit solution then follows:

$$\boxed{y = \frac{x}{3} + \frac{C}{x^2}}$$

OR

Try to re-write the ODE in another exact form $\frac{d}{dx}\left(u(x,y)\right) = v(x)$:

$$\frac{dy}{dx} + \frac{2}{x} y = 1 \qquad \Rightarrow \quad x^2 \frac{dy}{dx} + 2xy = x^2 \quad \Rightarrow \quad \frac{d}{dx}\left(x^2 y\right) = x^2$$

$$\Rightarrow \quad x^2 y = \frac{x^3}{3} + c \qquad \Rightarrow \quad y = \frac{x}{3} + \frac{c}{x^2}$$