1.06 Second Order Homogeneous Linear ODEs

The general second order linear ordinary differential equation with constant real coefficients may be written in the form

$$\frac{d^2 y}{dx^2} + p\frac{dy}{dx} + q\,y = r(x)$$

If, in addition, the right-side function $r(x)$ is identically zero, then the ODE is said to be **homogeneous**. Otherwise it is inhomogeneous.

The most general possible solution y_C to the homogeneous ODE $y'' + p\,y' + q\,y = 0$ is called the **complementary function**.
A solution y_P to the inhomogeneous ODE $y'' + p\,y' + q\,y = r(x)$ is called the **particular solution**.
The linearity of the ODE leads to the following two properties:
Any linear combination of two solutions to the homogeneous ODE is another solution to the homogeneous ODE; and
The sum of any solution to the homogeneous ODE and a particular solution is another solution to the inhomogeneous ODE.

It can be shown that the following is a valid method for obtaining the complementary function:

From the ODE $y'' + p\,y' + q\,y = r(x)$ form the **auxiliary equation** (or "characteristic equation")

$$\lambda^2 + p\lambda + q = 0$$

If the roots λ_1, λ_2 of this quadratic equation are distinct, then a basis for the entire set of possible complementary functions is $\{y_1, y_2\} = \left\{e^{\lambda_1 x}, e^{\lambda_2 x}\right\}$.

If the roots are not real (and therefore form a complex conjugate pair $a \pm bj$), then the basis can be expressed instead as the equivalent real set $\left\{e^{ax}\cos bx,\ e^{ax}\sin bx\right\}$.

If the roots are equal (and therefore real), then a basis for the entire set of possible complementary functions is $\{y_1, y_2\} = \left\{e^{\lambda x}, x e^{\lambda x}\right\}$.
The complementary function, in the form that captures all possibilities, is then

$$y_C = A\,y_1 + B\,y_2$$

where A and B are arbitrary constants.

Example 1.06.1

A simple unforced mass-spring system (with damping coefficient per unit mass = 6 s^{-1} and restoring coefficient per unit mass = 9 s^{-2}) is released from rest at an extension 1 m beyond its equilibrium position ($s = 0$). Find the position $s(t)$ at all subsequent times t.

The simple mass-spring system may be modelled by a second order linear ODE.
The $\dfrac{d^2s}{dt^2}$ term represents the acceleration of the mass, due to the net force.
The $\dfrac{ds}{dt}$ term represents the friction (damping) term.
The s term represents the restoring force.

The model is

$$\frac{d^2s}{dt^2} + 6\frac{ds}{dt} + 9s = 0$$

The auxiliary equation is

$$\lambda^2 + 6\lambda + 9 = 0 \quad \Rightarrow \quad (\lambda+3)^2 = 0 \quad \Rightarrow \quad \lambda = -3, -3$$

The roots are equal, so the basis functions for the complementary function are

$$\{s_1, s_2\} = \left\{e^{-3t}, t\,e^{-3t}\right\}$$

The ODE is homogeneous, so its **general solution** is also its complementary function:

$$s(t) = Ae^{-3t} + Bt\,e^{-3t} = (A+Bt)e^{-3t}$$

However, we have two additional items of information, (the **initial conditions**), which allow us to determine the values of the two arbitrary constants.

Initial displacement

$$s(0) = 1 \quad \Rightarrow \quad (A+0)e^0 = 1 \quad \Rightarrow \quad A = 1$$

$$s'(t) = (B-3A-3Bt)e^{-3t} = (B-3-3Bt)e^{-3t}$$

Initial speed (released from rest)

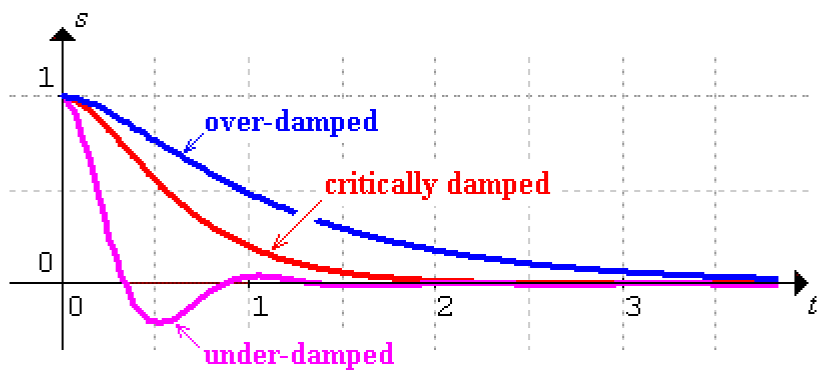
$$s'(0) = 0 \quad \Rightarrow \quad (B-3-0)e^0 = 0 \quad \Rightarrow \quad B = 3$$

Therefore the complete solution is

$$\boxed{s(t) = (1+3t)e^{-3t}}$$

Example 1.06.1 (continued)

This is an example of **critical damping**.
Real distinct roots for λ correspond to **over-damping**.
Complex conjugate roots for λ correspond to **under-damping** (damped oscillations).



Illustrated here are a critically damped case $s(t) = (1+3t)e^{-3t}$ (the solution to Example 1.06.1), an over-damped case $s(t) = \frac{1}{3}\left(4e^{-t} - e^{-4t}\right)$ and an under-damped case $s(t) = e^{-3t}\left(\cos 6t + \frac{1}{2}\sin 6t\right)$, all of which share the same initial conditions $s(0) = 1$ and $s'(0) = 0$.

1.07 Variation of Parameters

A particular solution y_P to the inhomogeneous ODE $y'' + p\,y' + q\,y = r(x)$ may be constructed from the set of basis functions $\{y_1, y_2\}$ for the complementary function by varying the parameters:

Try $y_P(x) = u(x)\,y_1(x) + v(x)\,y_2(x)$, where the functions $u(x)$ and $v(x)$ are such that

(i) y_P is a solution of $y'' + p\,y' + q\,y = r(x)$ and

(ii) one free constraint is imposed, to ease the search for $u(x)$ and $v(x)$.

Substituting $y_P = u\,y_1 + v\,y_2$ into the ODE,

$$(u\,y_1 + v\,y_2)'' + p(u\,y_1 + v\,y_2)' + q(u\,y_1 + v\,y_2) = r$$

$$\Rightarrow \left((u'\,y_1 + v'\,y_2)' + (u\,y_1' + v\,y_2')'\right)$$

$$+ \; p\left((u'\,y_1 + v'\,y_2) + (u\,y_1' + v\,y_2')\right) + q(u\,y_1 + v\,y_2) = r$$

Imposing the free constraint $u'\,y_1 + v'\,y_2 \equiv 0$ simplifies the above expression to

$$(0 + u'\,y_1' + v'\,y_2' + u\,y_1'' + v\,y_2'') + p(0 + u\,y_1' + v\,y_2') + q(u\,y_1 + v\,y_2) = r$$

$$\Rightarrow u(y_1'' + p\,y_1' + q\,y_1) + v(y_2'' + p\,y_2' + q\,y_2) + u'\,y_1' + v'\,y_2' = r$$

But y_1 and y_2 are solutions of the homogeneous ODE $y'' + p\,y' + q\,y = 0$.

Therefore $0 + 0 + u'\,y_1' + v'\,y_2' = r$ is our other constraint.

Rewrite the two constraints together as a matrix equation:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

Using Cramer's rule to solve this matrix equation for u' and v' we obtain

$$u' = \frac{W_1}{W} \quad \text{and} \quad v' = \frac{W_2}{W}, \quad \text{where} \quad W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \text{ (the Wronskian),}$$

$$\text{and} \quad W_1 = \begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix} = -y_2 r, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix} = +y_1 r$$

Integrate to find $u(x)$ and $v(x)$, then construct $y_P(x) = u(x)\,y_1(x) + v(x)\,y_2(x)$.

[space to continue the derivation of the method of variation of parameters]

Example 1.07.1

A mass spring system is at rest until the instant $t = 3$, when a sudden hammer blow, of impulse 10 Ns, sets the system into motion. No further external force is applied to the system, which has a mass of 1 kg, a restoring force coefficient of 26 kg s^{-2} and a friction coefficient of 2 kg s^{-1}. The response $x(t)$ at any time $t > 0$ is governed by the differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 26x = 10\,\delta(t-3)$$

(where $\delta(t-a)$ is the Dirac delta function),
together with the initial conditions $x(0) = x'(0) = 0$.
Find the complete solution to this initial value problem.

A.E.: $\lambda^2 + 2\lambda + 26 = 0 \quad \Rightarrow \quad \lambda = \dfrac{-2 \pm \sqrt{4 - 4\times 26}}{2} = -1 \pm 5j$

C.F.: $x_C = Ax_1 + Bx_2$, where $x_1 = e^{-t}\cos 5t$ and $x_2 = e^{-t}\sin 5t$

Define the abbreviations
$c = \cos 5t$, $s = \sin 5t$ $\Delta = \delta(t-3)$, and $E = e^{-t}$, then $r(t) = 10\,\Delta$.

P.S.: $r(t)$ is such that the method of undetermined coefficients cannot be used.

$$W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} Ec & Es \\ E(-5s-c) & E(5c-s) \end{vmatrix} = E^2\left(5c^2 - cs + 5s^2 + cs\right) = 5E^2$$

$$W_1 = \begin{vmatrix} 0 & x_2 \\ r & x_2' \end{vmatrix} = -x_2 r = -Es\cdot 10\Delta$$

$$u' = \frac{W_1}{W} = \frac{-Es\cdot 10\Delta}{5E^2} = -2E^{-1}s\,\Delta$$

$$\Rightarrow \quad u = -2\int e^{+t}\sin 5t\,\delta(t-3)\,dt = -2e^3(\sin 15)\,H(t-3)$$

(using the sifting property of the Dirac delta function in integrals,

$$\int_c^d f(t)\,\delta(t-a)\,dt = \begin{cases} f(a) & (\text{if} \quad c < a < d) \\ 0 & (a < c \ \text{ or } \ a > d) \end{cases}$$

and where $H(t-a) = \begin{cases} 0 & (t < a) \\ 1 & (t \ge a) \end{cases}$ is the Heaviside (unit step) function.)

$$W_2 = \begin{vmatrix} x_1 & 0 \\ x_1' & r \end{vmatrix} = +x_1 r = Ec\cdot 10\Delta$$

$$v' = \frac{W_2}{W} = \frac{Ec\cdot 10\Delta}{5E^2} = 2E^{-1}c\,\Delta$$

$$\Rightarrow \quad v = 2\int e^{+t}\cos 5t\,\delta(t-3)\,dt = 2e^3(\cos 15)\,H(t-3)$$

Example 1.07.1 (continued)

Using the trigonometric identity sin (*A* – *B*) = sin *A* cos *B* – cos *A* sin *B*,

$$x_P = u\,x_1 + v\,x_2 = 2e^3 H(t-3)\,e^{-t}\left(-(\sin 15)\cos 5t + (\cos 15)\sin 5t\right)$$

$$= 2e^{-(t-3)} H(t-3)\sin(5t-15)$$

G.S. $x = x_C + x_P$:

$$x(t) = e^{-t}\left(A\cos 5t + B\sin 5t\right) + 2e^{-(t-3)}\sin\left(5(t-3)\right)H(t-3)$$

But, for $t < 3$, the system is undisturbed, at rest at equilibrium, so that

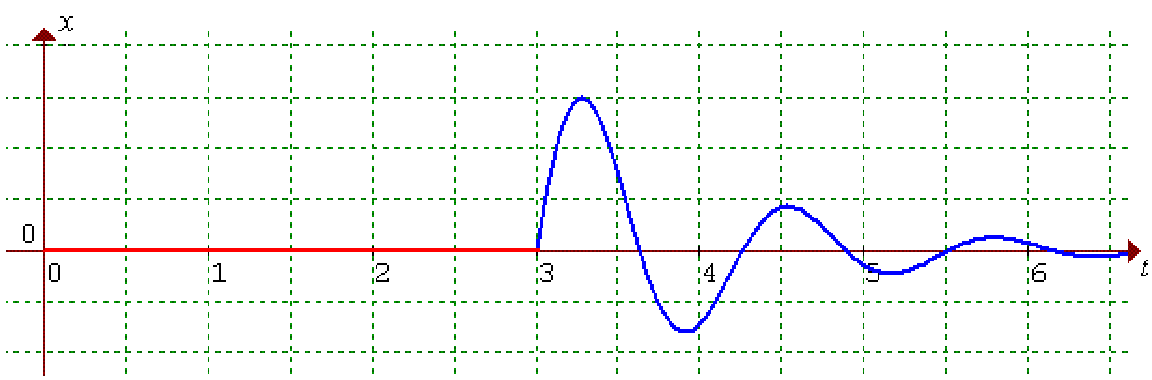
$$x(t) = e^{-t}\left(A\cos 5t + B\sin 5t\right) + 0\,; \qquad x(0) = x'(0) = r(t) = 0$$

$\Rightarrow \quad A = B = 0.$

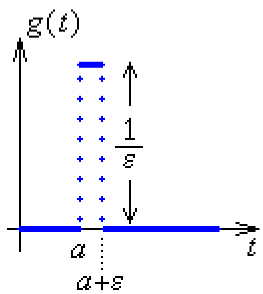
The complete solution is therefore

$$\boxed{x(t) = 2e^{-(t-3)}\sin 5(t-3)\,H(t-3)}$$

This complete solution is continuous at $t = 3$.
It is not differentiable at $t = 3$, because of the infinite discontinuity of the Dirac delta function inside $r(t)$ at $t = 3$.



Note: $\delta(t-a) = \lim\limits_{\varepsilon \to 0} g(t;a,\varepsilon)$ $\qquad H(t-a) = \begin{cases} 0 & (t < a) \\ 1 & (t \ge a) \end{cases}$



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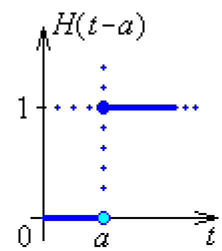

Example 1.07.2

Find the general solution of the ODE $y'' + 2y' - 3y = x^2 + e^{2x}$.

A.E.: $\lambda^2 + 2\lambda - 3 = 0$

$\Rightarrow \quad (\lambda + 3)(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda = -3, 1$

$y_1 = e^{-3x}, \quad y_2 = e^x, \quad r = x^2 + e^{2x}$

Particular Solution by Variation of Parameters:

$$W(x) = \det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \det \begin{bmatrix} e^{-3x} & e^x \\ -3e^{-3x} & e^x \end{bmatrix} = 4e^{-2x}$$

$$W_1 = \det \begin{bmatrix} 0 & y_2 \\ r & y_2' \end{bmatrix} = -y_2 r = -e^x\left(x^2 + e^{2x}\right)$$

$$\Rightarrow \quad u' = \frac{W_1}{W} = \frac{-\left(x^2 e^x + e^{3x}\right)}{4e^{-2x}} = -\left(\frac{x^2 e^{3x} + e^{5x}}{4}\right)$$

$$\Rightarrow \quad u = -\frac{1}{4}\int\left(x^2 e^{3x} + e^{5x}\right)dx$$

D		**I**
x^2		e^{3x}
	$+$	
$2x$		$\frac{1}{3}e^{3x}$
	$-$	
2		$\frac{1}{9}e^{3x}$
	$+$	
0		$\frac{1}{27}e^{3x}$

$$\Rightarrow \quad u = -\frac{1}{4}\left(\frac{e^{3x}}{27}\left(9x^2 - 6x + 2\right) + \frac{1}{5}e^{5x}\right)$$

$$W_2 = \det \begin{bmatrix} y_1 & 0 \\ y_1' & r \end{bmatrix} = +y_1 r = e^{-3x}\left(x^2 + e^{2x}\right)$$

Example 1.07.2 (continued)

$$\Rightarrow \quad v' = \frac{W_2}{W} = \frac{x^2e^{-3x} + e^{-x}}{4e^{-2x}} = \frac{x^2e^{-x} + e^x}{4}$$

$$\Rightarrow \quad v = \frac{1}{4}\int \left(x^2e^{-x} + e^x\right)dx$$

$$\Rightarrow \quad v = \frac{1}{4}\left(e^{-x}\left(-x^2 - 2x - 2\right) + e^x\right)$$

D		**I**
x^2		e^{-x}
	$+$	
$2x$		$-e^{-x}$
	$-$	
2		$+e^{-x}$
	$+$	
0		$-e^{-x}$

$$y_P = u \cdot y_1 + v \cdot y_2 =$$

$$\frac{1}{4}\left\{\left(-\frac{e^{3x}}{27}\left(9x^2 - 6x + 2\right) - \frac{1}{5}e^{5x}\right)e^{-3x} + \frac{27}{27}\left(-e^{-x}\left(x^2 + 2x + 2\right) + e^x\right)e^x\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{27}\left(-9x^2 + 6x - 2 - 27x^2 - 54x - 54\right) + \left(-\frac{1}{5} + 1\right)e^{2x}\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{27}\left(-36x^2 - 48x - 56\right) + \frac{4}{5}e^{2x}\right\}$$

Therefore

$$y_P = \frac{1}{5}e^{2x} - \frac{1}{27}\left(9x^2 + 12x + 14\right)$$

and the general solution is

$$y(x) = \underline{\underline{A\,e^{-3x} + B\,e^x + \frac{1}{5}e^{2x} - \frac{1}{27}\left(9x^2 + 12x + 14\right)}}$$

1.08 Method of Undetermined Coefficients

When trying to find the particular solution of the inhomogeneous ODE

$$\frac{d^2 y}{dx^2} + p\frac{dy}{dx} + q\,y = r(x)$$

an alternative method to variation of parameters is available only when $r(x)$ is one of the following special types:

e^{kx}, $\cos kx$, $\sin kx$, $\sum_{k=1}^{n} a_k x^k$ and any linear combinations of these types and any products of these types. When it is available, this method is often faster than the method of variation of parameters.

The method involves the substitution of a form for y_P that resembles $r(x)$, with coefficients yet to be determined, into the ODE.

If $r(x) = c\,e^{kx}$, then try $y_P = d\,e^{kx}$, with the coefficient d to be determined.

If $r(x) = a\cos kx$ or $b\sin kx$, then try $y_P = c\cos kx + d\sin kx$, with the coefficients c and d to be determined.

If $r(x)$ is an n^{th} order polynomial function of x, then set y_P equal to an n^{th} order polynomial function of x, with all $(n + 1)$ coefficients to be determined.

However, if $r(x)$ contains a constant multiple of either part of the complementary function $(y_1 \text{ or } y_2)$, then that part must be multiplied by x in the trial function for y_P.

Example 1.08.1 (Example 1.07.2 again)

Find the general solution of the ODE $y'' + 2y' - 3y = x^2 + e^{2x}$.

A.E.: $\lambda^2 + 2\lambda - 3 = 0$

$\Rightarrow \quad (\lambda + 3)(\lambda - 1) = 0 \quad \Rightarrow \quad \lambda = -3, 1$

C.F.: $y_C = A\,e^{-3x} + B\,e^x$

Particular Solution by Undetermined Coefficients:

$r(x) = x^2 + e^{2x}$, so try $y_P = ax^2 + bx + c + d\,e^{2x}$

Then $y'' + 2y' - 3y =$

$$\begin{array}{ccccccccccc}
 & & & & & 2a & + & 4d\,e^{2x} & \leftarrow & y_P'' \\
+ & & & 4a\,x & + & 2b & + & 4d\,e^{2x} & \leftarrow & +2y_P' \\
+ & -3a\,x^2 & - & 3b\,x & - & 3c & - & 3d\,e^{2x} & \leftarrow & -3y_P \\
\hline
= & 1\,x^2 & + & 0\,x & + & 0 & + & 1\,e^{2x} & \leftarrow & = r
\end{array}$$

Matching coefficients:

$x^2: \quad -3a = 1 \quad \Rightarrow \quad a = -\tfrac{1}{3}$

$x^1: \quad 4\left(-\tfrac{1}{3}\right) - 3b = 0 \quad \Rightarrow \quad b = -\tfrac{4}{9}$

$x^0: \quad 2\left(-\tfrac{1}{3}\right) + 2\left(-\tfrac{4}{9}\right) - 3c = 0 \quad \Rightarrow \quad c = -\dfrac{2}{3}\left(\dfrac{3+4}{9}\right) = -\tfrac{14}{27}$

$e^{2x}: \quad (4+4-3)d = 1 \quad \Rightarrow \quad d = \tfrac{1}{5}$

G.S.: $y(x) = y_C(x) + y_P(x)$

Therefore

$$y(x) = \underline{\underline{A\,e^{-3x} + B\,e^x + \tfrac{1}{5}e^{2x} - \tfrac{1}{27}\left(9x^2 + 12x + 14\right)}}$$

Example 1.08.2

Find the general solution of the ODE

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$$

A.E.: $\lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \quad (\lambda+2)^2 = 0 \quad \Rightarrow \quad \lambda = -2, -2$

C.F.: $y_C = (Ax+B)e^{-2x}$

P.S.:

$r(x) = e^{-2x}$, but both e^{-2x} and $x\,e^{-2x}$ are in the C.F.

Therefore try $y_P = c\,x^2 e^{-2x}$.

$$y_P'' + 4y_P' + 4y_P = c\Big(\big(4x^2 - 8x + 2\big) + 4\big(-2x^2 + 2x\big) + 4\big(x^2\big)\Big)e^{-2x} = e^{-2x}$$

$$\Rightarrow \quad c\Big((4-8+4)x^2 + (-8+8)x + 2\Big) = 1$$

$$\Rightarrow \quad c = \tfrac{1}{2}$$

Therefore the general solution is

$$\boxed{y(x) = \left(\tfrac{1}{2}x^2 + Ax + B\right)e^{-2x}}$$

Again, this is much faster than variation of parameters.
However, the method of variation of parameters may be employed regardless of the form of the right side $r(x)$, while the method of undetermined coefficients may be used only for a narrow range of forms of $r(x)$.