1.11 The Gamma Function

The gamma function $\Gamma(x)$ is a special function that will be needed in the solution of Bessel's ODE. $\Gamma(x)$ is a generalisation of the factorial function $n!$ from positive integers to most real numbers. For any positive integer n,
$n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$ (with $0!$ defined to be 1)

When x is a positive integer n, $\boxed{\Gamma(n) = (n-1)!}$

We know that $n! = n \times (n-1)!$
The gamma function has a similar recurrence relationship: $\Gamma(x+1) = x \cdot \Gamma(x)$

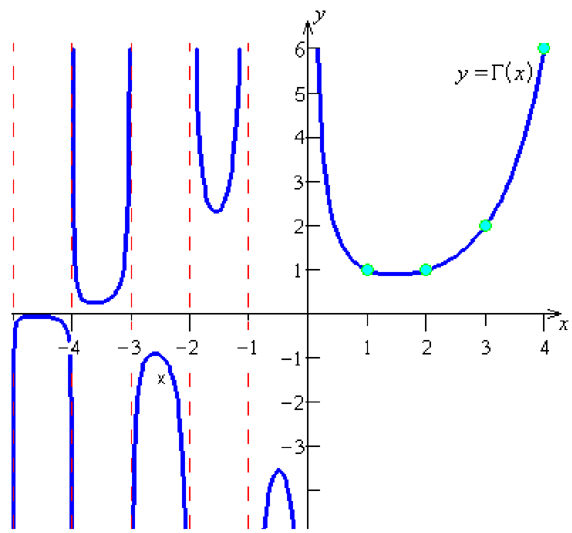
This allows $\Gamma(x)$ to be defined for non-integer negative x, using $\boxed{\Gamma(x) = \dfrac{\Gamma(x+1)}{x}}$

For example,
it can be shown that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\Rightarrow \quad \Gamma\left(-\tfrac{1}{2}\right) = \frac{\Gamma\left(+\frac{1}{2}\right)}{-\frac{1}{2}} = -2\sqrt{\pi} \qquad \Rightarrow \quad \Gamma\left(-\tfrac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} = +\frac{4\sqrt{\pi}}{3}, \text{ etc.}$$

$\Gamma(x)$ is infinite when x is a negative integer or zero. It is well defined for all other real numbers x.

In this graph of $y = \Gamma(x)$, values of the factorial function (at positive integer values of x) are highlighted.



There are several ways to define the gamma function, such as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \qquad (x > 0)$$

and

$$\Gamma(x) = \lim_{n \to \infty} \frac{n! \, n^x}{x(x+1)\ldots(x+n)}$$

A related special function is the **beta function**:

$$B(m,n) = \int_0^1 t^{m-1}(1-t)^{n-1}\,dt = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Among the many results involving the gamma function are:

For the closed region V in the first octant, bounded by the coordinate planes and the surface $\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta + \left(\frac{z}{c}\right)^\gamma = 1$, with all constants positive,

$$I = \iiint_V x^{p-1} y^{q-1} z^{r-1} dx\,dy\,dz = \frac{a^p b^q c^r}{\alpha\beta\gamma} \cdot \frac{\Gamma\left(\frac{p}{\alpha}\right)\Gamma\left(\frac{q}{\beta}\right)\Gamma\left(\frac{r}{\gamma}\right)}{\Gamma\left(1+\frac{p}{\alpha}+\frac{q}{\beta}+\frac{r}{\gamma}\right)}$$

For the closed area A in the first quadrant, bounded by the coordinate axes and the curve $\left(\frac{x}{a}\right)^\alpha + \left(\frac{y}{b}\right)^\beta = 1$, with all constants positive,

$$I = \iint_A x^{p-1} y^{q-1} dx\,dy = \frac{a^p b^q}{\alpha\beta} \cdot \frac{\Gamma\left(\frac{p}{\alpha}\right)\Gamma\left(\frac{q}{\beta}\right)}{\Gamma\left(1+\frac{p}{\alpha}+\frac{q}{\beta}\right)}$$

<u>Example 1.11.1</u>

Establish the formula for the area enclosed by an ellipse.

The Cartesian equation of a standard ellipse is $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$.

Set $\alpha = \beta = 2$ and $p = q = 1$, then

$$A = 4I = 4\iint_A 1\,dx\,dy = 4\frac{a^1 b^1}{2\times 2} \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(1+\frac{1}{2}+\frac{1}{2}\right)} = ab \cdot \frac{\left(\Gamma\left(\frac{1}{2}\right)\right)^2}{\Gamma(2)} = ab\frac{\left(\sqrt{\pi}\right)^2}{1!} = \pi ab$$

1.12 Bessel and Legendre ODEs

Frobenius Series Solution of an ODE

If the ODE

$$P(x)\,y'' + Q(x)\,y' + R(x)\,y = F(x)$$

is such that $P(x_o) = 0$, but $(x - x_o)\dfrac{Q(x)}{P(x)}$, $(x - x_o)^2 \dfrac{R(x)}{P(x)}$ and $\dfrac{F(x)}{P(x)}$ are all analytic at x_o, then $x = x_o$ is a **regular singular point** of the ODE.

A Frobenius series solution of the ODE about $x = x_o$ exists:

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_o)^{n+r}$$

for some real number(s) r and for some set of values $\{ c_n \}$.

Example 1.12.1

Find a solution of Bessel's ordinary differential equation of order v, ($v \geq 0$),

$$x^2 y'' + x\,y' + (x^2 - v^2)\,y = 0$$

$P(0) = 0 \;\Rightarrow\; x_o = 0$ is a singular point.

$$(x - x_o)\frac{Q(x)}{P(x)} = x\frac{x}{x^2} = 1 \quad , \quad (x - x_o)^2 \frac{R(x)}{P(x)} = x^2 \frac{(x^2 - v^2)}{x^2} = x^2 - v^2$$

and $\dfrac{F(x)}{P(x)} = 0$

Therefore $x_o = 0$ is a regular singular point of Bessel's equation.

Substitute the Frobenius series $y(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$ into the ODE:

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2+2} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1+1}$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} v^2 c_n x^{n+r} = 0$$

Example 1.12.1 (continued)

Adjust the index on the third summation so that the exponents of x match:

$$\sum_{n=0}^{\infty} c_n \left[(n+r)(n+r-1+1) - \nu^2 \right] x^{n+r} + \sum_{n=2}^{\infty} c_{n-2} x^{n+r} = 0$$

The summations can be combined for $n = 2$ onwards:

$$\left(r^2 - \nu^2\right) c_0 x^r + \left[(r+1)^2 - \nu^2 \right] c_1 x^{r+1} + \sum_{n=2}^{\infty} \left[\left((n+r)^2 - \nu^2 \right) c_n + c_{n-2} \right] x^{n+r} = 0$$

Setting the coefficient of x^r (the lowest exponent present) to zero generates the **indicial equation** $r^2 - \nu^2 = 0 \quad \Rightarrow \quad r = \pm \nu$.

Examining the positive root, the series now becomes

$$0 + \left[(\nu+1)^2 - \nu^2 \right] c_1 x^{r+1} + \sum_{n=2}^{\infty} \left[\left((n+\nu)^2 - \nu^2 \right) c_n + c_{n-2} \right] x^{n+r} = 0$$

$$\Rightarrow \quad (2\nu+1) c_1 x^{r+1} + \sum_{n=2}^{\infty} \left[\left(2n\nu + n^2 \right) c_n + c_{n-2} \right] x^{n+r} = 0$$

But $\nu \ge 0 \quad \Rightarrow \quad 2\nu + 1 \ne 0 \quad \Rightarrow \quad c_1 = 0$

For $n \ge 2, \quad c_n = \dfrac{-c_{n-2}}{n(n+2\nu)}$

It then follows that this series must be even: $0 = c_1 = c_3 = c_5 = \ldots$ or $c_{2k-1} = 0 \quad \forall k \in \mathbb{N}$

For the even order terms, replace the index n by the even index $2k$ (where k is any natural number) and pursue the recurrence relation down to c_0:

$$c_{2k} = \frac{-c_{2k-2}}{2k(2k+2\nu)} = \frac{(-1)}{2^2 k(k+\nu)} c_{2(k-1)} = \frac{(-1)}{2^2 k(k+\nu)} \cdot \frac{(-1)}{2^2 (k-1)(k-1+\nu)} c_{2(k-2)}$$

$$= \frac{(-1)}{2^2 k(k+\nu)} \cdot \frac{(-1)}{2^2 (k-1)(k-1+\nu)} \cdot \frac{(-1)}{2^2 (k-2)(k-2+\nu)} c_{2(k-3)} = \cdots$$

$$= \frac{(-1)^k}{2^{2k} k(k-1)(k-2)\ldots(k-[k-1]) \cdot (k+\nu)(k-1+\nu)(k-2+\nu)\ldots(k-[k-1]+\nu)} c_{2(k-k)}$$

$$\Rightarrow \quad c_{2k} = \frac{(-1)^k}{2^{2k} k!(\nu+k)(\nu+k-1)\ldots(\nu+1)} c_0$$

$$\Rightarrow \quad c_{2k} = \frac{(-1)^k}{2^{2k} k!(\nu+k)(\nu+k-1)\ldots(\nu+1)} \cdot \frac{\Gamma(\nu+1)}{\Gamma(\nu+1)} c_0$$

$$\Rightarrow \quad c_{2k} = \frac{(-1)^k}{2^{2k} k!\Gamma(\nu+k+1)} \cdot \Gamma(\nu+1) c_0 \quad \left(= \frac{(-1)^k}{2^{2k} k!(\nu+k)!} \cdot \nu! c_0 \text{ if } \nu = 0,1,2,\ldots \right)$$

Example 1.12.1 (continued)

One Frobenius solution of Bessel's equation of order ν is therefore

$$y(x) = c_0\,\Gamma(\nu+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}\,k!\,\Gamma(\nu+k+1)}\,x^{2k+\nu} = c_0\,\Gamma(\nu+1)\,J_\nu(x)$$

where $J_\nu(x)$ is the **Bessel function of the first kind of order** $\boldsymbol{\nu}$.

It turns out that the Frobenius series found by setting $r = -\nu$ generates a second linearly independent solution $J_{-\nu}(x)$ of the Bessel equation only if ν is not an integer.

The Bessel ODE in standard form,

$$x^2 y'' + x y' + \left(x^2 - \nu^2\right) y = 0$$

has the general solution

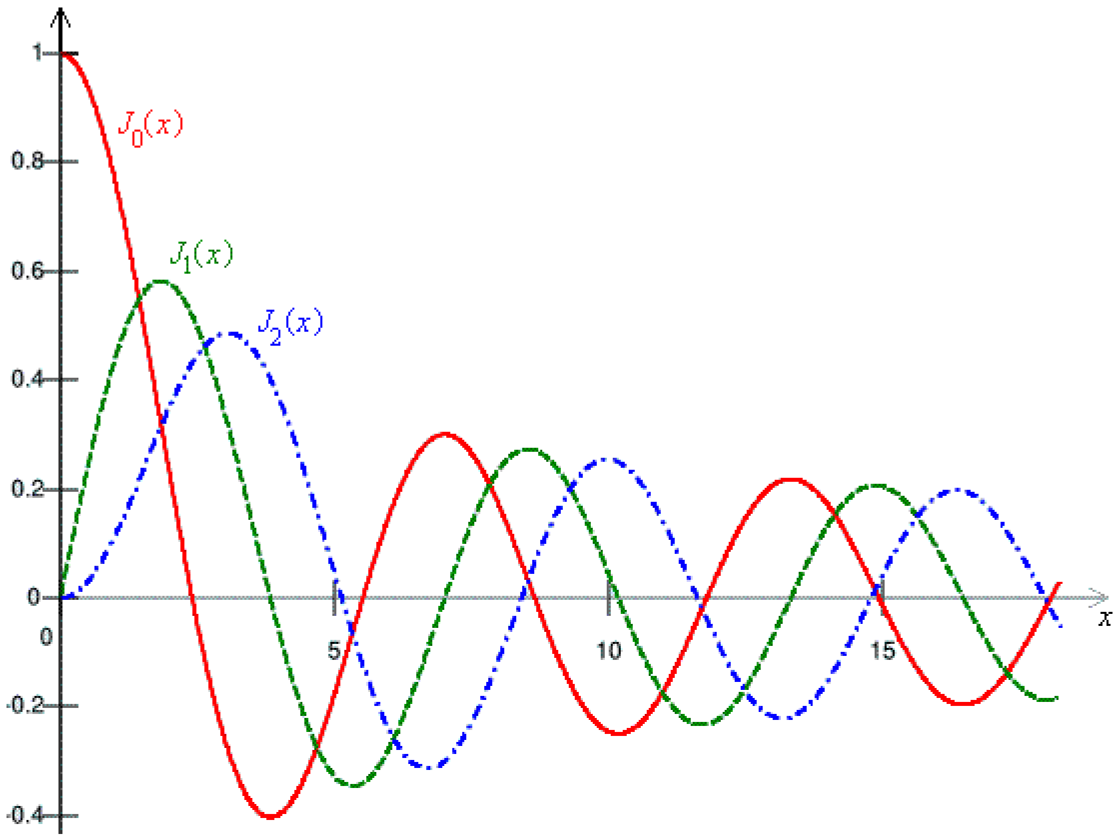
$$y(x) = A J_\nu(x) + B Y_\nu(x)$$

unless ν is not an integer, in which case $Y_\nu(x)$ can be replaced by $J_{-\nu}(x)$.

$Y_\nu(x)$ is the Bessel function of the second kind.

When ν is an integer, $J_{-\nu}(x) = (-1)^\nu J_\nu(x)$.

Graphs of Bessel functions of the first kind, for $\nu = 0, 1, 2$:



The series expression for the Bessel function of the first kind is

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

This function has a simpler form when ν is an odd half-integer. For example,

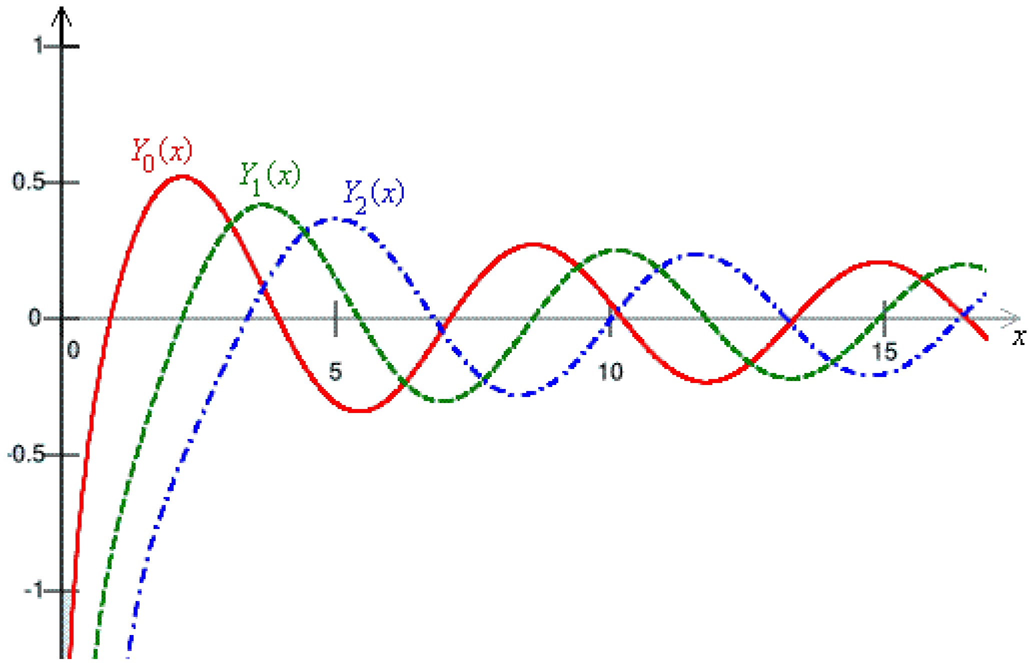
$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x , \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$$

The Bessel function of the second kind is

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

$Y_\nu(x)$ is unbounded as $x \to 0$: $\lim_{x\to 0^+} Y_\nu(x) = -\infty$

Bessel functions of the second kind (all of which have a singularity at $x = 0$):



Bessel functions arise frequently in situations where cylindrical or spherical polar coordinates are used.

A generalised Bessel ODE is

$$x^2 \frac{d^2y}{dx^2} + (1-2a)x\frac{dy}{dx} + \left(b^2c^2x^{2c} + \left(a^2 - c^2\nu^2\right)\right)y = 0$$

whose general solution is

$$y(x) = x^a\left(A\,J_\nu\left(bx^c\right) + B\,Y_\nu\left(bx^c\right)\right)$$

For a generalised Bessel ODE with $a \ge 0$, whenever the solution must remain bounded as $x \to 0$, the general solution simplifies to $y(x) = A\,x^a J_\nu\left(bx^c\right)$.

Example 1.12.2

Find a Maclaurin series solution to Legendre's ODE

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p\left(p+1\right)y = 0$$

in the case when p is a non-negative integer.

$P(x)=1-x^2 \quad\Rightarrow\quad P(0)=1\neq 0 \quad\Rightarrow\quad x=0$ is a regular point of the ODE.

Let the general solution be $y(x) = \sum_{n=0}^{\infty} a_n x^n$.

Then $y'(x) = \sum_{n=1}^{\infty} n\,a_n x^{n-1}$ and $y''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$.

Substitute into the ODE:

$$\sum_{n=0}^{\infty} n(n-1)a_n\left(x^{n-2}-x^n\right) - \sum_{n=0}^{\infty} 2n\,a_n x^n + \sum_{n=0}^{\infty} p(p+1)a_n x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty}\left(-n(n-1) - 2n + p(p+1)\right)a_n x^n = 0$$

But the first two terms (n = 0 and n = 1) of the first series are both zero.

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = 0 + 0 + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

after a shift in indices. Returning to the full ODE,

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty}\left(-n^2+n-2n + p^2+p\right)a_n x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty}\Big((n+2)(n+1)a_{n+2} + \left(-\left(n^2-p^2\right) - (n-p)\right)a_n\Big)x^n = 0$$

Matching coefficients of x^n, $(n\geq 0)$:

$$(n+2)(n+1)a_{n+2} = \left(n^2-p^2+n-p\right)a_n \quad\Rightarrow\quad a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)}a_n$$

Shifting indices, $a_n = \dfrac{(n+p-1)(n-p-2)}{n(n-1)}a_{n-2} \quad (n\geq 2)$, with a_0 and a_1 arbitrary.

Rearranging slightly, $a_n = \dfrac{-\left(p-(n-2)\right)\left(p+(n-1)\right)}{n(n-1)}a_{n-2} \quad (n\geq 2)$

Example 1.12.2 (continued)

$$\Rightarrow \quad a_2 = \frac{-p(p+1)}{2\times 1}a_0\,, \quad a_3 = \frac{-(p-1)(p+2)}{3\times 2}a_1\,,$$

$$a_4 = \frac{-(p-2)(p+3)}{4\times 3}a_2 = \frac{+p(p+1)(p-2)(p+3)}{4\times 3\times 2\times 1}a_0\,,$$

$$a_5 = \frac{-(p-3)(p+4)}{5\times 4}a_3 = \frac{+(p-1)(p+2)(p-3)(p+4)}{5\times 4\times 3\times 2\times 1}a_1\,,$$

$$a_6 = \frac{-(p-4)(p+5)}{6\times 5}a_4 = \frac{-p(p+1)(p-2)(p+3)(p-4)(p+5)}{6\times 5\times 4\times 3\times 2\times 1}a_0\,, \quad \text{etc.}$$

It then follows that the general solution to Legendre's ODE is

$$y_p(x) = a_0\left(1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p+1)(p-2)(p+3)}{4!}x^4 - \frac{p(p+1)(p-2)(p+3)(p-4)(p+5)}{6!}x^6 + \dots\right)$$
$$+ a_1\left(x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p+2)(p-3)(p+4)}{5!}x^5 - \frac{(p-1)(p+2)(p-3)(p+4)(p-5)(p+6)}{7!}x^7 + \dots\right)$$

where a_0 and a_1 are arbitrary constants. This series converges on [–1, 1] .

These solutions $y(x) = \sum_{n=0}^{\infty} a_n x^n$ are Legendre functions of order p .

If p is a non-negative integer then

$$a_{p+2} = \frac{(p-p)(p+p+1)}{(p+2)(p+1)}a_p = 0 \quad \Rightarrow \quad a_{p+4} = 0 \quad \Rightarrow \quad \dots$$

$$\Rightarrow \quad a_{p+2k} = 0 \quad \forall k \in \mathbb{N}$$

If we set $a_1 = 0$ when p is even, then the series solution terminates as a p^{th} order polynomial (and therefore converges for all x).

If we set $a_0 = 0$ when p is odd, then the series solution terminates as a p^{th} order polynomial (and therefore converges for all x).

With suitable choices of a_0 and a_1, so that $P_n(1) = 1$,
we have the set of **Legendre polynomials**:
$P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2 - 1)$,
$P_3(x) = \frac{1}{2}(5x^3 - 3x)$, $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$, $P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$,
$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$, etc.
Each $P_n(x)$ is a solution of Legendre's ODE with $p = n$.

Rodrigues' formula generates all of the Legendre polynomials:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left(\left(x^2 - 1\right)^n\right)$$

Among the properties of Legendre polynomials is their orthogonality on [–1, 1]:

$$\int_{-1}^{+1} P_m(x) P_n(x)\, dx = \begin{cases} 0 & (m \neq n) \\ \dfrac{2}{2n+1} & (m = n) \end{cases}$$

[Space for any additional notes]

END OF CHAPTER 1