2.02 Summary of Matrix Algebra

Some rules of matrix algebra are summarized here.

The **dimensions** of a matrix are (# rows × #columns) [in that order].

Addition and subtraction are defined only for matrices of the same dimensions as each other. The sum of two matrices is found by adding the corresponding entries.

Example 2.02.1

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

Scalar multiplication:
The product cA of matrix A with scalar c is obtained by multiplying every element in the matrix by c.

Example 2.02.2

$$5\begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 0 \\ 0 & 15 & 10 \end{bmatrix}$$

Matrix multiplication:
The product C = AB of a $(p \times q)$ matrix A with an $(r \times s)$ matrix B is defined if and only if $q = r$. The product C has dimensions $(p \times s)$ and entries

$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$$

or c_{ij} = (i^{th} row of A) • (j^{th} column of B) [usual Cartesian dot product]

Example 2.02.3

$$\text{AB} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} (1\times 3 + 2\times 2 + 0\times 1) \\ (0\times 3 + 3\times 2 + 2\times 1) \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

Note that matrix multiplication is, in general, **not commutative**: BA $\neq$ AB.
In this example, BA is not even defined!

The **transpose** of the $(m \times n)$ matrix $A = \{ a_{ij} \}$ is the $(n \times m)$ matrix $A^T = \{ a_{ji} \}$.
The transpose of the product AB is $(AB)^T = B^T A^T$.

Example 2.02.4

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 0 & 2 \end{bmatrix}, \quad B^T = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad B^T A^T = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 8 \end{bmatrix} = (AB)^T$$

A matrix is **symmetric** if and only if $A^T = A$ (which requires $a_{ji} = a_{ij}$ for all (i, j)).
A matrix is **skew-symmetric** if and only if $A^T = -A$.

A **square matrix** has equal numbers of rows and columns.

If a matrix is symmetric or skew-symmetric, then it must be a square matrix.

If a matrix is skew-symmetric, then it must be a square matrix whose leading diagonal elements are all zero.

Example 2.02.5

$$A = \begin{bmatrix} 1 & 5 & 0 & -2 \\ 5 & 2 & -1 & 7 \\ 0 & -1 & 3 & 1 \\ -2 & 7 & 1 & 4 \end{bmatrix} \quad \text{is symmetric.}$$

$$B = \begin{bmatrix} 0 & 5 & 0 & -2 \\ -5 & 0 & -1 & 7 \\ 0 & 1 & 0 & -1 \\ 2 & -7 & 1 & 0 \end{bmatrix} \quad \text{is skew-symmetric.}$$

Any square matrix may be written as the sum of a symmetric matrix and a skew-symmetric matrix.

A square matrix is **upper triangular** if all entries below the leading diagonal are zero.
A square matrix is **lower triangular** if all entries above the leading diagonal are zero.
A square matrix that is both upper and lower triangular is **diagonal**.

Example 2.02.6

$$\mathrm{A} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & \frac{1}{5} \\ 0 & 0 & 3 \end{bmatrix}$$ is upper triangular.

$$\mathrm{A}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 0 & \frac{1}{5} & 3 \end{bmatrix}$$ is lower triangular.

$$\mathrm{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$ is diagonal.

The **trace** of a diagonal matrix is the sum of its elements. $\Rightarrow$ trace(B) = 6.

The diagonal matrix whose diagonal entries are all one is the **identity matrix** I.
Let I_n represent the $(n \times n)$ identity matrix.
$\mathrm{I}_m \mathrm{A} = \mathrm{A}\,\mathrm{I}_n = \mathrm{A}$ for all $(m \times n)$ matrices A.

If it exists, the **inverse** A^{-1} of a square matrix A is such that

$$\mathrm{A}^{-1}\mathrm{A} = \mathrm{A}\mathrm{A}^{-1} = \mathrm{I}$$

If the inverse A^{-1} exists, then A^{-1} is unique and A is **invertible**.
If the inverse A^{-1} does not exist, then A is **singular**.

Important distinctions between matrix algebra and scalar algebra:

$ab = ba$ for all scalars a, b; but
AB = BA is true only for some special choices of matrices A, B.

$ab = 0 \Rightarrow a = 0$ and/or $b = 0$, but
AB = 0 can happen when neither A nor B is the zero matrix.

Example 2.02.7

$$\mathrm{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathrm{B} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \mathrm{AB} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathrm{O}$$

2.03 Determinants and Inverse Matrices

The determinant of the trivial 1 × 1 matrix is just its sole entry:
det [a] = a .

The determinant of a 2 × 2 matrix A is

$$\det(\mathrm{A}) = |\mathrm{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For higher order ($n \times n$) matrices A = { a_{ij} }, the determinant can be evaluated as follows:

The **minor** M_{ij} of element a_{ij} is the determinant of order (n – 1) formed from matrix A by deleting the row and column through the element a_{ij}.

The **cofactor** C_{ij} of element a_{ij} is found from $\mathrm{C}_{ij} = (-1)^{i+j}\,\mathrm{M}_{ij}$

The determinant of A is the sum, along any one row or down any one column, of the product of each element with its cofactor:

$$\det(A) = \sum_{j=1}^{n} a_{ij} C_{ij} \qquad (i = \text{any one of } 1, 2, \ldots, n)$$

or

$$\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} \qquad (j = \text{any one of } 1, 2, \ldots, n)$$

If one row or column has more zero entries than the others, then one usually chooses to expand along that row or column.

The determinant of a triangular matrix is just the product of its diagonal entries.
det (I) = 1

Example 2.03.1

Evaluate the vector (cross) product of the vectors $\vec{\mathbf{a}} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\vec{\mathbf{b}} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

Expanding along the top row,

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 2 & 4 & 3 \end{vmatrix} = +\hat{\mathbf{i}}\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} - \hat{\mathbf{j}}\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} + \hat{\mathbf{k}}\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$$

$$= +(2\times 3 - 4\times 3)\hat{\mathbf{i}} - (1\times 3 - 2\times 3)\hat{\mathbf{j}} + (1\times 4 - 2\times 2)\hat{\mathbf{k}}$$

$$\therefore \quad \vec{\mathbf{a}} \times \vec{\mathbf{b}} = -6\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$$

$$\det(AB) = \det(BA) = \det(A)\det(B)$$

$$\det(A^T) = \det(A)$$

$\det(A) = 0 \Rightarrow$ A is singular.

$$\det(A) \neq 0 \quad \Rightarrow \quad A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)}$$

where adj(A) is the adjoint matrix of A, which is the transpose of the matrix of cofactors of A. For a (2×2) matrix, the formula for the inverse follows quickly:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (ad \neq bc)$$

<u>Example 2.03.2</u>

$$A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{5}\begin{bmatrix} 4 & -1 \\ -3 & 2 \end{bmatrix}$$

For higher order matrices, this adjoint/determinant method of obtaining the inverse matrix becomes very tedious and time-consuming. A much faster method of finding the inverse involves Gaussian elimination to transform the augmented matrix [A | I] into the augmented matrix in reduced echelon form [I | A^{-1}].

<u>Example 2.03.3</u>

Find the inverse of the matrix $A = \begin{bmatrix} -1 & 1 & 0 \\ 3 & 2 & 1 \\ -2 & -1 & -1 \end{bmatrix}$.

$$[A|I] = \left[\begin{array}{rrr|rrr} -1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Multiply Row 1 by (–1):

$$\xrightarrow{R_1 \times -1} \left[\begin{array}{rrr|rrr} \boxed{1} & -1 & 0 & -1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ -2 & -1 & -1 & 0 & 0 & 1 \end{array}\right]$$

Example 2.03.3 (continued)

From Row 2 subtract (3 × Row 1) and
to Row 3 add (2 × Row 1):

$$\xrightarrow[R_3+2R_1]{R_2-3R_1} \left[\begin{array}{ccc|ccc} \boxed{1} & -1 & 0 & -1 & 0 & 0 \\ 0 & 5 & 1 & 3 & 1 & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

All entries below the first leading one are now zero.
The next leading entry is a '5'. Scale it down to a '1'.
Multiply Row 2 by (1/5):

$$\xrightarrow{R_2\times\frac{1}{5}} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & \boxed{1} & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & 0 \\ 0 & -3 & -1 & -2 & 0 & 1 \end{array}\right]$$

Clear the entry below the new leading one.
To Row 3 add (3 × Row 2):

$$\xrightarrow[R_3+3R_2]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & \boxed{1} & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{2}{5} & -\frac{1}{5} & \frac{3}{5} & 1 \end{array}\right]$$

The next leading entry is a '–2/5'. Scale it down to a '1'.
Multiply Row 3 by (–5/2):

$$\xrightarrow[R_3\times-\frac{5}{2}]{} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{3}{5} & \frac{1}{5} & 0 \\ 0 & 0 & \boxed{1} & \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \end{array}\right]$$

From Row 2 subtract (1/5 × Row 3):

$$\xrightarrow{R_2-\frac{1}{5}R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \boxed{1} & \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \end{array}\right]$$

To Row 1 add Row 2:

$$\xrightarrow{R_1+R_2} \left[\mathrm{I}|\mathrm{A}^{-1}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \end{array}\right] \Rightarrow \mathrm{A}^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & -5 \end{bmatrix}$$

Example 2.03.3 (continued)

As a check on the answer,

$$A^{-1}A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & -5 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 3 & 2 & 1 \\ -2 & -1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = I$$

Determinants may be evaluated in a similar manner:

Every row operation that subtracts a multiple of a row from another row produces a matrix whose determinant is the same as the previous matrix.

Every interchange of rows changes the sign of the determinant.

Every multiplication of a row by a constant multiplies the determinant by that constant.

Tracking the operations performed in Example 2.03.3 above (that reduced matrix A to the identity matrix I),

Operations	Net factor to date:
Multiply Row 1 by (–1):	**× (–1)**
From Row 2 subtract (3 × Row 1) and	× (–1)
to Row 3 add (2 × Row 1):	× (–1)
Multiply Row 2 by (1/5):	**× (–1/5)**
To Row 3 add (3 × Row 2):	× (–1/5)
Multiply Row 3 by (–5/2):	**× (+1/2)**
From Row 2 subtract (1/5 × Row 3):	× (+1/2)
To Row 1 add Row 2:	× (+1/2)

Therefore

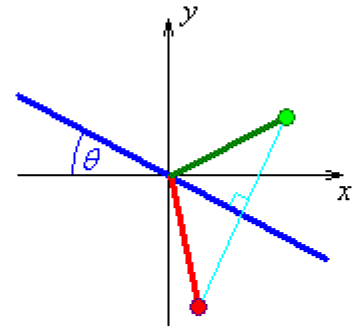
$$\det I = \frac{1}{2}\times\det A \quad\Rightarrow\quad \det A = \begin{vmatrix} -1 & 1 & 0 \\ 3 & 2 & 1 \\ -2 & -1 & -1 \end{vmatrix} = 2\left(\det I\right) = 2$$

One can also show that

$$\operatorname{adj}\left(\begin{bmatrix} -1 & 1 & 0 \\ 3 & 2 & 1 \\ -2 & -1 & -1 \end{bmatrix}\right) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & -5 \end{bmatrix} \quad\Rightarrow\quad A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & -5 \end{bmatrix}$$

2.04 Eigenvalues and Eigenvectors

Example 2.04.1



In $\mathbb{R}^3$, the effect of reflection, in a vertical plane mirror through the origin that makes an angle θ with the x-z coordinate plane, on the values of the Cartesian coordinates (x, y, z), may be represented by the matrix equation

$$\vec{\mathbf{x}}_{\text{new}} = \mathrm{R}_\theta \vec{\mathbf{x}}_{\text{old}} \quad \text{or} \quad \begin{bmatrix} x_{\text{new}} \\ y_{\text{new}} \\ z_{\text{new}} \end{bmatrix} = \begin{bmatrix} +\cos 2\theta & -\sin 2\theta & 0 \\ -\sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\text{old}} \\ y_{\text{old}} \\ z_{\text{old}} \end{bmatrix}$$

The reflection matrix R_θ may be constructed from the composition of three consecutive operations:
rotate all of $\mathbb{R}^3$ about the z axis, so that the mirror is rotated into the x-z plane; then
reflect the y coordinate to its negative; then
rotate all of $\mathbb{R}^3$ about the z axis, so that the mirror is rotated back to its starting position.
With the help of some trigonometric identities, one can show that

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) & 0 \\ \sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} +\cos 2\theta & -\sin 2\theta & 0 \\ -\sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Obviously, any point on the mirror does not move as a result of the reflection.
Points on the mirror have coordinates $(r \cos\theta, -r \sin\theta, z)$, where r and z are any real numbers.
[Note that two free parameters are needed to describe a two-dimensional surface.]

$$\vec{\mathbf{x}} = \begin{bmatrix} r\cos\theta \\ -r\sin\theta \\ z \end{bmatrix} \Rightarrow \mathrm{R}_\theta \vec{\mathbf{x}} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ -\sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r\cos\theta \\ -r\sin\theta \\ z \end{bmatrix}$$

$$= \begin{bmatrix} r(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta) \\ r(-\sin 2\theta\cos\theta + \cos 2\theta\sin\theta) \\ z \end{bmatrix} = \begin{bmatrix} r\cos(2\theta-\theta) \\ -r\sin(2\theta-\theta) \\ z \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ -r\sin\theta \\ z \end{bmatrix} = \vec{\mathbf{x}}$$

Therefore any member of the two dimensional vector space

$$\vec{\mathbf{x}} = \left\{ r \begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad (r, z \in \mathbb{R})$$

is invariant under the reflection, $\left(\mathrm{R}_\theta \vec{\mathbf{x}} = \vec{\mathbf{x}} \right)$. The basis vectors of this vector space,

$\begin{bmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, are the **eigenvectors** of R_θ for the **eigenvalue** +1,

(as is any non-zero combination of them).

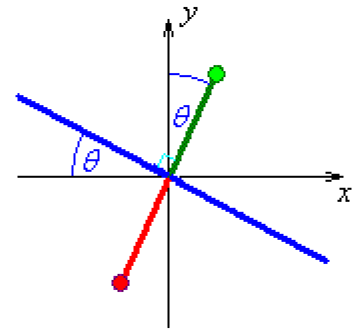
Any point on the line through the origin that is at right angles to the mirror, (r sin θ, r cos θ, 0), will be reflected to –(r sin θ, r cos θ, 0).

For these points, $\mathrm{R}_\theta \vec{\mathbf{x}} = -1\vec{\mathbf{x}}$.

The basis vector of this one-dimensional vector space,

$\begin{bmatrix} \sin\theta \\ \cos\theta \\ 0 \end{bmatrix}$, is the eigenvector of R_θ for the eigenvalue –1,

(as is any non-zero multiple of it).



The zero vector is always a solution of any matrix equation of the form A**x** = λ **x**.
$\vec{\mathbf{x}} = \vec{\mathbf{0}}$ is known as the **trivial solution**.

Non-trivial solutions of A **x** = λ **x** are possible only for λ = +1 and for λ = –1 in this example (with $\mathrm{A} = \mathrm{R}_\theta$).

The eigenvectors for λ = +1 correspond to points on the mirror that map to themselves under the reflection operation R_θ.

The eigenvectors for λ = –1 correspond to points on the normal line that map to their own negatives under the reflection operation R_θ.

No other non-zero vectors will map to simple multiples of themselves under R_θ.

We can summarize the results by displaying the unit eigenvectors as the columns of one matrix and their corresponding eigenvalues as the matching entries in a diagonal matrix:

$$\mathrm{X} = \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the matrix X is **orthogonal**, ($X^{-1} = X^T$ - its inverse is the same as its transpose)
[In this case, X happens to be symmetric also, so that $X^{-1} = X^T = X$.]

Also note that $X^{-1} R_\theta X = \Lambda$:

$$\begin{bmatrix} \sin\theta & \cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 2\theta & -\sin 2\theta & 0 \\ -\sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sin\theta & \cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore the matrix X of unit eigenvectors of R_θ diagonalizes the matrix R_θ.

This is generally true of any ($n \times n$) matrix that possesses n linearly independent eigenvectors (some ($n \times n$) matrices do not).

Note that

$$A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}} \quad \Rightarrow \quad (A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$$

The solution to this square matrix equation will be unique if and only if $\det(A - \lambda I) \neq 0$.
That unique solution is the trivial solution $\vec{\mathbf{x}} = \vec{\mathbf{0}}$.
Therefore eigenvectors can be found if and only if λ is such that $\det(A - \lambda I) = 0$.

General method to find eigenvalues and eigenvectors

$\det(A - \lambda I) = 0$ is the **characteristic equation** from which all of the eigenvalues of the matrix A can be found. For each value of λ, the corresponding eigenvectors are determined by finding the non-trivial solutions to the matrix equation $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

Example 2.04.2

Find all eigenvalues and unit eigenvectors for the matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$.

Characteristic equation:

$$\det(A - \lambda I) = 0 \quad \Rightarrow \quad \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (-2-\lambda)^2 - 1 = 0$$

$\Rightarrow \lambda^2 + 4\lambda + 4 - 1 = 0 \quad \Rightarrow \lambda^2 + 4\lambda + 3 = 0 \quad \Rightarrow (\lambda + 3)(\lambda + 1) = 0$

Therefore the eigenvalues are

$$\boxed{\lambda = -3 \text{ and } \lambda = -1 .}$$

$\underline{\lambda = -3}$:

$$(A - (-3)I)\vec{\mathbf{x}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \begin{bmatrix} -2+3 & 1 \\ 1 & -2+3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x + y = 0$ (only one independent equation)

$\Rightarrow y = -x$

$\Rightarrow$ any non-zero multiple of $\begin{bmatrix} +1 \\ -1 \end{bmatrix}$ is an eigenvector for $\lambda = -3$.

The unit eigenvector is $\dfrac{\sqrt{2}}{2}\begin{bmatrix} +1 \\ -1 \end{bmatrix}$ (or its negative).

$\underline{\lambda = -1}$:

$$(A - (-1)I)\mathbf{x} = \mathbf{0} \quad \Rightarrow \quad \begin{bmatrix} -2+1 & 1 \\ 1 & -2+1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -x + y = 0$ (only one independent equation)

$\Rightarrow y = x$

$\Rightarrow$ any non-zero multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$.

The unit eigenvector is $\dfrac{\sqrt{2}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (or its negative).

A matrix X that diagonalizes A (by $X^T A X = \Lambda$) is

$$X = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \rightarrow \quad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}.$$

One can quickly show that

$$X^T A X = \frac{\sqrt{2}}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\frac{\sqrt{2}}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} = \Lambda .$$

END OF CHAPTER 2

3. Numerical Methods

The majority of equations of interest in actual practice do not admit any analytic solution. Even equations as simple as $x = e^{-x}$ and $I = \int e^{-x^2} dx$ have no exact solution. Such cases require numerical methods. Only a very brief survey is presented here.

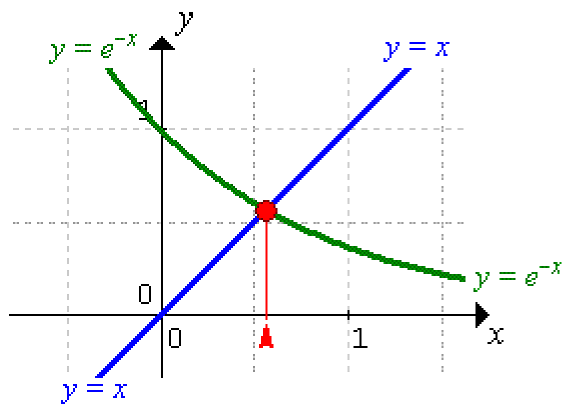
Sections in this Chapter:

3.01 Bisection

Example 3.01.1

Find the solution of $x = e^{-x}$, correct to 4 decimal places.

From a sketch of the two curves $y = x$ and $y = e^{-x}$, it is obvious that the only solution is somewhere in the interval (0, 1).



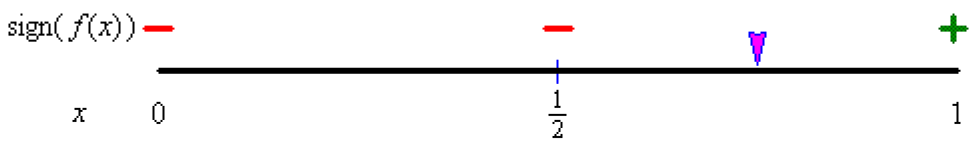
Let $f(x) = x - e^{-x}$.

Clearly $f(0) = -1 < 0$

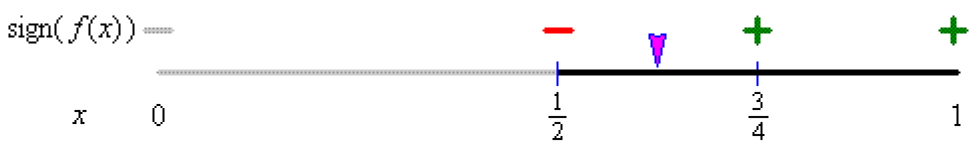
and $f(1) = 1 - 1/e > 0$

$f(x)$ is continuous and changes sign only once inside (0, 1).

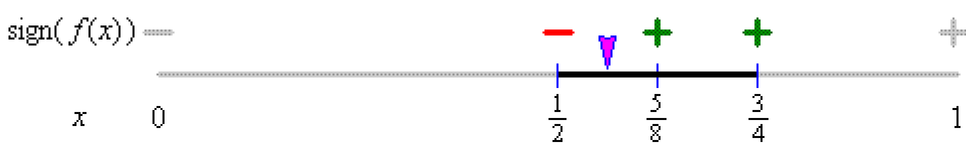
Halve the interval repeatedly and retain the half with a sign change:



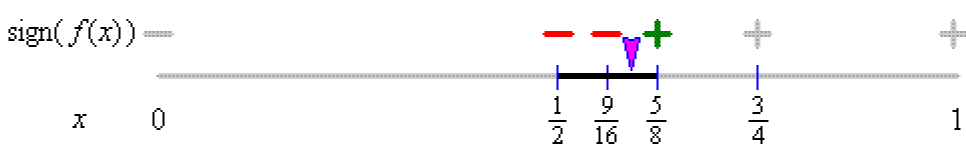
$f(0.50000) = -0.1065... < 0 \quad \Rightarrow \quad$ root is in (0.50000, 1.00000)



$f(0.75000) = +0.2776... > 0 \quad \Rightarrow \quad$ root is in (0.50000, 0.75000)



$f(0.62500) = +0.0897... > 0 \quad \Rightarrow \quad$ root is in (0.50000, 0.62500)

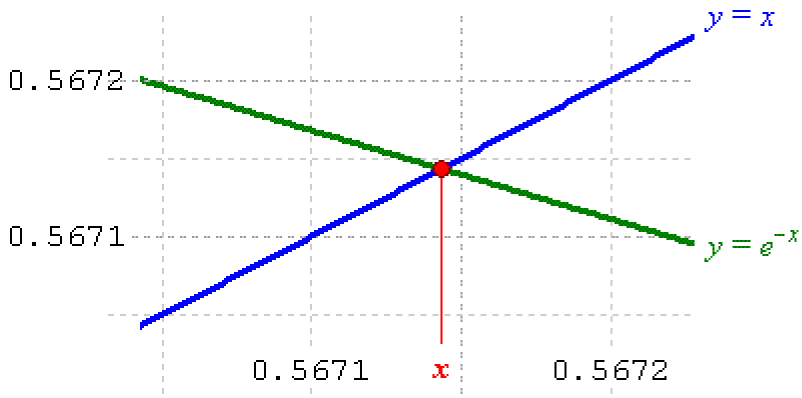


$f(0.56250) = -0.0072... < 0 \quad \Rightarrow \quad$ root is in (0.56250, 0.62500)

Example 3.01.1 (continued)

This method is slow and requires eighteen steps before the change in x is small enough to leave the fourth decimal place undisturbed with certainty:
$f(0.567142) = -0.0000... < 0 \quad \Rightarrow \quad$ root is in (0.567142, 0.567146)

This method is equivalent to zooming in graphically by repeated factors of 2 until the desired accuracy is obtained. The result of a faster graphical zoom, sufficient to determine the solution to five decimal places, is displayed here:



Correct to four decimal places, the solution to $x = e^{-x}$ is $\boldsymbol{x} = \mathbf{0.5671}$.

A calculator quickly confirms that $e^{-0.5671} \approx 0.5671$.

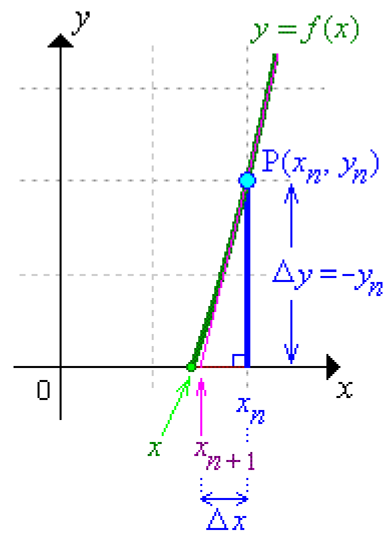
A spreadsheet to demonstrate the bisection method for this example is available from the course web site, at "`www.engr.mun.ca/~ggeorge/9420/demos/`".

3.02 Newton's Method

From the definition of the derivative, $\dfrac{dy}{dx} = \lim_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$,

we obtain $\Delta y \approx \dfrac{dy}{dx} \Delta x$ or, equivalently,

$\Delta x \approx \dfrac{\Delta y}{f'(x)}$.



The tangent line to the curve $y = f(x)$ at the point $P(x_n, y_n)$ has slope $= f'(x_n)$.

Follow the tangent line down to its x axis intercept.
That intercept is the next approximation x_{n+1} .

$\Delta y = y_{n+1} - y_n = 0 - y_n = -f(x_n)$ and

$\Delta x = x_{n+1} - x_n$

$$\Rightarrow \quad x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$$

If x_n is the n^{th} approximation to the equation $f(x) = 0$, then a better approximation may be

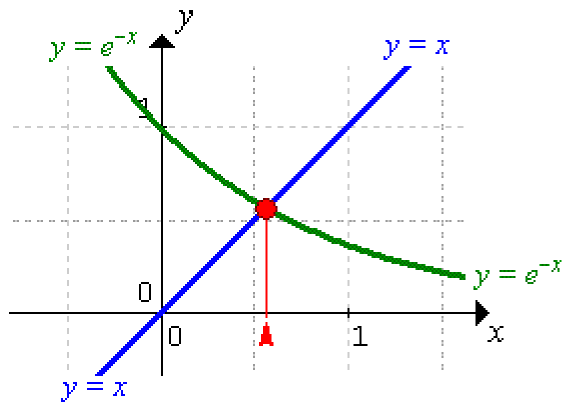
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

which is Newton's method.

Example 3.02.1

Find the solution of $x = e^{-x}$, correct to 4 decimal places.

From a sketch of the two curves $y = x$ and $y = e^{-x}$, it is obvious that the only solution is somewhere in the interval (0, 1). A reasonable first guess is $x_0 = \frac{1}{2}$.



$$f(x) = x - e^{-x} \quad \Rightarrow \quad f'(x) = 1 + e^{-x}$$

$$\Rightarrow \quad x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n - e^{-x_n}}{1 + e^{-x_n}}.$$

Table of consecutive values:

x_n	$f(x_n) = x_n - e^{-x_n}$	$f'(x_n) = 1 + e^{-x_n}$	$\dfrac{f(x_n)}{f'(x_n)}$
0.500000	–0.106531	1.606531	–0.066311
0.566311	–0.001305	1.567616	–0.000832
0.567143	0.000000	1.567143	0.000000
0.567143			

Correct to four decimal places, the solution to $x = e^{-x}$ is $\boldsymbol{x = 0.5671}$.
In fact, we have the root correct to six decimal places, $x = 0.567143$.

A spreadsheet to demonstrate Newton's method for this example is available from the course web site, at "`www.engr.mun.ca/~ggeorge/9420/demos/`".

This method converges much more rapidly than bisection, but requires more computational effort.

Note that Newton's method can fail if $f'(x) = 0$ in the neighbourhood of the root. A shallow tangent line could result in a sequence of approximations that fails to converge to the correct value.

3.03 Euler's Method for First Order ODEs

One of the simplest methods for obtaining the numerical values of solutions of initial value problems of the form

$$y' = f(x, y), \qquad y(x_0) = y_0$$

is Euler's method.

From the definition of the derivative, $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$, we obtain $\Delta y \approx \dfrac{dy}{dx} \Delta x$.
If we seek values of the solution $y(x)$ at successive evenly spaced values of x, then we have $\Delta y = y_{n+1} - y_n \Rightarrow y_{n+1} = y_n + \Delta y \approx y_n + f(x_n, y_n) \Delta x$.
With (by convention) $h = \Delta x$, we have the iterative scheme

$$\boxed{y_{n+1} = y_n + h f(x_n, y_n)}$$

However, errors propagate rapidly unless the step size h is very small, which requires a proportionate increase in the number of computations. Several modifications to Euler's method have been proposed, that replace the derivative $y' = f(x_n, y_n)$ by a weighted average of values of f at points around (x_n, y_n).

One of the most popular modifications is the fourth order Runge-Kutta method (RK4).

3.04 Fourth Order Runge-Kutta Procedure (RK4)

Values (x_n, y_n) [with $x_n = x_0 + nh$] of the solution $y(x)$ to the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0$$

are given by the iterative scheme

$$\boxed{\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h\,k_1\right) \\ k_3 &= f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h\,k_2\right) \\ k_4 &= f\left(x_n + h,\ y_n + h\,k_3\right) \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned}}$$

Example 3.04.1

Use the RK4 procedure with step size $h = 0.1$ to obtain an approximation to $y(1.5)$ for the solution of the initial value problem $y' = 2xy$, $y(1) = 1$.

$x_0 = 1$, $h = 0.1$ and we want $y(1.5)$. $1.5 = 1 + 5\times0.1$, so we need to find y_5.
$(x_0, y_0) = (1, 1)$ and $f(x, y) = 2xy$.

For $n = 0$:

$$k_1 = f(x_0, y_0) = 2x_0 y_0 = 2\times1\times1 = 2$$

$$k_2 = 2\left(1+\tfrac{1}{2}(0.1)\right)\left(1+\tfrac{1}{2}(0.1)2\right) = 2.31$$

$$k_3 = 2\left(1+\tfrac{1}{2}(0.1)\right)\left(1+\tfrac{1}{2}(0.1)2.31\right) = 2.34255$$

$$k_4 = 2(1+0.1)\left(1+(0.1)2.34255\right) = 2.715361$$

$$y_1 = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{0.1}{6}\left(2+2(2.31)+2(2.34255)+2.715361\right)$$

Therefore $y(1.1) \approx y_1 = 1.23367435$

We can proceed with a similar chain of calculations to find y_2, y_3, y_4 and finally y_5 .

Example 3.04.1 (continued)

n	x_n	y_n	k_1	k_2	k_3	k_4
0	1.000000	1.000000	2.000000	2.310000	2.342550	2.715361
1	1.100000	1.233674	2.714084	3.149571	3.199652	3.728735
2	1.200000	1.552695	3.726469	4.347547	4.425182	5.187555
3	1.300000	1.993687	5.183586	6.082738	6.204124	7.319478
4	1.400000	2.611633	7.312573	8.634059	8.825675	10.482602
5	**1.500000**	**3.490211**				

Therefore $y(1.5) \approx \mathbf{3.4902}$.

This initial value problem happens to have an exact solution, $y = e^{x^2-1}$.
We can therefore test the accuracy of the RK4 procedure in this case.

The exact value of $y(1.5)$ is 3.4903..., an absolute error of less than 0.0002 and a relative error of less than 0.01%. Euler's method, in contrast, has an error exceeding 16%!

END OF CHAPTER 3