Let (α_i, β_i) be the eigenvector associated with the eigenvalue λ_i of the coefficient matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let c_1, c_2 be arbitrary constants.

Case of **real, distinct, negative eigenvalues** (with $\lambda_2 < \lambda_1 < 0$):

Two linearly independent solutions are

$$(x(t), y(t)) = \left(\alpha_1 e^{\lambda_1 t},\ \beta_1 e^{\lambda_1 t}\right) \quad \text{and} \quad (x(t), y(t)) = \left(\alpha_2 e^{\lambda_2 t}, \beta_2 e^{\lambda_2 t}\right)$$

The general solution is

$$(x(t), y(t)) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\ c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)$$

One can see that $\lim_{t \to \infty} (x(t), y(t)) = (0, 0)$.

All orbits therefore terminate at the critical point at the origin.
The system is **asymptotically stable**.

If both arbitrary constants are zero, then we have the trivial solution ($x = y = 0$ for all t).

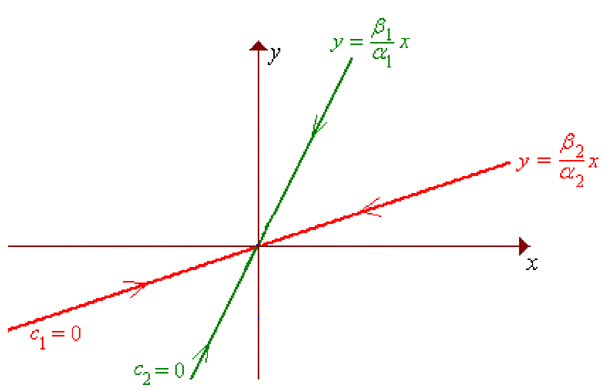
If one of the arbitrary constants is zero (say c_1), then

$$(x(t), y(t)) = \left(c_2\alpha_2 e^{\lambda_2 t},\ c_2\beta_2 e^{\lambda_2 t}\right) \quad \Rightarrow \quad y(t) = \frac{\beta_2}{\alpha_2} x(t)$$

which is a straight line through the origin, of slope $\frac{\beta_2}{\alpha_2}$.

[The situation is similar if c_2 is zero.]

We therefore obtain straight-line trajectories ending at the singular point, when exactly one of the arbitrary constants is zero.



If neither arbitrary constant is zero, then

$$\frac{y(t)}{x(t)} = \frac{c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}}{c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t}} = \frac{c_1\beta_1 + c_2\beta_2 e^{(\lambda_2-\lambda_1)t}}{c_1\alpha_1 + c_2\alpha_2 e^{(\lambda_2-\lambda_1)t}} = \frac{c_1\beta_1 e^{-(\lambda_2-\lambda_1)t} + c_2\beta_2}{c_1\alpha_1 e^{-(\lambda_2-\lambda_1)t} + c_2\alpha_2}$$

Because $\lambda_2 < \lambda_1 < 0$,

$$\lim_{t\to-\infty} \frac{y(t)}{x(t)} = \lim_{t\to-\infty} \frac{c_1\beta_1 e^{-(\lambda_2-\lambda_1)t} + c_2\beta_2}{c_1\alpha_1 e^{-(\lambda_2-\lambda_1)t} + c_2\alpha_2} = \frac{\beta_2}{\alpha_2}$$

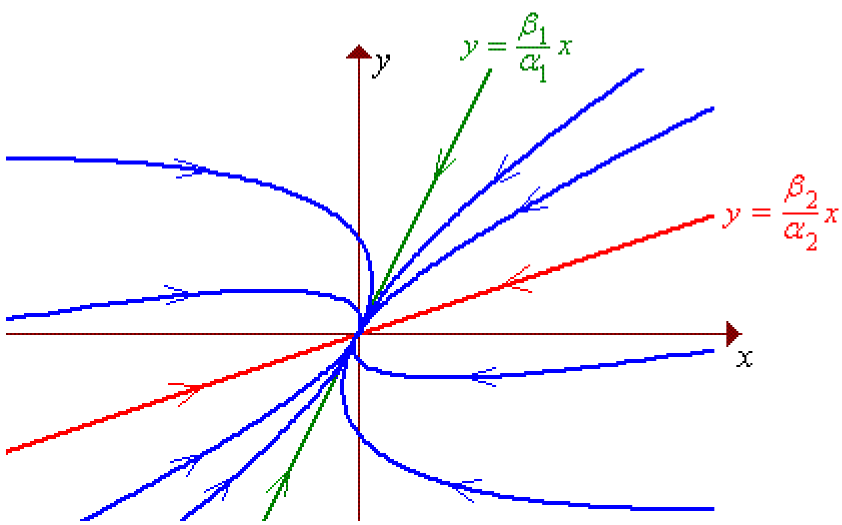
and

$$\lim_{t\to\infty} \frac{y(t)}{x(t)} = \lim_{t\to\infty} \frac{c_1\beta_1 + c_2\beta_2 e^{(\lambda_2-\lambda_1)t}}{c_1\alpha_1 + c_2\alpha_2 e^{(\lambda_2-\lambda_1)t}} = \frac{\beta_1}{\alpha_1}$$

All orbits therefore come in from infinity parallel to the line $y = \frac{\beta_2}{\alpha_2}x$.

All orbits share the same tangent at the origin, $y = \frac{\beta_1}{\alpha_1}x$.

We obtain a **stable node** that is also asymptotically stable.



[The case illustrated here is $\alpha_1 = 1,\ \alpha_2 = 3,\ \beta_1 = 2,\ \beta_2 = 1,\ \lambda_1 = -5,\ \lambda_2 = -10$, which is generated from $A = \begin{bmatrix} -11 & +3 \\ -2 & -4 \end{bmatrix}$.]

Case of **real, distinct, positive eigenvalues** (with $\lambda_2 > \lambda_1 > 0$):

The analysis leads to the same phase space, except that the arrows are reversed.
The result is an **unstable node**.

Case of **real, distinct eigenvalues of opposite sign** (with $\lambda_2 < 0 < \lambda_1$):

The general solution is

$$\left(x(t), y(t)\right) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\ c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)$$

$\lambda_2 < 0 < \lambda_1 \quad \Rightarrow \quad \lim\limits_{t\to-\infty}\left(x(t), y(t)\right)$ and $\lim\limits_{t\to\infty}\left(x(t), y(t)\right)$ do not exist (infinite),
(with the exception of the orbit for $c_1 = 0$).
All orbits (except $c_1 = 0$) therefore move away from the critical point at the origin.
The system is **unstable**.

If both arbitrary constants are zero, then we have the trivial solution ($x = y = 0$ for all t).

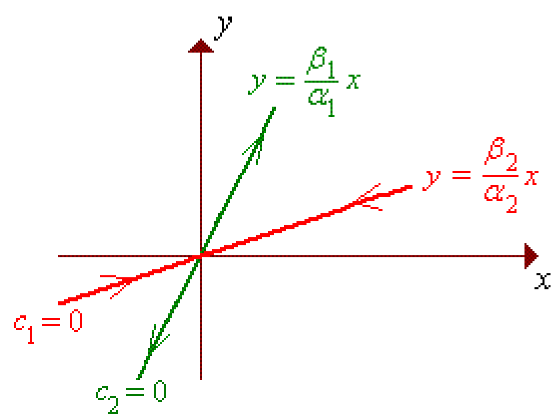
If one of the arbitrary constants is zero (say c_1), then

$$\left(x(t), y(t)\right) = \left(c_2\alpha_2 e^{\lambda_2 t},\ c_2\beta_2 e^{\lambda_2 t}\right) \quad \Rightarrow \quad y(t) = \frac{\beta_2}{\alpha_2}x(t)$$

which is a straight line through the origin, of slope $\dfrac{\beta_2}{\alpha_2}$.

[The situation is similar if c_2 is zero.]

We therefore obtain straight-line trajectories when one of the arbitrary constants is zero. One of them ($c_1 = 0$) ends at the singular point while the other begins there.



If neither arbitrary constant is zero, then

$$\frac{y(t)}{x(t)} = \frac{c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}}{c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t}} = \frac{c_1\beta_1 + c_2\beta_2 e^{(\lambda_2-\lambda_1)t}}{c_1\alpha_1 + c_2\alpha_2 e^{(\lambda_2-\lambda_1)t}} = \frac{c_1\beta_1 e^{-(\lambda_2-\lambda_1)t} + c_2\beta_2}{c_1\alpha_1 e^{-(\lambda_2-\lambda_1)t} + c_2\alpha_2}$$

Because $\lambda_2 < 0 < \lambda_1$,

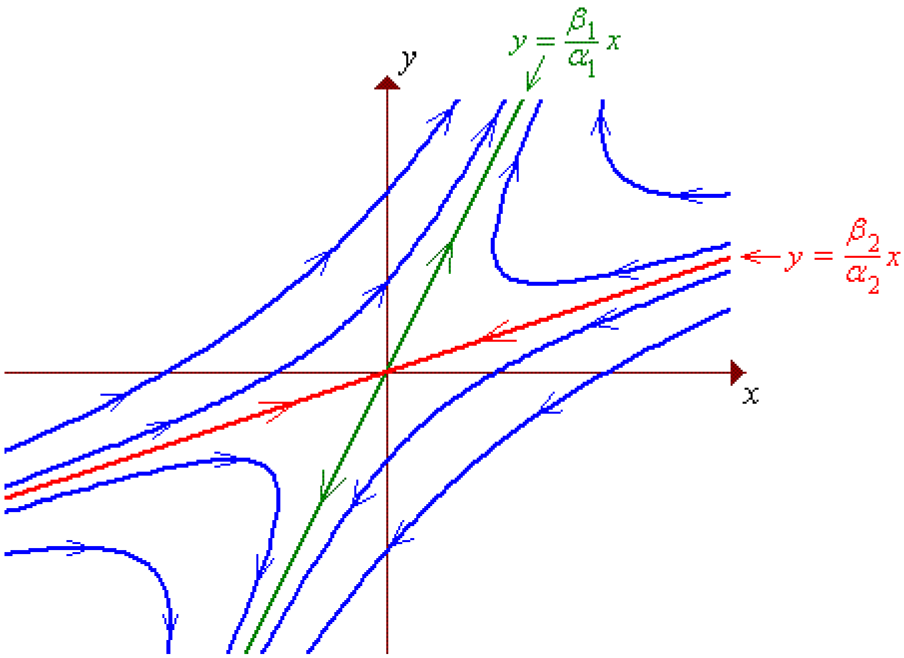
$$\lim_{t\to-\infty}\frac{y(t)}{x(t)} = \lim_{t\to-\infty}\frac{c_1\beta_1 e^{-(\lambda_2-\lambda_1)t} + c_2\beta_2}{c_1\alpha_1 e^{-(\lambda_2-\lambda_1)t} + c_2\alpha_2} = \frac{\beta_2}{\alpha_2}$$

and

$$\lim_{t\to\infty}\frac{y(t)}{x(t)} = \lim_{t\to\infty}\frac{c_1\beta_1 + c_2\beta_2 e^{(\lambda_2-\lambda_1)t}}{c_1\alpha_1 + c_2\alpha_2 e^{(\lambda_2-\lambda_1)t}} = \frac{\beta_1}{\alpha_1}$$

All orbits therefore share the same asymptotes, $y = \frac{\beta_2}{\alpha_2}x$ (incoming) and $y = \frac{\beta_1}{\alpha_1}x$ (outgoing).

We obtain a **saddle point**, which is an unstable critical point.



[The case illustrated here is $\alpha_1 = 1,\ \alpha_2 = 3,\ \beta_1 = 2,\ \beta_2 = 1,\ \lambda_1 = +5,\ \lambda_2 = -5$, which is generated from $A = \begin{bmatrix} 7 & -6 \\ 4 & -7 \end{bmatrix}$.]

Case of **real, equal, negative eigenvalues** ($\lambda_1 = \lambda_2 < 0$) **and** $b = c = 0$:

The system is uncoupled:

$$\frac{dx}{dt} = ax$$
$$\frac{dy}{dt} = dy$$

and equal eigenvalues now require $a = d = \lambda$.

The general solution is $\left(x(t), y(t)\right) = \left(c_1 e^{\lambda t}, c_2 e^{\lambda t}\right)$.

$\lambda < 0 \quad \Rightarrow \quad \lim_{t \to -\infty} \left(|x(t)|, |y(t)|\right) = (\infty, \infty)$ and $\lim_{t \to \infty} \left(x(t), y(t)\right) = (0, 0)$.

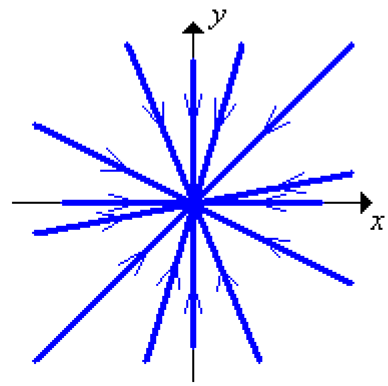
All orbits therefore terminate at the critical point at the origin.
The system is asymptotically stable.

If both arbitrary constants are zero, then we have the trivial solution ($x = y = 0$ for all t).

$c_1 \neq 0 \quad \Rightarrow \quad \dfrac{y(t)}{x(t)} = \dfrac{c_2}{c_1} \quad \forall t$

and $c_1 = 0, c_2 \neq 0 \quad \Rightarrow \quad x(t) = 0 \quad \forall t$

The orbits are straight lines
ending at the critical point at the origin.



The critical point is an **asymptotically stable star-shaped node**.

Additional Note:

The eigenvalues of *any* triangular matrix are the diagonal entries of that matrix:

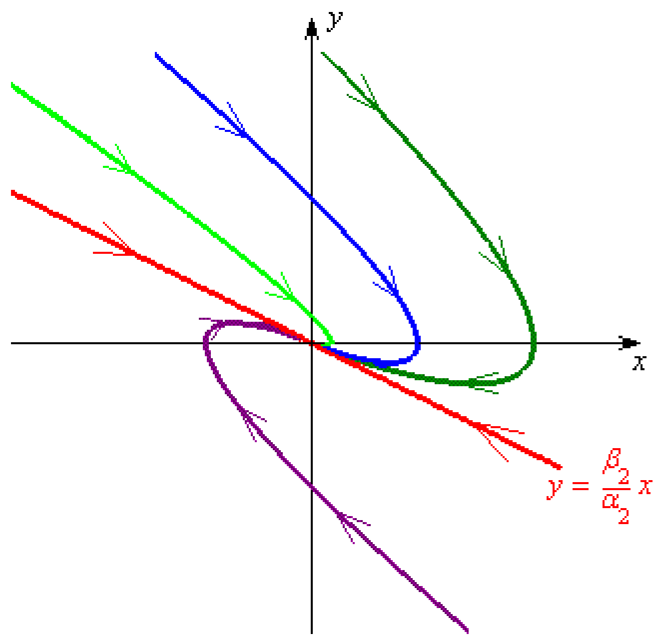
The characteristic equation of $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} a-\lambda & b \\ 0 & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) = 0 \qquad \Rightarrow \quad \lambda = a \text{ or } d$$

Case of **real, equal, negative eigenvalues** ($\lambda_1 = \lambda_2 < 0$) **and** b, c not both zero:

The characteristic equation $\lambda^2 - (a+d)\lambda + (ad - bc) = 0$
has the discriminant $(a+d)^2 - 4(ad-bc) = (a-d)^2 + 4bc = 0$.
The solution of the characteristic equation simplifies to $\lambda = \dfrac{a+d}{2}$.
The general solution is $\left(x(t), y(t)\right) = \left((c_1\alpha_1 + c_2\alpha_2 t)e^{\lambda t},\ (c_1\beta_1 + c_2\beta_2 t)e^{\lambda t}\right)$.
$\lambda < 0 \quad \Rightarrow \quad \lim\limits_{t\to -\infty}\left(|x(t)|, |y(t)|\right) = (\infty, \infty)$ and $\lim\limits_{t\to\infty}\left(x(t), y(t)\right) = (0,0)$.
All orbits therefore terminate at the critical point at the origin.
The system is **asymptotically stable**.

If both arbitrary constants are zero, then we have the trivial solution ($x = y = 0$ for all t).
If $c_2 \neq 0$, then $\dfrac{y(t)}{x(t)} = \dfrac{c_1\beta_1 + c_2\beta_2 t}{c_1\alpha_1 + c_2\alpha_2 t} \to \dfrac{\beta_2}{\alpha_2}$ as $t \to \pm\infty$
All orbits (except for $c_2 = 0$) therefore come in from infinity parallel to the line $y = \dfrac{\beta_2}{\alpha_2}x$, which is also a tangent at the origin. It can be shown that $\dfrac{\beta_1}{\alpha_1} = \dfrac{\beta_2}{\alpha_2}$ when $c_2 = 0$, so that the trajectories for $c_1 = 0$ and $c_2 = 0$ are both $y = \dfrac{\beta_2}{\alpha_2}x$.



Neither eigenvalue can be zero, otherwise (0, 0) is not the only critical point (as shown on page 4.14).

Case of **real, equal, positive eigenvalues** ($\lambda_1 = \lambda_2 > 0$)

The analysis leads to the same phase planes as in the case of real equal negative eigenvalues, but the signs of the arrows are reversed and the result is an **unstable node**.

Case of **complex conjugate pair of eigenvalues with negative real part**

The eigenvalues (roots of the characteristic equation) are

$$\lambda_1 = a + jb, \quad \lambda_2 = a - jb, \quad (a < 0).$$

The general solution has the form

$$x(t) = \left[c_1\left(A_1 \cos bt - A_2 \sin bt\right) + c_2\left(A_1 \sin bt + A_2 \cos bt\right)\right]e^{at}$$

$$y(t) = \left[c_1\left(B_1 \cos bt - B_2 \sin bt\right) + c_2\left(B_1 \sin bt + B_2 \cos bt\right)\right]e^{at}$$

Using the definitions

$$A = \sqrt{\left(c_2 A_1 - c_1 A_2\right)^2 + \left(c_1 A_1 + c_2 A_2\right)^2}, \quad B = \sqrt{\left(c_2 B_1 - c_1 B_2\right)^2 + \left(c_1 B_1 + c_2 B_2\right)^2}$$

$$\cos\alpha = \frac{c_1 A_1 + c_2 A_2}{A}, \quad \sin\alpha = \frac{c_2 A_1 - c_1 A_2}{A}, \quad \cos\beta = \frac{c_1 B_1 + c_2 B_2}{B}, \quad \sin\beta = \frac{c_2 B_1 - c_1 B_2}{B}$$

the general solution can be written more compactly as

$$\left(x(t), y(t)\right) = \left(A\, e^{at} \cos\left(bt + \alpha\right),\ B\, e^{at} \cos\left(bt + \beta\right)\right)$$

$$a < 0 \quad \Rightarrow \quad \lim_{t \to -\infty}\left(|x(t)|, |y(t)|\right) = (\infty, \infty) \quad \text{and} \quad \lim_{t \to \infty}\left(x(t), y(t)\right) = (0, 0).$$

If $x(t) = 0$ then $bt + \alpha = \dfrac{\pi}{2} + n\pi \quad (n \in \mathbb{Z})$

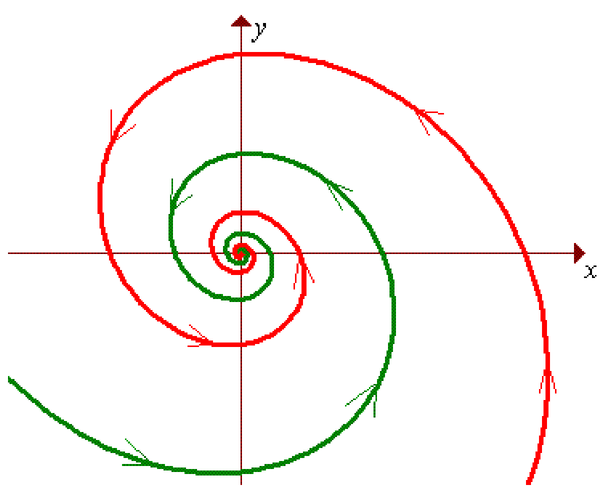
If $y(t) = 0$ then $bt + \beta = \dfrac{\pi}{2} + n\pi \quad (n \in \mathbb{Z})$

$$\frac{y(t)}{x(t)} = \frac{B\cos\left(bt + \beta\right)}{A\cos\left(bt + \alpha\right)}$$

$\dfrac{y(t)}{x(t)}$ is periodic, with period $\dfrac{2\pi}{b}$.

The orbits spiral in to the origin.

We have an asymptotically stable spiral, also known as a **stable focus**.



Case of **complex conjugate pair of eigenvalues with positive real part**

The analysis leads to the same phase planes as in the case of negative real part, but the signs of the arrows are reversed and the result is an **unstable focus**.

Case of **complex conjugate pair of eigenvalues with zero real part** (pure imaginary)

The eigenvalues (roots of the characteristic equation) are

$$\lambda_1 = -jb, \quad \lambda_2 = +jb.$$

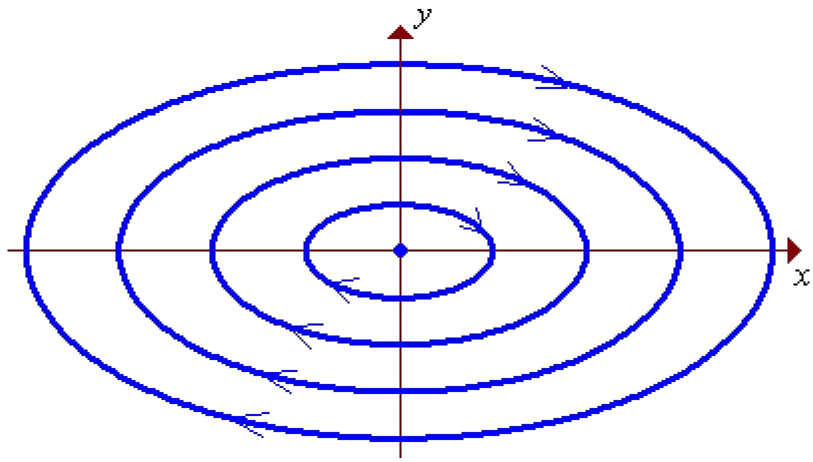
The general solution has the compact form

$$\big(x(t), y(t)\big) = \big(A\cos(bt+\alpha),\ B\cos(bt+\beta)\big)$$

If $\alpha = 0$ and $\beta = -\frac{\pi}{2}$, then

$$\big(x(t), y(t)\big) = \big(A\cos bt,\ B\sin bt\big) \quad \Rightarrow \quad \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

so that the orbits are ellipses, centred on the critical point at the origin.
This is a **stable centre**.



Other choices of α and β also lead to concentric sets of ellipses, but rotated with respect to the coordinates axes.

Note that this is the only case of a stable critical point that is ***not*** asymptotically stable.

Summary for the Linear System

$$\frac{dx}{dt} = ax + by, \quad \frac{dy}{dt} = cx + dy, \qquad (a, b, c, d = \text{constants})$$

Characteristic equation:

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

Discriminant

$$D = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc$$

Roots of characteristic equation (= eigenvalues of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$):

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2}$$

Cases:

$a + d$	D	other condition	λ	Type of point
$a+d < 0$	$D > 0$	$ad - bc > 0$	real, distinct negative	**Stable** node
$a+d < 0$	$D = 0$	$b = c = 0$	real, equal negative	**Stable** star shape
$a+d < 0$	$D = 0$	b, c not both 0	real, equal negative	**Stable** node
$a+d < 0$	$D < 0$		complex conjugate pair	**Stable** focus [spiral]
$a+d = 0$	$D < 0$		Pure imaginary pair	**Stable** centre
$a+d > 0$	$D > 0$	$ad - bc > 0$	real, distinct positive	**Unstable** node
(*any*)	$D > 0$	$ad - bc < 0$	real, distinct opposite signs	**Unstable** saddle point
$a+d > 0$	$D = 0$	$b = c = 0$	real, equal positive	**Unstable** star shape
$a+d > 0$	$D = 0$	b, c not both 0	real, equal positive	**Unstable** node
$a+d > 0$	$D < 0$		complex conjugate pair	**Unstable** focus [spiral]

Note that $ad - bc = \det A$ and that $a + d$ = the trace of the matrix A.

In brief, if the real parts of both eigenvalues are negative (or both zero), then the origin is stable. Otherwise it is unstable.

[See also the example at "`www.engr.mun.ca/~ggeorge/9420/demos/phases.html`".]

Example 4.05.1

Find the nature of the critical point of the system

$$\frac{dx}{dt} = 4x - 3y, \quad \frac{dy}{dt} = 5x - 4y$$

and find the general solution.

The coefficient matrix is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 5 & -4 \end{pmatrix}$.

trace(A) $= a + d = 4 + (-4) = 0$

$D = (a-d)^2 + 4bc = (4+4)^2 + 4(-3)(5) = 64 - 60 = +4 > 0$

$\det A = \begin{vmatrix} 4 & -3 \\ 5 & -4 \end{vmatrix} = -16 + 15 < 0$

$D > 0$ and $ad - bc < 0 \Rightarrow \lambda$ are real with opposite signs and the critical point is a **saddle point** (**unstable**).

Solving the system:

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{0 \pm \sqrt{4}}{2} = \pm 1$$

$$\left(x(t), y(t)\right) = \left(c_1\alpha_1 e^{-t} + c_2\alpha_2 e^{t},\ c_1\beta_1 e^{-t} + c_2\beta_2 e^{t}\right)$$

where $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -1$

and $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = +1$.

To find the eigenvectors, find non-zero solutions to the equation

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

At $\lambda = -1$:

$$\begin{pmatrix} 4+1 & -3 \\ 5 & -4+1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $5\alpha - 3\beta = 0$ will provide an eigenvector.

Select $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

Example 4.05.1 (continued)

At $\lambda = +1$:

$$\begin{pmatrix} 4-1 & -3 \\ 5 & -4-1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $\alpha - \beta = 0$ will provide an eigenvector.

Select $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The general solution is

$$\boxed{\left(x(t), y(t)\right) = \left(3c_1 e^{-t} + c_2 e^t,\; 5c_1 e^{-t} + c_2 e^t\right)}$$

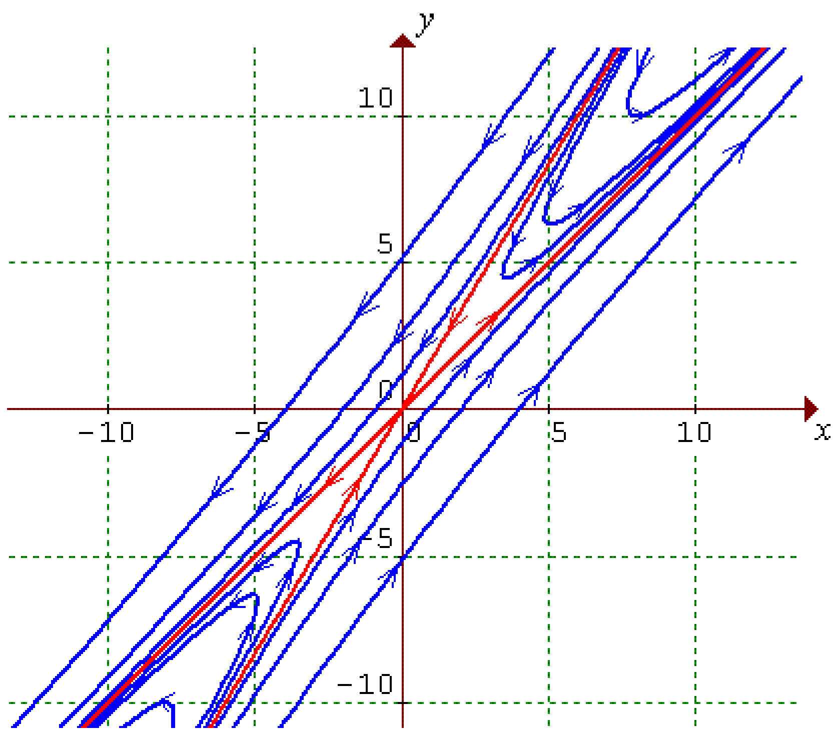
[It is simple to check that ($4x - 3y$, $5x - 4y$) is indeed equal to $\left(\dot{x}, \dot{y}\right)$].

Also note that

$$\frac{y(t)}{x(t)} = \frac{5c_1 e^{-t} + c_2 e^t}{3c_1 e^{-t} + c_2 e^t} \Rightarrow \lim_{t \to -\infty} \frac{y(t)}{x(t)} = \frac{5}{3} \quad (c_1 \neq 0) \quad \text{and} \quad \lim_{t \to +\infty} \frac{y(t)}{x(t)} = 1 \quad (c_2 \neq 0)$$

so that all orbits for which both c_1 and c_2 are non-zero share the same asymptotes,
$3y = 5x$ (which is the incoming orbit, when $c_2 = 0$) and
$y = x$ (which is the outgoing orbit, when $c_1 = 0$).

A few representative orbits and the two asymptotes are plotted in this phase space diagram:



Example 4.05.2

Find the nature of the critical point of the system

$$\frac{dx}{dt} = -2x + y\,, \quad \frac{dy}{dt} = x - 2y$$

and find the general solution.

The coefficient matrix is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$.

trace(A) $= a + d = -2 + -2 = -4 < 0$.

$D = (a-d)^2 + 4bc = (-2+2)^2 + 4(1)(1) = 0 + 4 = 4 > 0$

$\Rightarrow$ λ are real, distinct and negative and

$\det A = ad - bc = 4 - 1 = 3 > 0 \quad \Rightarrow$ the critical point is a **stable node**.

Solving the system:

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-4 \pm \sqrt{4}}{2} = -2 \pm 1 = -3, -1$$

$$\left(x(t), y(t)\right) = \left(c_1\alpha_1 e^{-3t} + c_2\alpha_2 e^{-t},\ c_1\beta_1 e^{-3t} + c_2\beta_2 e^{-t}\right)$$

where $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -3$

and $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -1$.

To find the eigenvectors, find non-zero solutions to the equation

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

At $\lambda = -3$:

$$\begin{pmatrix} -2+3 & 1 \\ 1 & -2+3 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $\alpha + \beta = 0$ will provide an eigenvector.

Select $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

At $\lambda = -1$:

$$\begin{pmatrix} -2+1 & 1 \\ 1 & -2+1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $-\alpha + \beta = 0$ will provide an eigenvector.

Select $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Example 4.05.2 (continued)

The general solution is

$$\big(x(t), y(t)\big) = \left(c_1 e^{-3t} + c_2 e^{-t},\ -c_1 e^{-3t} + c_2 e^{-t}\right)$$

[It is simple to check that $(-2x + y,\ x - 2y)$ is indeed equal to $(\dot{x}, \dot{y})$].

Also note that

$$\frac{y(t)}{x(t)} = \frac{-c_1 e^{-3t} + c_2 e^{-t}}{c_1 e^{-3t} + c_2 e^{-t}} = \frac{-c_1 e^{-2t} + c_2}{c_1 e^{-2t} + c_2} = \frac{-c_1 + c_2 e^{2t}}{c_1 + c_2 e^{2t}}$$

$$\Rightarrow \lim_{t \to -\infty} \frac{y(t)}{x(t)} = -1 \quad (c_1 \neq 0) \quad \text{and} \quad \lim_{t \to +\infty} \frac{y(t)}{x(t)} = 1 \quad (c_2 \neq 0)$$

and $\lim_{t \to \infty} \big(x(t), y(t)\big) = \lim_{t \to \infty} \left(c_1 e^{-3t} + c_2 e^{-t},\ -c_1 e^{-3t} + c_2 e^{-t}\right) = (0,0)$

so that all orbits for which both c_1 and c_2 are non-zero come in from a direction parallel to $y = -x$ (which is the orbit when $c_2 = 0$) and share the same tangent at the origin, $y = x$ (which is the orbit when $c_1 = 0$).

A few representative orbits and the common tangent are plotted in this phase space diagram: