Example 4.05.3

Find the nature of the critical point of the system

$$\frac{dx}{dt} = x - 5y, \quad \frac{dy}{dt} = x - 3y$$

and find the general solution.

The coefficient matrix is $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$.

trace(A) $= a + d = 1 + -3 = -2 < 0$.

$D = (a - d)^2 + 4bc = (1 + 3)^2 + 4(-5)(1) = 16 - 20 = -4 < 0$

$\Rightarrow$ λ are a complex conjugate pair with negative real part
and the critical point is a **stable focus** (spiral).

Solving the system:

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm j$$

$$\big(x(t), y(t)\big) = \left(c_1 \alpha_1 e^{(-1-j)t} + c_2 \alpha_2 e^{(-1+j)t},\ c_1 \beta_1 e^{(-1-j)t} + c_2 \beta_2 e^{(-1+j)t}\right)$$

where $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -1 - j$

and $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}$ is the eigenvector associated with the eigenvalue $\lambda = -1 + j$.

To find the eigenvectors, find non-zero solutions to the equation

$$\begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

At $\lambda = -1 - j$:

$$\begin{pmatrix} 1-(-1-j) & -5 \\ 1 & -3-(-1-j) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2+j & -5 \\ 1 & -2+j \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $\alpha = (2-j)\,\beta$ will provide an eigenvector.
Note that the two rows of the matrix equation are equivalent:

$$(2+j)\alpha - 5\beta = 0 \quad \Rightarrow \quad \alpha = \frac{5\beta}{2+j} = \frac{5\beta}{2+j} \cdot \frac{2-j}{2-j} = \frac{5(2-j)\beta}{4+1} = (2-j)\beta$$

Select $\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} 2-j \\ 1 \end{pmatrix}$.

At $\lambda = -1 + j$:

$$\begin{pmatrix} 1-(-1+j) & -5 \\ 1 & -3-(-1+j) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 2-j & -5 \\ 1 & -2-j \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Any non-zero choice such that $\alpha = (2+j)\,\beta$ will provide an eigenvector.

Example 4.05.3 (continued)

Again the two rows of the matrix equation are equivalent.

Select $\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 2+j \\ 1 \end{pmatrix}$.

The general solution is

$$\left(x(t), y(t)\right) = \left(c_1(2-j)e^{(-1-j)t} + c_2(2+j)e^{(-1+j)t},\; c_1 e^{(-1-j)t} + c_2 e^{(-1+j)t}\right)$$

But $c_1(2-j)e^{(-1-j)t} + c_2(2+j)e^{(-1+j)t}$

$$= e^{-t}\left(c_1(2-j)(\cos t - j\sin t) + c_2(2+j)(\cos t + j\sin t)\right)$$

$$= e^{-t}\begin{pmatrix} c_1\left((2\cos t - \sin t) - j(\cos t + 2\sin t)\right) \\ + c_2\left((2\cos t - \sin t) + j(\cos t + 2\sin t)\right) \end{pmatrix}$$

$$= e^{-t}\left((c_1 + c_2)(2\cos t - \sin t) + j(-c_1 + c_2)(\cos t + 2\sin t)\right)$$

$$= e^{-t}\left(c_3(2\cos t - \sin t) + c_4(\cos t + 2\sin t)\right)$$

where new real arbitrary constants c_3, c_4 are defined in terms of the complex arbitrary constants c_1, c_2 by $c_3 = c_1 + c_2$, $c_4 = j(-c_1 + c_2)$

Similarly, $c_1 e^{(-1-j)t} + c_2 e^{(-1+j)t} = e^{-t}\left(c_1(\cos t - j\sin t) + c_2(\cos t + j\sin t)\right)$

$$= e^{-t}\left((c_1 + c_2)\cos t + j(-c_1 + c_2)\sin t\right)$$

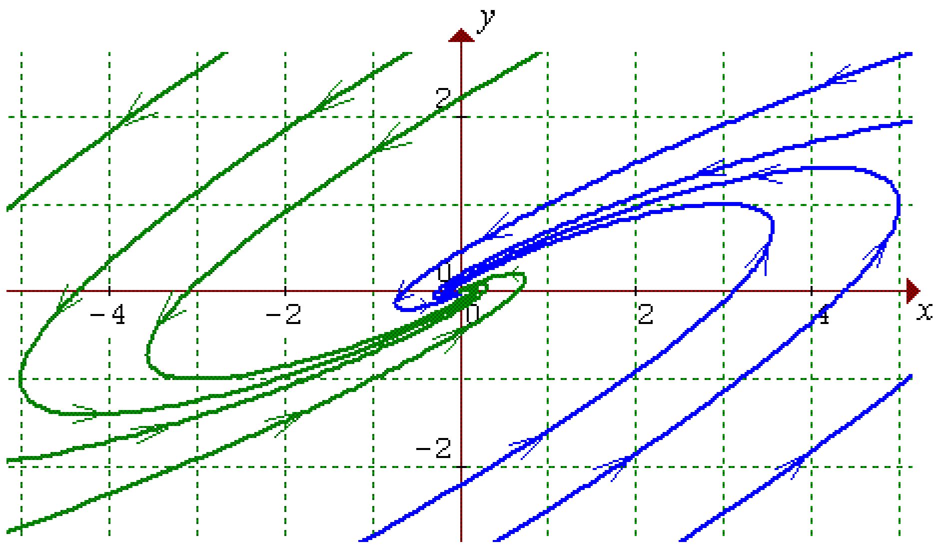
Therefore, in terms of purely real quantities, the general solution is

$$\boxed{\left(x(t), y(t)\right) = e^{-t}\left(c_3(2\cos t - \sin t) + c_4(\cos t + 2\sin t),\; c_3\cos t + c_4\sin t\right)}$$

One can show that this solution does satisfy the original system of ODEs

and $\lim_{t\to\infty}\left(x(t), y(t)\right) = \lim_{t\to\infty}\left(e^{-t} \times [\text{finite vector}]\right) = (0,0)$.

The orbits spiral in to the origin. A few representative orbits are plotted in this phase space diagram:



General Form for the General Solution

From the linear system of ODEs

$$\frac{dx}{dt} = ax + by$$
$$\frac{dy}{dt} = cx + dy$$

calculate the discriminant

$$D = (a+d)^2 - 4(ad - bc) = (a-d)^2 + 4bc$$

If $D > 0$ then the general solution is

$$\boxed{(x(t), y(t)) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\ c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)},$$

where

$$\lambda_1 = \frac{(a+d) - \sqrt{D}}{2}, \quad \lambda_2 = \frac{(a+d) + \sqrt{D}}{2},$$

$$\begin{pmatrix}\alpha_1 \\ \beta_1\end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix}\dfrac{(a-d) - \sqrt{D}}{2} \\ c\end{pmatrix},$$

$$\begin{pmatrix}\alpha_2 \\ \beta_2\end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix}\dfrac{(a-d) + \sqrt{D}}{2} \\ c\end{pmatrix}$$ and c_1, c_2 are arbitrary constants.

[An exception occurs if $c = 0$: use $\begin{pmatrix} b \\ \dfrac{(d-a) \pm \sqrt{D}}{2}\end{pmatrix}$ instead.]

If $D < 0$ then the general solution is

$$\boxed{\begin{aligned}&(x(t), y(t)) = \\ &e^{ut}\left(c_3\left((u-d)\cos vt - v\sin vt\right) + c_4\left(v\cos vt + (u-d)\sin vt\right),\ c\left(c_3\cos vt + c_4\sin vt\right)\right)\end{aligned}}$$

where $u = \dfrac{a+d}{2}$ $\left(\Rightarrow\ u - d = \dfrac{a-d}{2}\right)$ and $v = \dfrac{\sqrt{-(a-d)^2 - 4bc}}{2} = \dfrac{\sqrt{-D}}{2}$

and c_3, c_4 are [real] arbitrary constants.

[The derivation of this general result follows steps similar to those of Example 4.05.3.]

The situation for $D = 0$ is more complicated.
The general solution is

$$\boxed{\left(x(t), y(t)\right) = \left(\left(c_1\left(\frac{a-d}{2}\right) + c_2\left(1 + \left(\frac{a-d}{2}\right)(1+t)\right)\right)e^{\lambda t},\ c\left(c_1 + c_2(1+t)\right)e^{\lambda t}\right)},$$

unless $a = d$ and $c = 0$ but $b \neq 0$, in which case

$$\boxed{\left(x(t), y(t)\right) = \left(\left(c_1 + c_2 t\right)e^{at},\ \frac{c_2}{b}e^{at}\right)}$$

or the decoupled system $a = d$ and $b = c = 0$, in which case

$$\boxed{\left(x(t), y(t)\right) = \left(c_1 e^{at},\ c_2 e^{at}\right)}$$

where the sole distinct eigenvalue and eigenvector are

$$\lambda = \frac{(a+d)}{2},$$

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \text{any non-zero multiple of} \begin{pmatrix} \frac{a-d}{2} \\ c \end{pmatrix} \quad \left(\text{or} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ if } a = d \text{ and } c = 0\right).$$

Outline derivation of the general solution:

The one eigenvalue and eigenvector generate part of the complementary function:

$$\left(x_1(t), y_1(t)\right) = \left(\alpha_1 e^{\lambda t},\ \beta_1 e^{\lambda t}\right)$$

x_2 and y_2 must be of the form $e^{\lambda t}$ multiplied by a linear function of t :

$$\left(x_2(t), y_2(t)\right) = \left(\left(\alpha_2 + \alpha_3 t\right)e^{\lambda t},\ \left(\beta_2 + \beta_3 t\right)e^{\lambda t}\right)$$

But, upon substituting (x_2, y_2) into the system of ODEs, we find that $\begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$

and we obtain the singular linear system

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}$$

so that

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \text{any non-zero multiple of} \begin{pmatrix} \lambda - d + 1 \\ c \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} b \\ \lambda - a + 1 \end{pmatrix}.$$