4.06 Linear Approximation to a System of Non-Linear ODEs (2)

From sections 4.02 and 4.03, the non-linear system

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y) \qquad \textbf{(1)}$$

with critical point at (0, 0) may be expressed as

$$\begin{aligned} \dot{x} &= ax + by + P_1(x,y) \\ \dot{y} &= cx + dy + Q_1(x,y) \end{aligned} \qquad \textbf{(2)}$$

where a, b, c, d are all constants and

$$\lim_{(x,y)\to(0,0)} \frac{P_1(x,y)}{\sqrt{x^2+y^2}} = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} \frac{Q_1(x,y)}{\sqrt{x^2+y^2}} = 0 .$$

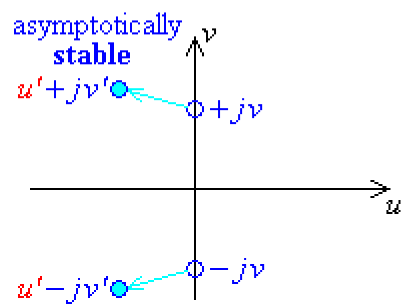
Near the critical point (0, 0), this system may be approximated by the linear system

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned} \qquad \textbf{(3)}$$

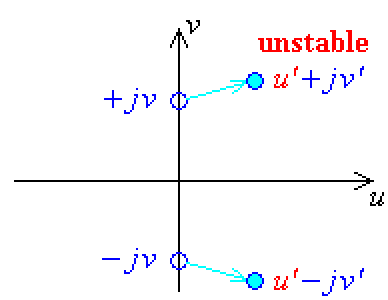
Effect of Small Perturbations

Small perturbations in the values of the coefficients a, b, c, d are reflected in small changes in the eigenvalues λ.

If the eigenvalues are a pure imaginary pair, $\lambda = \pm jv$, then the critical point is a centre. The effect of small changes in the coefficients will change the eigenvalues to the complex conjugate pair $\lambda' = u' \pm jv'$, where u' is small in magnitude and v' is close to v.

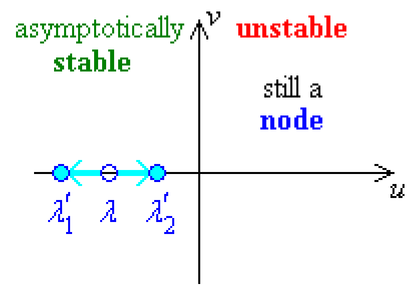


The trajectories of the new system are likely to be spirals. The critical point will be an asymptotically stable focus if $u' < 0$, a stable (but not asymptotically stable) centre if $u' = 0$, but it will be an unstable focus if $u' > 0$.

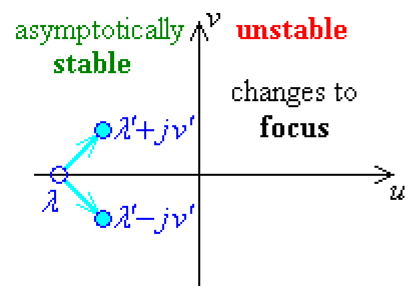


Therefore small perturbations in a linear system with pure imaginary eigenvalues are likely to result in radical changes in the trajectories (orbits) and may change the stable system into an unstable system.

If the eigenvalues are a real equal pair, $\lambda_1 = \lambda_2$, then a slight perturbation is likely to separate the roots into distinct values. If those values are still real, then the critical point remains a node.



If the perturbed eigenvalues are a complex conjugate pair, then the nature of the trajectories will change into spirals and the critical point changes from a node into a focus.



However, in both cases, an asymptotically stable critical point remains asymptotically stable after a small perturbation, while an unstable critical point remains unstable.

In all other cases, a slight perturbation leaves the sign of the real part of both eigenvalues unchanged and affects neither the type of critical point nor the overall type of the orbits.

These results are summarized in the two following theorems.

Poincaré's Theorem:

The singularities of the non-linear system **(2)** are identical to the singularities of the linear system **(3)**, except for the cases

$D < 0$ and $a + d = 0$, which is a centre in the linear case, but may be a centre or a focus in the non-linear case; and
$D = 0$, which is either a node or a focus in the non-linear case.

Theorem on stability of the singularity at (0, 0):

Linear approximation	Non-linear system
asymptotically stable	asymptotically stable
unstable	unstable
stable but not asymptotically stable	any (unstable, stable or asymptotically stable)

Example 4.06.1

Perform a stability analysis on the system

$$\frac{dx}{dt} = x - x^2 + xy, \quad \frac{dy}{dt} = 2y - xy - 6y^2$$

Find the critical points:

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \quad \Rightarrow \quad x - x^2 + xy = 0 \quad \text{and} \quad 2y - xy - 6y^2 = 0$$

$$\Rightarrow \quad x(1 - x + y) = 0 \quad \text{and} \quad y(2 - x - 6y) = 0$$

This generates four solutions:

$$x = 0 \quad \text{and} \quad y = 0 \quad \Rightarrow \quad (x, y) = (0, 0)$$

$$x = 0 \quad \text{and} \quad 2 - x - 6y = 0 \quad \Rightarrow \quad (x, y) = \left(0, \tfrac{1}{3}\right)$$

$$1 - x + y = 0 \quad \text{and} \quad y = 0 \quad \Rightarrow \quad (x, y) = (1, 0)$$

$$1 - x + y = 0 \quad \text{and} \quad 2 - x - 6y = 0 \quad \Rightarrow \quad (x, y) = \left(\tfrac{8}{7}, \tfrac{1}{7}\right)$$

Linearize the system near each critical point.

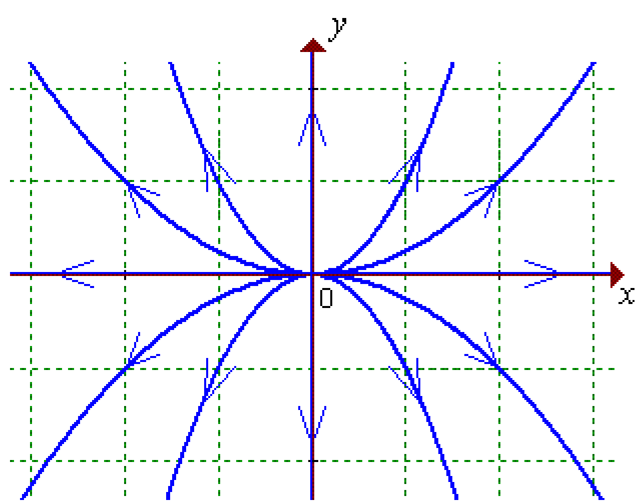
Near the critical point (0, 0), it is obvious that the linear approximation to the system is

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = 2y$$

This is an uncoupled system of ODEs, whose general solution is quickly found (by direct integration or by the results on page 4.30) to be

$$\left(x(t), y(t)\right) = \left(c_1 e^t, c_2 e^{2t}\right)$$

The critical point (0, 0) is an **unstable node**.



Also note that $y = k x^2$, so that the solutions are parabolas all sharing the same vertex at the origin, except for the lines $x = 0$ (from $c_1 = 0$) and $y = 0$ (from $c_2 = 0$).

The orbits all begin at the origin and move away, (a hallmark of an unstable node).

This diagram is valid for the linear approximation everywhere, but is valid only in the immediate neighbourhood of (0, 0) for the non-linear system.

Example 4.06.1 (continued)

$$P(x,y) = x - x^2 + xy\,, \qquad Q(x,y) = 2y - xy - 6y^2$$

$$\Rightarrow \quad \frac{\partial P}{\partial x} = 1 - 2x + y \qquad \frac{\partial P}{\partial y} = x$$

and $\displaystyle \frac{\partial Q}{\partial x} = -y \qquad \frac{\partial Q}{\partial y} = 2 - x - 12y$

Near a critical point (a, b), the linear approximation to the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \left.\frac{\partial P}{\partial x}\right|_{(a,b)} & \left.\frac{\partial P}{\partial y}\right|_{(a,b)} \\ \left.\frac{\partial Q}{\partial x}\right|_{(a,b)} & \left.\frac{\partial Q}{\partial y}\right|_{(a,b)} \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix} = \begin{pmatrix} 1-2a+b & a \\ -b & 2-a-12b \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

Near the critical point $\left(0, \frac{1}{3}\right)$, the linear approximation to the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1+\frac{1}{3} & 0 \\ -\frac{1}{3} & 2-\frac{12}{3} \end{pmatrix} \begin{pmatrix} x \\ y-\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 0 \\ -\frac{1}{3} & -2 \end{pmatrix} \begin{pmatrix} x \\ y-\frac{1}{3} \end{pmatrix}$$

A simple change of variables to $(x, z) = \left(x, y-\frac{1}{3}\right)$ leads to

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} & 0 \\ -\frac{1}{3} & -2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}$$

The matrix is triangular, allowing us to identify the eigenvalues immediately.
The eigenvalues are of opposite sign (+4/3 and –2).
Therefore the critical point at $\left(0, \frac{1}{3}\right)$ is an **unstable saddle point**.

From page 4.30,

$$D = (a-d)^2 + 4bc = \left(\frac{4}{3}+2\right)^2 + 0 = \left(\frac{10}{3}\right)^2$$

The eigenvector corresponding to eigenvalue $\lambda = -2$ is

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \text{ any non-zero multiple of } \begin{pmatrix} \frac{(a-d)-\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \frac{\frac{10}{3}-\frac{10}{3}}{2} \\ -\frac{1}{3} \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

The eigenvector corresponding to eigenvalue $\lambda = +4/3$ is

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \text{ any non-zero multiple of } \begin{pmatrix} \frac{(a-d)+\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \frac{\frac{10}{3}+\frac{10}{3}}{2} \\ -\frac{1}{3} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 10 \\ -1 \end{pmatrix}$$

Example 4.06.1 (continued)

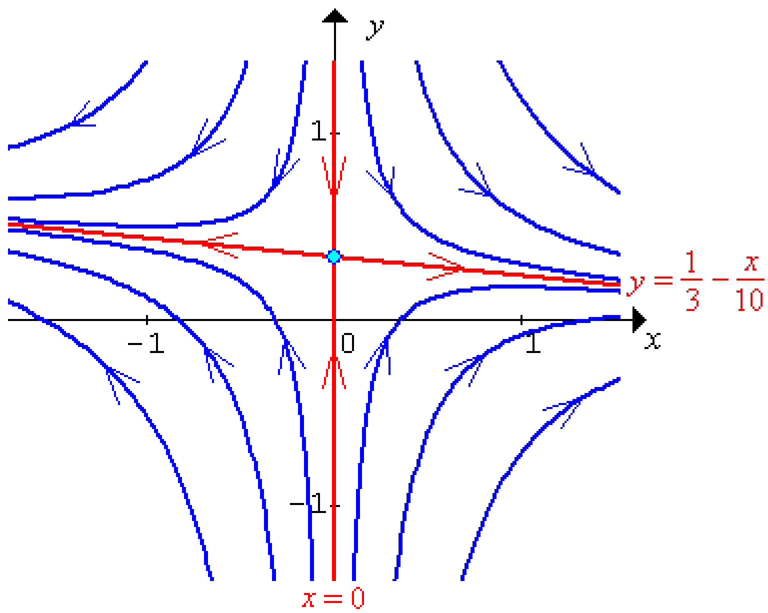
The asymptotes are therefore $x = 0$ (corresponding to $c_1 = 0$; inward because $\lambda_1 < 0$) and $x = -10\left(y - \frac{1}{3}\right)$, (corresponding to $c_2 = 0$; outward because $\lambda_2 > 0$). This is sufficient to sketch the phase portrait, even without the general solution.

The general solution near the critical point $\left(0, \frac{1}{3}\right)$ is

$$\boxed{\left(x(t), z(t)\right) = \left(x(t), y(t) - \tfrac{1}{3}\right) = \left(10c_1 e^{\frac{4}{3}t},\ -c_1 e^{\frac{4}{3}t} + c_2 e^{-2t}\right)}$$

where c_1, c_2 are arbitrary constants.

All trajectories (except for $c_2 = 0$) therefore come in from infinity near the asymptote $x = 0$ (where $c_1 = 0$)
and
all trajectories (except for $c_1 = 0$) return to infinity near the asymptote
$y - \frac{1}{3} = -\frac{1}{10}x$
(where $c_2 = 0$).



The diagram is valid for the linear approximation everywhere, but is valid only in the immediate neighbourhood of $\left(0, \frac{1}{3}\right)$ for the non-linear system.

Example 4.06.1 (continued)

Near the critical point (1, 0), the linear approximation to the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1-2 & 1 \\ 0 & 2-1 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x-1 \\ y \end{pmatrix}$$

Change variables to $(w, y) = (x-1, y)$. Then

$$\begin{pmatrix} \dot{w} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix}$$

The matrix is triangular, with eigenvalues of opposite sign (-1 and $+1$).
Therefore the critical point at (1, 0) is also a[n **unstable**] **saddle point**.

$$D = (a-d)^2 + 4bc = (-1-1)^2 + 0 = 4$$

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \text{any non-zero multiple of} \begin{pmatrix} \dfrac{(a-d)-\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{-2-2}{2} \\ 0 \end{pmatrix} = -2\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$c = 0$, so the alternative form is needed for the other eigenvector:

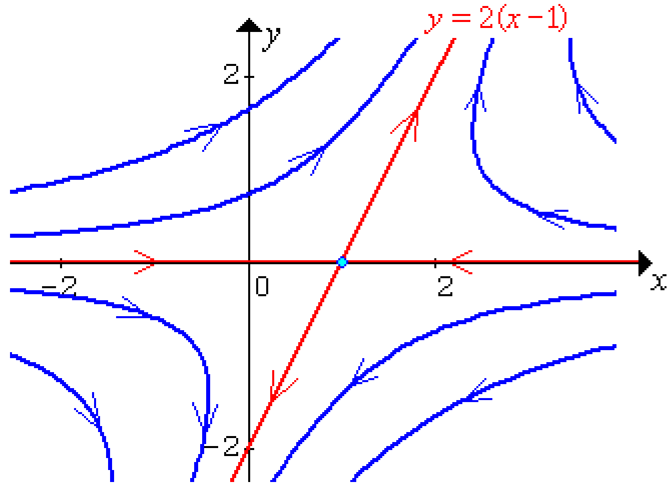
$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \text{any non-zero multiple of} \begin{pmatrix} b \\ \dfrac{(d-a)+\sqrt{D}}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ \dfrac{2+2}{2} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The general solution near the critical point (1, 0) is

$$\boxed{(w(t), y(t)) = (x(t)-1, y(t)) = \left(c_1 e^{-t} + c_2 e^t,\ 2c_2 e^t\right)}$$

where c_1, c_2 are arbitrary constants.

All trajectories (except for $c_1 = 0$) therefore come in from infinity near the asymptote $y = 0$ (where $c_2 = 0$)
and
all trajectories (except for $c_2 = 0$) return to infinity near the asymptote $y = 2(x - 1)$ (where $c_1 = 0$).



Again, the diagram is valid for the linear approximation everywhere, but is valid only in the immediate neighbourhood of (1, 0) for the non-linear system.

Example 4.06.1 (continued)

Near the critical point $\left(\frac{8}{7},\frac{1}{7}\right)$, the linear approximation to the system is

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}\frac{7-16+1}{7} & \frac{8}{7}\\ \frac{-1}{7} & \frac{14-8-12}{7}\end{pmatrix}\begin{pmatrix}x-\frac{8}{7}\\ y-\frac{1}{7}\end{pmatrix} = \begin{pmatrix}\frac{-8}{7} & \frac{8}{7}\\ \frac{-1}{7} & \frac{-6}{7}\end{pmatrix}\begin{pmatrix}x-\frac{8}{7}\\ y-\frac{1}{7}\end{pmatrix}$$

Change variables to $(w,z) = \left(x-\frac{8}{7}, y-\frac{1}{7}\right)$

$$\begin{pmatrix}\dot{w}\\ \dot{z}\end{pmatrix} = \frac{1}{7}\begin{pmatrix}-8 & 8\\ -1 & -6\end{pmatrix}\begin{pmatrix}w\\ z\end{pmatrix}$$

$$D = (a-d)^2 + 4bc = \left(\frac{-8+6}{7}\right)^2 + 4\left(\frac{8}{7}\right)\left(\frac{-1}{7}\right) = -\frac{28}{49} = -\frac{4}{7}$$

$D < 0$ and $(a+d) < 0$ $\Rightarrow$ the critical point is an **asymptotically stable focus**.

The eigenvalues, which are the solutions to $\det\begin{pmatrix}-\frac{8}{7}-\lambda & \frac{8}{7}\\ -\frac{1}{7} & -\frac{6}{7}-\lambda\end{pmatrix} = 0$, are

$$\lambda = \frac{(a+d)\pm\sqrt{D}}{2} = \frac{-14\pm\sqrt{-28}}{14} = -1\pm j\frac{\sqrt{7}}{7}$$

Using the formula on page 4.30, $u=-1,\ v=\frac{1}{\sqrt{7}}=\frac{\sqrt{7}}{7},\ u-d=-1+\frac{6}{7}=-\frac{1}{7}$.

$$(w(t),z(t)) = \left(x(t)-\tfrac{8}{7}, y(t)-\tfrac{1}{7}\right) =$$

$$e^{-t}\left(c_3\left(-\frac{1}{7}\cos\frac{t}{\sqrt{7}}-\frac{1}{\sqrt{7}}\sin\frac{t}{\sqrt{7}}\right)+c_4\left(\frac{1}{\sqrt{7}}\cos\frac{t}{\sqrt{7}}-\frac{1}{7}\sin\frac{t}{\sqrt{7}}\right),\ -\frac{1}{7}\left(c_3\cos\frac{t}{\sqrt{7}}+c_4\sin\frac{t}{\sqrt{7}}\right)\right)$$

Redefining the arbitrary constants as $c_5=-\frac{1}{7}c_3$ and $c_6=-\frac{1}{7}c_4$,

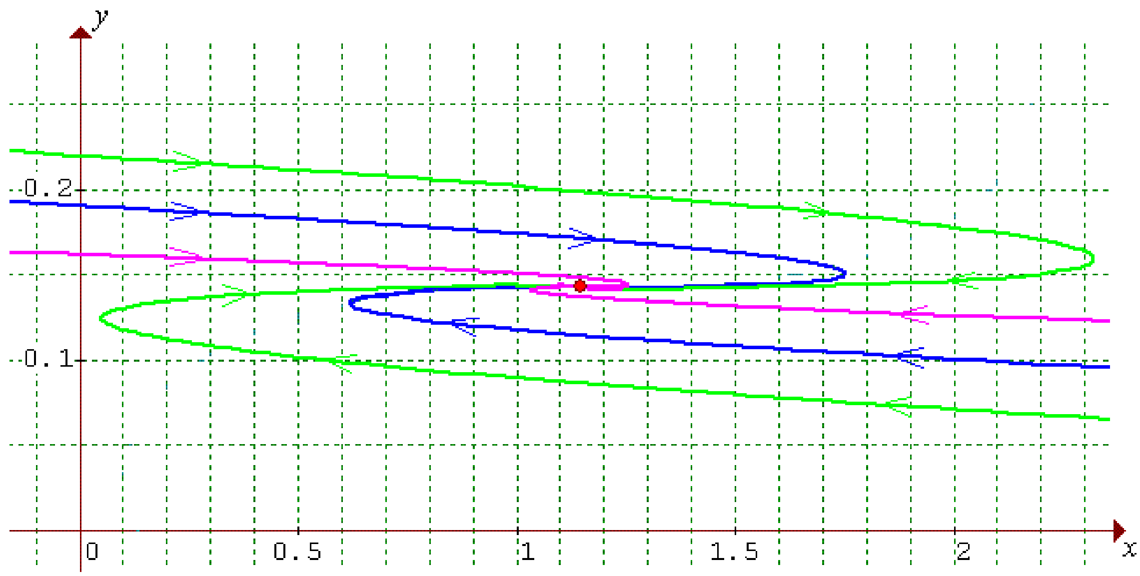
the general solution near the critical point $\left(\frac{8}{7},\frac{1}{7}\right)$ is

$$\boxed{\begin{array}{l}(w(t),z(t)) = \left(x(t)-\tfrac{8}{7}, y(t)-\tfrac{1}{7}\right) = \\ e^{-t}\left(c_5\left(\cos\frac{t}{\sqrt{7}}+\sqrt{7}\sin\frac{t}{\sqrt{7}}\right)+c_6\left(-\sqrt{7}\cos\frac{t}{\sqrt{7}}+\sin\frac{t}{\sqrt{7}}\right),\ \left(c_5\cos\frac{t}{\sqrt{7}}+c_6\sin\frac{t}{\sqrt{7}}\right)\right)\end{array}}$$

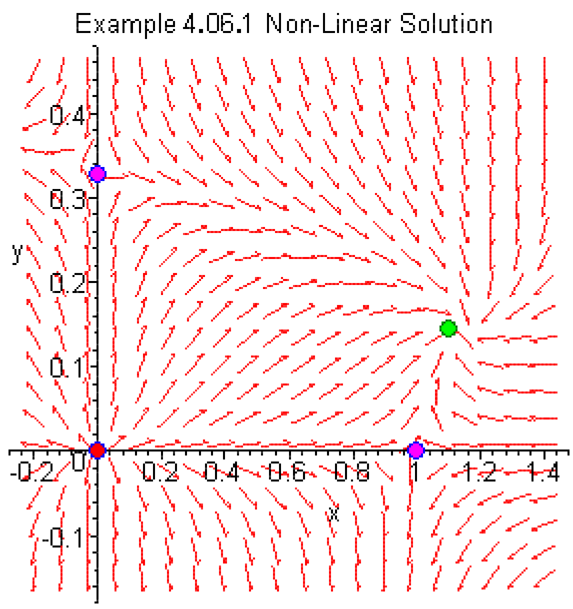
where c_5, c_6 are arbitrary constants.

All trajectories spiral in to the critical point [a phase portrait for the linear approximation is on the next page].

Example 4.06.1 (continued)

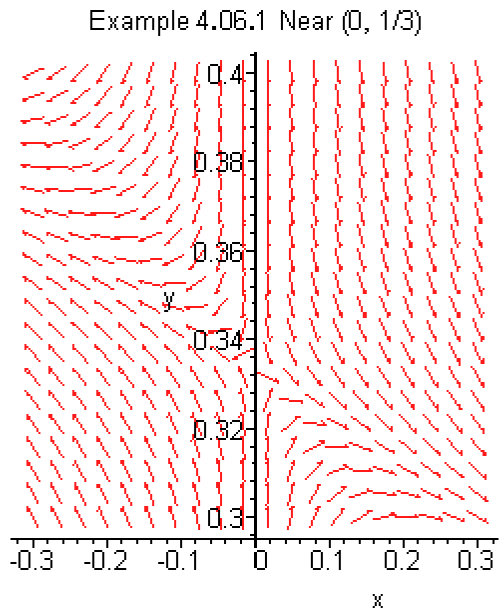


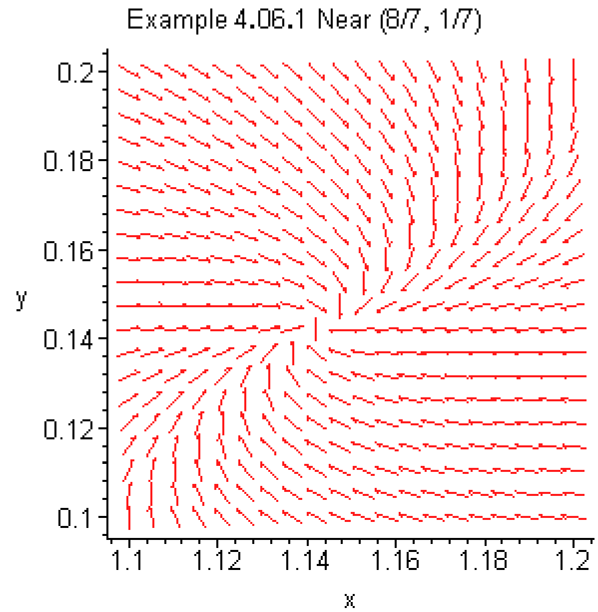
Maple produces the following direction field plots:



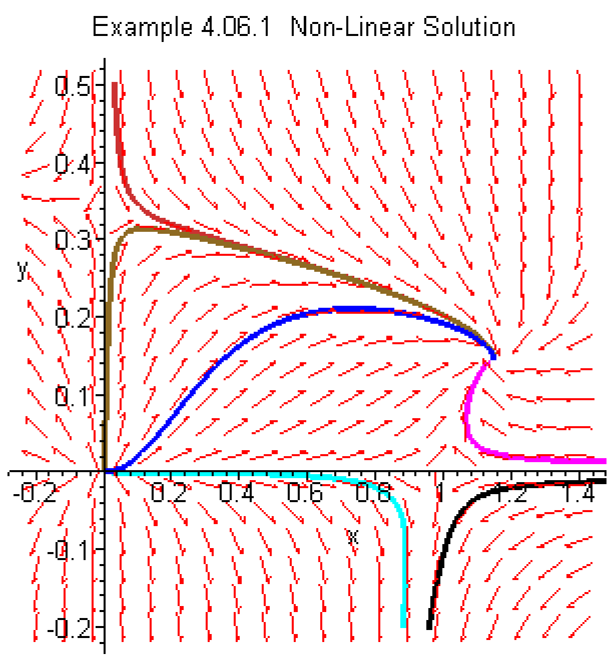
One can clearly see trajectories flowing outward from the unstable node at the origin in all directions. The natures of the other critical points are somewhat less obvious. Zooming in to the neighbourhood of one of the saddle points and to the neighbourhood of the stable focus produces the next pair of diagrams:

Example 4.06.1 (continued)





Maple can also superimpose some trajectories on these phase portraits:



Maple commands for this plot:

```
> with(DEtools):

> phaseportrait(
[diff(x(t),t) =
x(t) - x(t)^2 + x(t)*y(t),
diff(y(t),t) =
2*y(t) - x(t)*y(t) - 6*y(t)^2],
[x(t), y(t)], t=-10..10,
[[x(0)=0.1, y(0)=0.02],
[x(0)=0.8, y(0)=-0.02],
[x(0)=1.2, y(0)=0.02],
[x(0)=0.05, y(0)=0.3],
[x(0)=0.05, y(0)=0.4],
[x(0)=1.2, y(0)=-0.02]],
x=-0.2..1.4, y=-0.2..0.5,
stepsize=.01, colour=red,
linecolour=[blue, cyan, magenta, sienna, orange, black],
title=`Example 4.06.1  Non-Linear Solution`);
```

Example 4.06.2

Perform a stability analysis on the system

$$\frac{dx}{dt} = y + x\left(1 - x^2 - y^2\right), \quad \frac{dy}{dt} = -x + y\left(1 - x^2 - y^2\right)$$

Find the critical points:

Clearly $(x, y) = (0, 0)$ satisfies $\frac{dx}{dt} = \frac{dy}{dt} = 0$.

$$\frac{dx}{dt} = \frac{dy}{dt} = x = 0 \qquad \Rightarrow \quad y = 0$$

$$\frac{dx}{dt} = \frac{dy}{dt} = y = 0 \qquad \Rightarrow \quad x = 0$$

If $x \neq 0$ and $y \neq 0$, then at any critical point

$$y + x\left(1 - x^2 - y^2\right) = -x + y\left(1 - x^2 - y^2\right) = 0$$

$$\Rightarrow \quad y = -x\left(1 - x^2 - y^2\right) \quad \text{and} \quad x = y\left(1 - x^2 - y^2\right)$$

$$\Rightarrow \quad \left(1 - x^2 - y^2\right) = -\frac{x}{y} = \frac{y}{x} \qquad \Rightarrow \quad \left(\frac{y}{x}\right)^2 = -1$$

which has no solution for real (x, y).

Therefore the only critical point is (0, 0).

The linear approximation to the non-linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$D = (a - d)^2 + 4bc = (1 - 1)^2 + 4(1)(-1) = -4 < 0$$

$(a + d) = 2 > 0 \quad \Rightarrow$ the critical point is an **unstable focus**.

Using the formula on page 4.30,

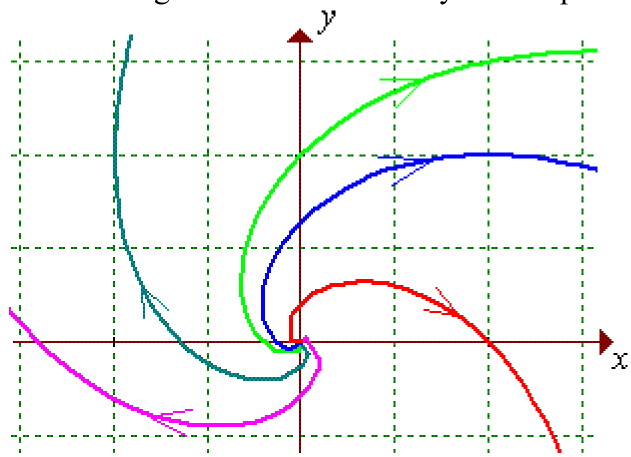
$$u = \frac{a + d}{2} = 1, \qquad u - d = \frac{a - d}{2} = 0, \quad v = \frac{\sqrt{-D}}{2} = 1$$

$$\boxed{\big(x(t), y(t)\big) = e^t\left(c_3 \sin t - c_4 \cos t,\ c_3 \cos t + c_4 \sin t\right)}$$

and c_3, c_4 are [real] arbitrary constants.

Example 4.06.2 (continued)

Solution in the neighbourhood of the only critical point (0, 0):



Now consider the distance r of any point (x, y) from the critical point (0, 0):

$$r^2 = x^2 + y^2 \quad \Rightarrow \quad 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad \text{[chain rule]}$$

From the original non-linear system:

$$r\frac{dr}{dt} = x\left(y + x\left(1-x^2-y^2\right)\right) + y\left(-x + y\left(1-x^2-y^2\right)\right)$$

$$= xy + x^2\left(1-r^2\right) - xy + y^2\left(1-r^2\right) = r^2\left(1-r^2\right)$$

$$\Rightarrow \quad \frac{dr}{dt} = r\left(1-r^2\right) \begin{cases} < 0 & (r>1) \\ > 0 & (r<1) \end{cases}$$

Therefore solutions starting closer than one unit to the critical point spiral out, but solutions starting further away than one unit approach the critical point. A solution on the circle $r = 1$ never changes its distance from the origin and stays on that circle, but is not stationary (because the only critical point is at (0, 0), not on that circle).

Note that $x^2 + y^2 = 1$ is a solution to the non-linear equation:

$$\frac{dx}{dt} = y + x\left(1-x^2-y^2\right) = y \quad \text{and} \quad \frac{dy}{dt} = -x + y\left(1-x^2-y^2\right) = -x$$

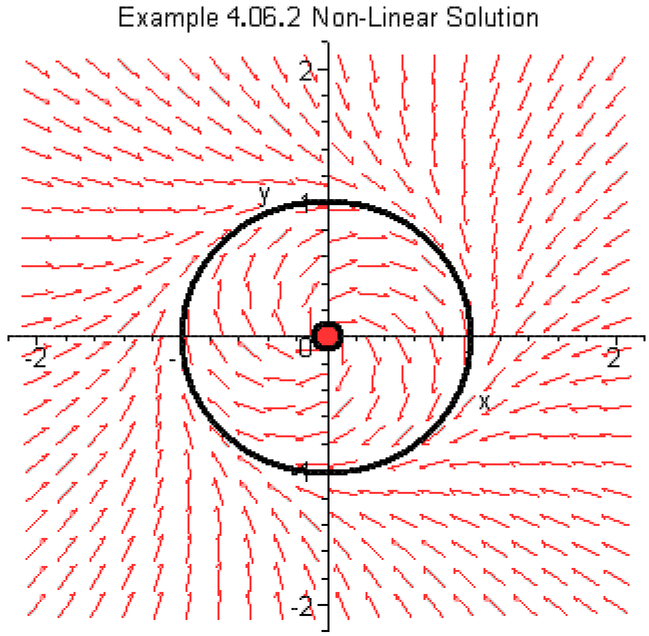
$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{x}{y}$$

But $x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2y\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

No other solution may cross this solution.
Therefore all non-trivial solutions approach the unit circle in the limit as $t \to \infty$.

The unit circle here is an example of a **limit cycle**.

Example 4.06.2 (continued)



Further notes:

Let $u = r^2$, then $du = 2r\,dr$ and

$$r\frac{dr}{dt} = r^2\left(1-r^2\right) \Rightarrow \frac{1}{2}\frac{du}{dt} = u\left(1-u\right) \Rightarrow \frac{du}{u\left(1-u\right)} = 2\,dt$$

$$\Rightarrow \int\left(\frac{1}{u} + \frac{1}{1-u}\right)du = 2\int 1\,dt \quad \Rightarrow \ln u - \ln\left(1-u\right) = 2t + C$$

$$\ln\left(\frac{u}{1-u}\right) = 2t + C \quad \Rightarrow \frac{u}{1-u} = e^{2t+C} = c_1 e^{2t}$$

$$\Rightarrow u = c_1 e^{2t} - u\,c_1 e^{2t} \Rightarrow \left(1 + c_1 e^{2t}\right)u = c_1 e^{2t}$$

$$\Rightarrow \left(c_2 e^{-2t} + 1\right)u = 1, \quad \text{where } c_2 = \frac{1}{c_1}$$

$$\Rightarrow u = \frac{1}{1 + c_2 e^{-2t}} \quad \Rightarrow r\left(t\right) = \frac{1}{\sqrt{1 + c_2 e^{-2t}}}$$

If $r(0) = r_o$, then

$$c_1 e^{2t} = \frac{u}{1-u} \quad \Rightarrow c_1 = \frac{r_o^{\,2}}{1-r_o^{\,2}} \quad \Rightarrow c_2 = \frac{1-r_o^{\,2}}{r_o^{\,2}}$$

Therefore

$$\boxed{r\left(t\right) = \frac{1}{\sqrt{1 + \left(\frac{1-r_o^{\,2}}{r_o^{\,2}}\right)e^{-2t}}}}$$

Example 4.06.2 (continued)

Also note that, from the polar coordinate system,

$$(x, y) = (r\cos\theta, r\sin\theta)$$

$$\Rightarrow \quad (\dot{x}, \dot{y}) = \left(\dot{r}\cos\theta - r\sin\theta\,\dot{\theta},\ \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}\right)$$

$$\Rightarrow \quad x\dot{y} - y\dot{x} = (r\cos\theta)\dot{r}\sin\theta + (r\cos\theta)r\cos\theta\,\dot{\theta} - (r\sin\theta)\dot{r}\cos\theta + (r\sin\theta)r\sin\theta\,\dot{\theta}$$

$$= r^2\left(\cos^2\theta + \sin^2\theta\right)\dot{\theta} = r^2\dot{\theta}$$

But the non-linear system is

$$(\dot{x}, \dot{y}) = \left(y + x\left(1 - r^2\right),\ -x + y\left(1 - r^2\right)\right)$$

$$\Rightarrow \quad x\dot{y} - y\dot{x} = -x^2 + xy\left(1 - r^2\right) - y^2 - xy\left(1 - r^2\right) = -\left(x^2 + y^2\right) = -r^2$$

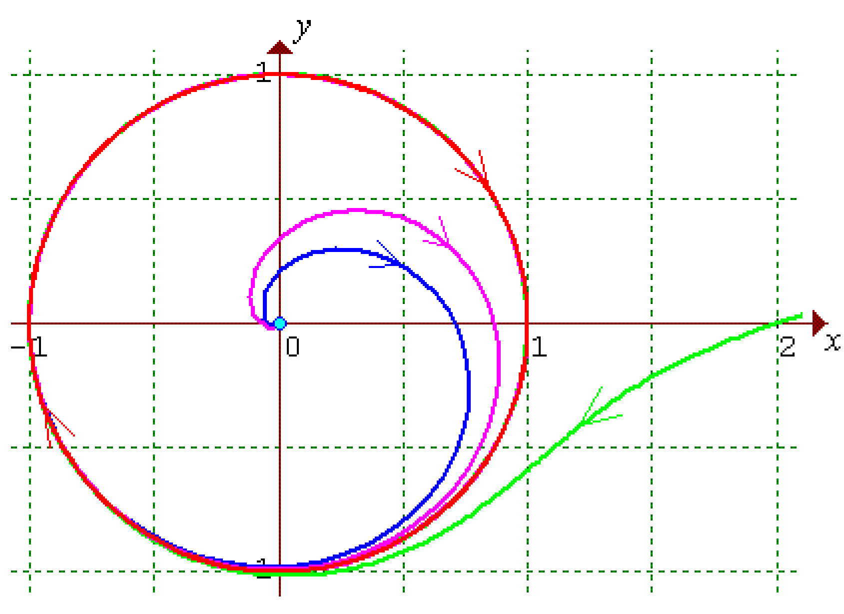
Therefore $\quad r^2\dot{\theta} = -r^2 \quad \Rightarrow \quad \dot{\theta} = -1 \quad \Rightarrow \quad \theta = -t + C$

All paths move clockwise with constant angular speed.

In Cartesian coordinates all orbits can therefore be described by

$$\boxed{(x(t), y(t)) = \frac{1}{\sqrt{1 + \left(\dfrac{1 - r_o^{\,2}}{r_o^{\,2}}\right)e^{-2t}}}\left(\cos(t - \theta_o),\ -\sin(t - \theta_o)\right)}$$

where $(x(0), y(0)) = r_o(\cos\theta_o,\ \sin\theta_o)$



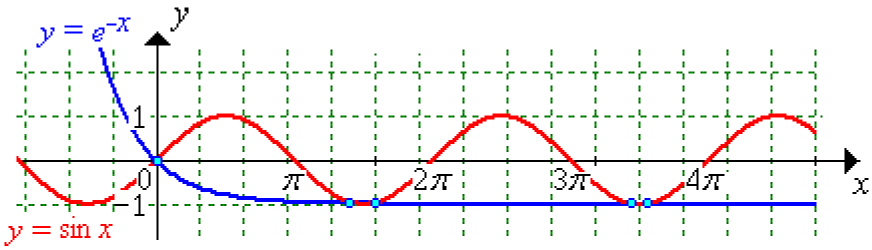
Example 4.06.3 (A more challenging and tedious case, for reference only)

Find and determine the nature of all critical points of the system

$$\frac{dx}{dt} = e^{-x} - y - 1, \qquad \frac{dy}{dt} = y - \sin x \qquad \textbf{(1)}$$

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \quad \Rightarrow \quad y = e^{-x} - 1 \quad \text{and} \quad y = \sin x$$

Critical points occur where the graphs of $y = e^{-x} - 1$ and $y = \sin x$ intersect.



The critical points are (0, 0) and (x_i, y_i), where

$$x_i = (4i-1)\tfrac{\pi}{2} - \delta_i,\ (4i-1)\tfrac{\pi}{2} + \varepsilon_i\ ; \quad (i = 1, 2, 3, \ldots)$$

Linearize for the critical point at (0, 0):

$$\left(e^{-x} - y - 1\right)\Big|_{\text{near}\,(0,0)} \approx \left.\frac{\partial P}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial P}{\partial y}\right|_{(0,0)} y = \left(-e^{-0}\right)x + (-1)\,y$$

$$\left(y - \sin x\right)\Big|_{\text{near}\,(0,0)} \approx \left.\frac{\partial Q}{\partial x}\right|_{(0,0)} x + \left.\frac{\partial Q}{\partial y}\right|_{(0,0)} y = (-\cos 0)\,x + (1)\,y$$

Therefore the linear system that models the non-linear system **(1)** near (0, 0) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \textbf{(2)}$$

Finding the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = -(1+\lambda)(1-\lambda) - (-1)(-1) = \lambda^2 - 1 - 1$$

$$\det(A - \lambda I) = 0 \quad \Rightarrow \quad \lambda^2 = 2 \quad \Rightarrow \quad \lambda = \pm\sqrt{2}$$

The eigenvalues are real and of opposite sign.
The critical point (0, 0) of **(2)** [and therefore also of **(1)**] is an **unstable saddle point**.

Example 4.06.3 (continued)

Using the results on page 4.30,

$$D = (a-d)^2 + 4bc = (-1-1)^2 + 4(-1)(-1) = 4+4 = 8$$

The eigenvectors are

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix} \dfrac{(a-d)-\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{-2-2\sqrt{2}}{2} \\ -1 \end{pmatrix} \quad \text{for} \quad \lambda = -\sqrt{2}$$

and

$$\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \text{any non-zero multiple of } \begin{pmatrix} \dfrac{(a-d)+\sqrt{D}}{2} \\ c \end{pmatrix} = \begin{pmatrix} \dfrac{-2+2\sqrt{2}}{2} \\ -1 \end{pmatrix} \quad \text{for} \quad \lambda = +\sqrt{2}$$

Choose a multiple of -1 in both cases.

The general solution of **(2)** is therefore

$$\boxed{(x(t), y(t)) = \left(c_1\left(1+\sqrt{2}\right)e^{-\sqrt{2}t} + c_2\left(1-\sqrt{2}\right)e^{\sqrt{2}t},\ c_1 e^{-\sqrt{2}t} + c_2 e^{\sqrt{2}t}\right)}$$

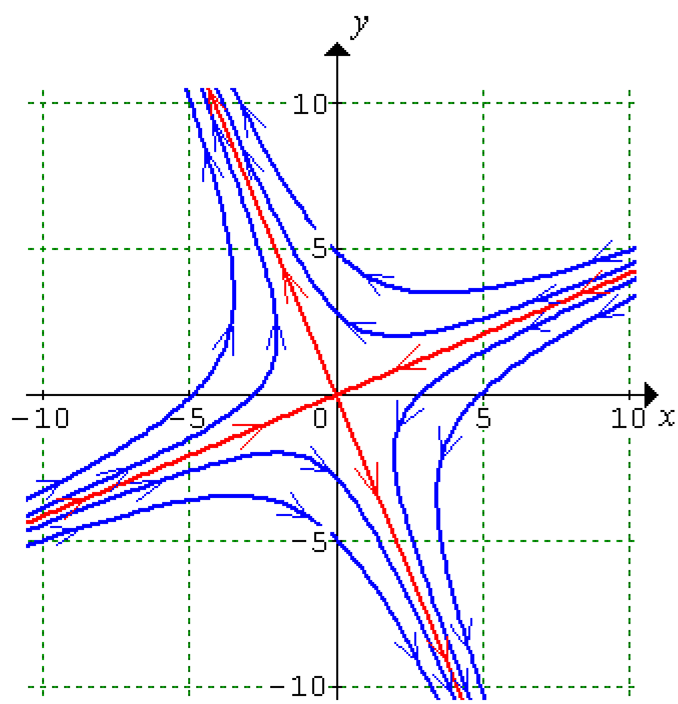
$$\lim_{t\to-\infty} \frac{y(t)}{x(t)} = \frac{1}{1+\sqrt{2}} = \frac{1-\sqrt{2}}{1-2} = \sqrt{2}-1 > 0 \quad \text{and}$$

$$\lim_{t\to+\infty} \frac{y(t)}{x(t)} = \frac{1}{1-\sqrt{2}} = \frac{1+\sqrt{2}}{1-2} = -1-\sqrt{2} < 0$$

The trajectories (except for $c_1 = 0$) therefore come in from infinity along the asymptote $y = \left(\sqrt{2}-1\right)x$.

The trajectories (except for $c_2 = 0$) return to infinity along the asymptote $y = -\left(1+\sqrt{2}\right)x$.

The phase space diagram for this solution (completely valid only for the linear system **(2)**) is



The phase space for the non-linear system **(1)** resembles this diagram only in the immediate neighbourhood of the critical point (0, 0).

Example 4.06.3 (continued)

At other critical points (k, l),

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \quad \Rightarrow \quad l = \sin k \quad \Rightarrow \quad e^{-k} - \sin k - 1 = 0$$

Linearizing (Taylor's series for $P(x, y)$ about $(x, y) = (k, l)$):

$$\left(e^{-x} - y - 1\right)\Big|_{\text{near }(k,l)} \approx \left.\frac{\partial P}{\partial x}\right|_{(k,l)} (x-k) + \left.\frac{\partial P}{\partial y}\right|_{(k,l)} (y-l) = \left(-e^{-k}\right)(x-k) + (-1)(y-l)$$

$$\left(y - \sin x\right)\Big|_{\text{near }(k,l)} \approx \left.\frac{\partial Q}{\partial x}\right|_{(k,l)} (x-k) + \left.\frac{\partial Q}{\partial y}\right|_{(k,l)} (y-l) = (-\cos k)(x-k) + (1)(y-l)$$

Therefore the linear system that models the non-linear system **(1)** near (k, l) is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -e^{-k} & -1 \\ -\cos k & 1 \end{pmatrix} \begin{pmatrix} x-k \\ y-l \end{pmatrix} \qquad \textbf{(3)}$$

Finding the eigenvalues:

$$\det(A - \lambda I) = \begin{vmatrix} -e^{-k} - \lambda & -1 \\ -\cos k & 1-\lambda \end{vmatrix} = -\left(e^{-k} + \lambda\right)(1-\lambda) - (-\cos k)(-1)$$

$$= \lambda^2 - \left(1 - e^{-k}\right)\lambda - \left(e^{-k} + \cos k\right) = 0$$

$$\Rightarrow \quad \lambda = \frac{\left(1-e^{-k}\right) \pm \sqrt{\left(1-e^{-k}\right)^2 + 4\left(e^{-k} + \cos k\right)}}{2}$$

$$\Rightarrow \quad \lambda = \frac{\left(1-e^{-k}\right) \pm \sqrt{\left(1+e^{-k}\right)^2 + 4\cos k}}{2}$$

Now $k > 0 \quad \Rightarrow \quad 0 < e^{-k} < 1 \quad \Rightarrow \quad 0 < 1 - e^{-k} < 1$

Recall that $k = x_i = (4i-1)\frac{\pi}{2} - \delta_i,\ (4i-1)\frac{\pi}{2} + \varepsilon_i\ ; \quad (i = 1, 2, 3, \ldots)$

Examining the right-hand critical point in each pair,

$k = (4i-1)\frac{\pi}{2} + \varepsilon_i\ , \quad (0 < \varepsilon_i \ll 1)$

$$\Rightarrow \quad \cos k = \cos\left((4i-1)\tfrac{\pi}{2} + \varepsilon_i\right) = \cos\left(\varepsilon_i - \tfrac{\pi}{2}\right) = \cos\left(\tfrac{\pi}{2} - \varepsilon_i\right) = \sin \varepsilon_i \approx \varepsilon_i > 0$$

Therefore $\sqrt{\left(1-e^{-k}\right)^2 + 4\left(e^{-k} + \cos k\right)} > \left(1 - e^{-k}\right) > 0$

and the two eigenvalues are real and of opposite sign.
These critical points are therefore all unstable **saddle points**.

Example 4.06.3 (continued)

Examining the left-hand critical point in each pair,
$k = (4i-1)\frac{\pi}{2} - \delta_i\,, \quad (0 < \delta_i \ll 1)$

$\Rightarrow \quad \cos k = \cos\left((4i-1)\frac{\pi}{2} - \delta_i\right) = \cos\left(-\delta_i - \frac{\pi}{2}\right) = \cos\left(\delta_i + \frac{\pi}{2}\right) = -\sin\delta_i$

But k is the solution to $e^{-k} - \sin k - 1 = 0$

$\Rightarrow \quad e^{-k} = 1 + \sin k = 1 + \sin\left((4i-1)\frac{\pi}{2} - \delta_i\right)$

$= 1 + \sin\left(-\delta_i - \frac{\pi}{2}\right) = 1 - \sin\left(\delta_i + \frac{\pi}{2}\right) = 1 - \cos\delta_i$

$\Rightarrow \quad e^{-k} + \cos k \approx (1 - \cos\delta_i) - \sin\delta_i \approx 1 - \left(1 - \frac{\delta_i^2}{2}\right) - \delta_i < 0 \quad$ (δ_i is small)

$\Rightarrow \quad \sqrt{\left(1 - e^{-k}\right)^2 + 4\left(e^{-k} + \cos k\right)} < 1 - e^{-k}$

But $\lambda = \dfrac{\left(1 - e^{-k}\right) \pm \sqrt{\left(1 - e^{-k}\right)^2 + 4\left(e^{-k} + \cos k\right)}}{2}$

Therefore the eigenvalues are a real distinct positive pair and the singularity is an **unstable node**.

The locations and nature of the first five critical points are listed here.

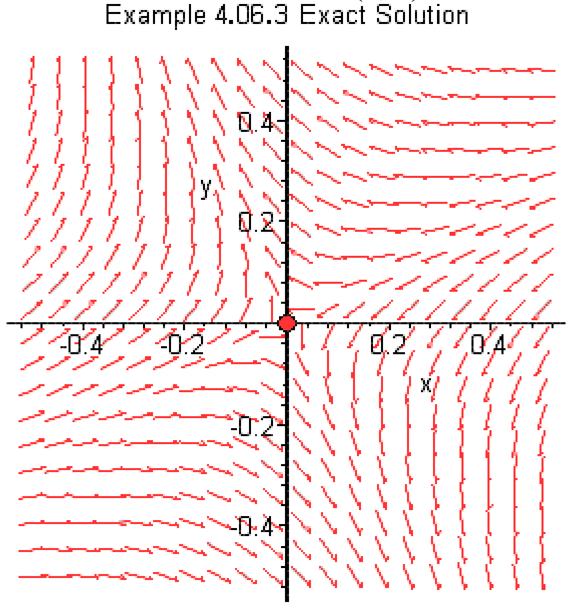
x	y	λ_1	λ_2	Type
0	0	–1.4142...	+1.4142...	unstable saddle point
4.56820...	–0.98962...	+0.1608...	+0.8287...	unstable node
4.83833...	–0.99208...	–0.1200...	+1.1121...	unstable saddle point
10.98977...	–0.99998...	+0.0058...	+0.9941...	unstable node
11.00135...	–0.99998...	–0.0057...	+1.0057...	unstable saddle point

Here is a Maple session to create direction field plots for the first three critical points of the non-linear system.

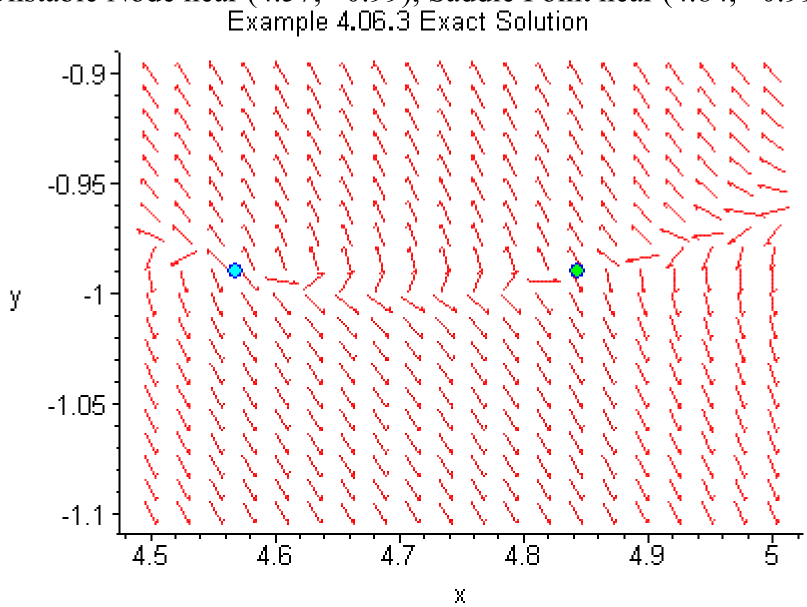
```
> with(DEtools):
> DEplot([diff(x(t),t) = -y(t) - 1 + exp(-x(t)),
  diff(y(t),t) = y(t) - sin(x(t))],
  [x(t),y(t)], t=-1..1, x=-0.5..0.5, y=-0.5..0.5,
  title=`Example 4.06.3 Exact Solution`);

> DEplot([diff(x(t),t) = -y(t) - 1 + exp(-x(t)),
  diff(y(t),t) = y(t) - sin(x(t))],
  [x(t),y(t)], t=-1..1, x=4.5..5, y=-1.1..-0.9,
  title=`Example 4.06.3 Exact Solution`);
```

<u>Example 4.06.3</u> (continued)

Saddle Point at (0, 0)



Unstable Node near (4.57, –0.99), Saddle Point near (4.84, –0.99)



Solution curves can be traced by following the arrows (which at every location point in the direction of $\frac{dy}{dx}$).