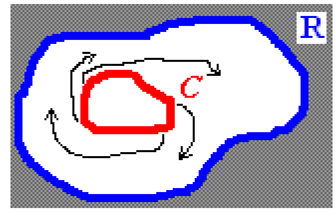
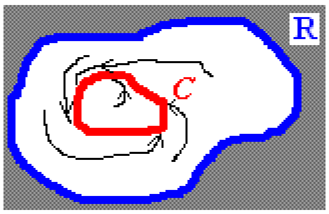
4.07 Limit Cycles

If, in some region, all trajectories begin on a closed curve inside that region, then that curve is an unstable limit cycle.





If all trajectories terminate on the curve, then it is a stable limit cycle.

More formally,
Let R be a bounded region in the *xy* plane.
Let C be a closed curve composed of interior points of R and bounding a region A.
Let C be a solution curve of the system

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y) \qquad \textbf{(1)}$$

where $P(x, y)$ and $Q(x, y)$ are differentiable with respect to x and y at all points of R.
C is a **limit cycle** of **(1)** if no other closed solution curve is close to C and if all orbits sufficiently near it approach it asymptotically as $t \to -\infty$ (unstable) or as $t \to +\infty$ (stable).

Bendixon Non-existence Theorem:

For system **(1)**, if the expression $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ does not change sign or vanish identically in a simply connected (= "no holes") region D inside R, then no closed trajectory can exist entirely within D.

The contrapositive statement is:
If C is a closed solution curve of **(1)** in R, then $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ must vanish for some subset of R.

Proof:

If C is a closed curve in R with interior region A, then Green's theorem in two dimensions states

$$\int_C (P\,dy - Q\,dx) = \iint_A \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\,dy \qquad \textbf{(2)}$$

But, for all points on C, (which is a solution curve of **(1)**),

$$P\,dy - Q\,dx = \left(P\frac{dy}{dx} - Q \right) dx = \left(P\frac{\dot{y}}{\dot{x}} - Q \right) dx = \left(P\frac{Q}{P} - Q \right) dx \equiv 0$$

It then follows that

$$\oint_C (P\,dy - Q\,dx) = \iint_A \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\,dy = 0$$

This is not possible unless the integrand $\dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y}$ changes sign or is identically zero inside region A.

Poincaré-Bendixon Theorem (Existence Theorem for Limit Cycles)

If the solution curve C of the system **(1)** is in and remains in a bounded region R for $t > t_0$ without approaching singular points and if $P(x, y)$ and $Q(x, y)$ are differentiable with respect to x and y at all points of R, then a limit cycle exists in R and either C is a limit cycle or it approaches a limit cycle as $t \to +\infty$.

4.08 Van der Pol's Equation

During an investigation of the properties of vacuum tubes, Van der Pol developed a second order non-linear ordinary differential equation to model the circuit:

$$\frac{d^2 x}{dt^2} - \mu\left(1-x^2\right)\frac{dx}{dt} + x = 0 \quad , \qquad (\mu > 0) \qquad \textbf{(1)}$$

The linear form resembles the linear ODE for the RLC circuit:

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0 \qquad \textbf{(2)}$$

The resistance term in **(2)** provides damping provided $R > 0$.
If $R < 0$, then the solution is unstable and the current would have an ever increasing amplitude, which is what the linear form of **(1)** predicts, ($-\mu < 0$).

However, experimental evidence suggests that, after some initial increase in amplitude, a periodic solution is attained. This is an indication that a limit cycle may exist for **(1)**.

The "resistance" term $-\mu\,(1 - x^2)$ in Van der Pol's equation is negative if $|\,x\,| < 1$, but is positive for $|\,x\,| > 1$. The non-linear term must be retained in order to find the periodic steady state solution.

Introduce a new variable y to Van der Pol's equation:

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= \mu\left(1-x^2\right)y - x \end{aligned} \qquad \textbf{(3)}$$

The linear version of **(3)** is:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \qquad \textbf{(4)}$$

Finding the critical points of **(3)**:

$$\frac{dx}{dt} = 0 \quad \Rightarrow \quad y = 0\,, \qquad \frac{dy}{dt} = y = 0 \quad \Rightarrow \quad x = 0$$

Thus (0, 0) is the only critical point of **(3)**.

Applying the formulae from page 4.30:

$D = (a-d)^2 + 4bc = (0-\mu)^2 + 4(1)(-1) = \mu^2 - 4$

so that $D < 0$ for $0 < \mu < 2$ and $D > 0$ for $\mu > 2$

$(a + d) = \mu > 0$

$0 < \mu < 2 \quad \Rightarrow$ the critical point (0, 0) is an **unstable focus**.

$\mu > 2 \quad \Rightarrow$ (0, 0) is an **unstable node**.

The eigenvalues are

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} \qquad \Rightarrow \quad \operatorname{Re}(\lambda) > 0$$

so that (0, 0) is unstable for all $\mu > 0$.

Searching for limit cycles:

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}\left(-x + \mu\left(1-x^2\right)y\right) = \mu\left(1-x^2\right)$$

$$|x| < 1 \quad \Rightarrow \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} > 0$$

Because $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ does not change sign anywhere in the region $|x| < 1$, there are no limit cycles contained entirely in that region (by the Bendixon Non-existence Theorem). There may be a limit cycle in a region that includes $x = -1$ and/or $x = +1$.

Transforming **(3)** to polar coordinates,

$r^2 = x^2 + y^2$

$$\Rightarrow \quad r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = x(y) + y\left(\mu\left(1-x^2\right)y - x\right) = \mu\left(1-x^2\right)y^2$$

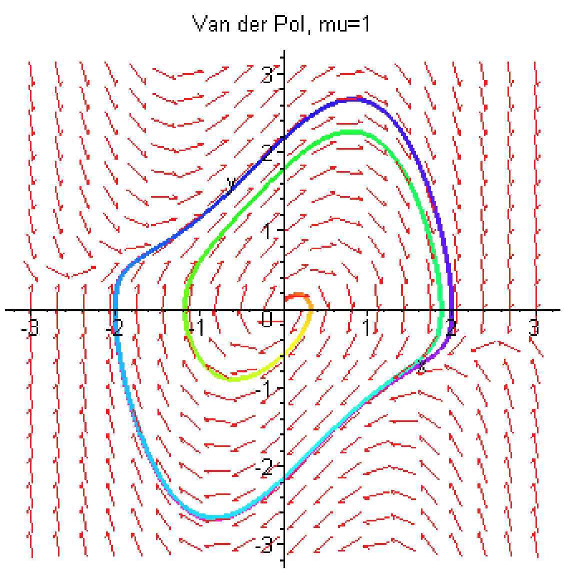
so that r is increasing with time for $|x| < 1$, but decreasing for $|x| > 1$ and not changing when $x = \pm 1$.

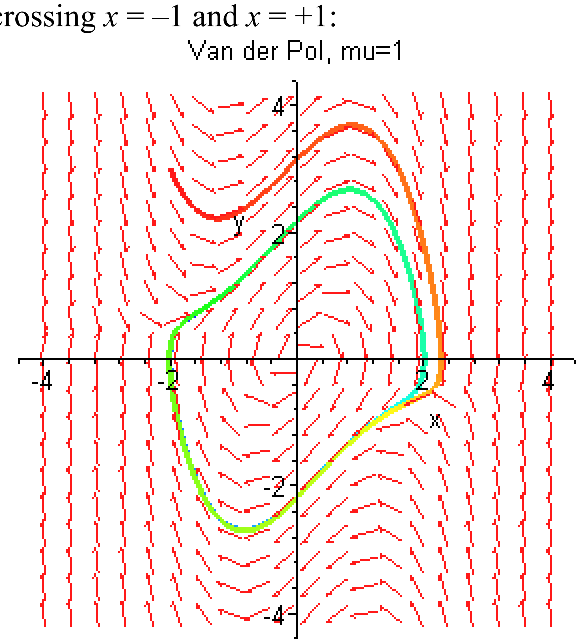
This suggests that a region extending to a sufficiently large x may contain a limit cycle.
When $0 < \mu < 2$ a closed periodic solution is possible and a limit cycle occurs.
When $\mu \geq 2$ a closed periodic solution is impossible and there is no limit cycle.

A Maple session that produces solution curves for the Van der Pol equation with $\mu = 1$ for two choices of starting point (one inside the limit cycle, one outside) is presented here.

```
> with(DEtools):
> phaseportrait([diff(x(t),t) = y(t), diff(y(t),t) =
y(t)*(1 - (x(t))^2) - x(t)],
[x(t),y(t)], t=0..20, [[x(0)=0,y(0)=0.1]], x=-3..3,
y=-3..3, stepsize=0.05, linecolour=t/2, title=`Van der Pol,
mu=1`);
> phaseportrait([diff(x(t),t) = y(t), diff(y(t),t) =
y(t)*(1 - (x(t))^2) - x(t)],
[x(t),y(t)], t=0..20, [[x(0)=-2,y(0)=3]], x=-4..4, y=-4..4,
stepsize=0.03, linecolour=t/2, title=`Van der Pol, mu=1`);
```

with output, clearly illustrating the limit cycle crossing $x = -1$ and $x = +1$:



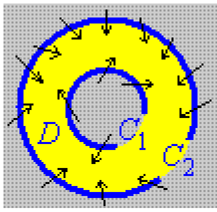


Again note how the trajectories move away from the origin only in the region $-1 < x < 1$.

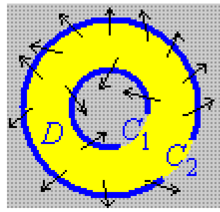
4.09 Theorem for Limit Cycles

Theorem (Extension of the Poincaré-Bendixon theorem):

Let D be an annular region between closed curves C_1 and C_2.



[stable]



[unstable]

If solution curves of the system

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y) \qquad \textbf{(1)}$$

enter D at every point of C_1 and C_2 (or leave at every point of C_1 and C_2), and there are no singularities of **(1)** in D or on C_1 or C_2, then
a limit cycle exists in D.

It also follows that a closed curve cannot be a limit cycle unless it encloses a singularity.

<u>Example 4.09.1</u>

Determine whether a limit cycle exists for the second order ODE $\dfrac{d^2x}{dt^2} + x^2 + 1 = 0$.

The ODE can be rewritten as the first order non-linear system

$$\dot{x} = y$$
$$\dot{y} = -x^2 - 1$$

But $-x^2 - 1 < 0$ for all real x.
No critical point exists for real (x, y).
But a limit cycle must enclose a singularity.
Therefore no limit cycle exists for this system.

Example 4.09.2

Perform a stability analysis and determine whether a limit cycle exists for the system

$$\begin{aligned} \frac{dx}{dt} &= x\left(1-x^2-y^2\right) + 5y \\ \frac{dy}{dt} &= -5x + y\left(1-x^2-y^2\right) \end{aligned} \qquad \textbf{(1)}$$

One critical point occurs where $x = y = 0$.
Substitution of $x = 0$ into **(1)** leads to $y = 0$ and vice versa.
If $x \neq 0$ and $y \neq 0$, then **(1)** $\Rightarrow$ at a critical point

$$\left(1-x^2-y^2\right) = \frac{-5y}{x} = \frac{5x}{y} \qquad \Rightarrow \quad \left(\frac{y}{x}\right)^2 = -1$$

which has no real solution for (x, y). Therefore $(0, 0)$ is the only critical point of **(1)**.

Near $(0, 0)$, the linear approximation to **(1)** is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \textbf{(2)}$$

The eigenvalues may be found either by solving $\begin{vmatrix} 1-\lambda & 5 \\ -5 & 1-\lambda \end{vmatrix} = 0$ or by use of the formula on page 4.30:

$$D = (a-d)^2 + 4bc = (1-1)^2 + 4(-5)(5) = -100$$

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{2 \pm \sqrt{-100}}{2} = 1 \pm 5j$$

The eigenvalues are a complex conjugate pair with positive real part
$\Rightarrow$ the critical point of **(2)** (and therefore also of **(1)**) is an **unstable focus**.

Example 4.09.2 (continued)

Checking for a limit cycle,

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x}\left(x\left(1-x^2-y^2\right)+5y\right) + \frac{\partial}{\partial y}\left(-5x+y\left(1-x^2-y^2\right)\right)$$

$$= 1-3x^2-y^2 + 1-x^2-3y^2 = 2\left(1-2\left(x^2+y^2\right)\right)$$

$$\Rightarrow \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \begin{cases} >0 & \left(x^2+y^2<\frac{1}{2}\right) \\ <0 & \left(x^2+y^2>\frac{1}{2}\right) \end{cases}$$

There may therefore be a limit cycle in a region bounded by $x^2+y^2=r^2$, where $r^2>\frac{1}{2}$, but it cannot exist entirely inside $x^2+y^2=\frac{1}{2}$.

Changing to polar coordinates,

$$r^2 = x^2 + y^2 \qquad \Rightarrow \quad 2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

From the original non-linear system:

$$r\frac{dr}{dt} = x\left(x\left(1-x^2-y^2\right) + 5y\right) + y\left(-5x + y\left(1-x^2-y^2\right)\right)$$

$$= +5xy + x^2\left(1-r^2\right) - 5xy + y^2\left(1-r^2\right) = r^2\left(1-r^2\right)$$

$$\Rightarrow \quad \frac{dr}{dt} = r\left(1-r^2\right) \begin{cases} <\ 0 & (r>1) \\ >\ 0 & (r<1) \end{cases}$$

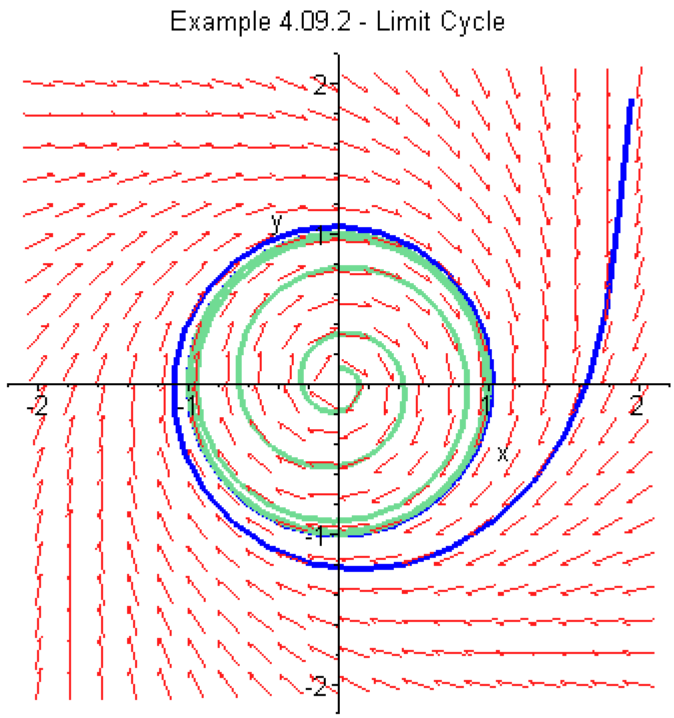
Therefore solutions that start closer than one unit to the critical point spiral out, but solutions that start further away than one unit approach the critical point. A solution on the circle $r=1$ never changes its distance from the origin and stays on that circle, but is not stationary.

Therefore $x^2+y^2 = 1$ is the limit cycle.

Consider the region D bounded by the circles $x^2 + y^2 = 1/100$ and $x^2 + y^2 = 2$, inside which $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$ and $\frac{dr}{dt}$ both change sign. All trajectories crossing the inner circle must be moving away from the origin into region D and all trajectories crossing the outer circle must be moving towards the origin, also into region D.

Thus, a solution path that enters D can never leave D.
There are no singularities in the region or its boundaries.
Therefore, by the Poincaré-Bendixon theorem, a limit cycle exists in the region.

Example 4.09.2 (continued)



Checking that $x^2 + y^2 = 1$ is a solution to the non-linear equation:

$$\frac{dx}{dt} = x\left(1 - x^2 - y^2\right) + 5y = 5y \quad \text{and} \quad \frac{dy}{dt} = -5x + y\left(1 - x^2 - y^2\right) = -5x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = -\frac{x}{y}$$

But $x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2y\dfrac{dy}{dx} = 0 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{x}{y}$

Therefore the limit cycle $x^2 + y^2 = 1$ is a solution to the non-linear system **(1)**.

4.10 Lyapunov Functions [for reference only - ***not*** examinable]

The equation of motion for an unforced damped elastic mass-spring system is

$$\frac{d^2x}{dt^2} + \varepsilon \frac{dx}{dt} + \mu x = 0 \qquad \textbf{(1)}$$

Consider the case where the restoring force (per unit mass) coefficient $\mu = 1$ and the damping (per unit mass) coefficient ε is small and positive. The equivalent first order system is

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x - \varepsilon y \end{aligned} \qquad \textbf{(2)}$$

The coefficient matrix is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -\varepsilon \end{pmatrix}$$

Using the results on page 4.30,

$D = (a-d)^2 + 4bc = (0+\varepsilon)^2 + 4(-1)(1) = \varepsilon^2 - 4 < 0$

$\lambda = \dfrac{(a+d) \pm \sqrt{D}}{2} = \dfrac{-\varepsilon \pm j\sqrt{4 - \varepsilon^2}}{2}$

[or solve the characteristic equation $\det(A - \lambda I) = 0$:

$(0-\lambda)(-\varepsilon-\lambda) - (1)(-1) = 0 \quad \Rightarrow \quad \lambda^2 - \varepsilon\lambda + 1 = 0$.]

The single critical point at the origin is therefore a **stable focus** (asymptotically stable).

The kinetic energy is $\frac{1}{2}mv^2 = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \frac{1}{2}my^2$.

The potential energy of a mass-spring system is proportional to the square of the extension x. Therefore the function $V(x, y) = \frac{1}{2}(x^2 + y^2)$ is related to the total energy of the system. $V(x, y)$ has an absolute minimum value of 0 at the origin, which should therefore be a stable equilibrium point.

From the chain rule and **(2)**,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = x \cdot y + y(-x - \varepsilon y) = -\varepsilon y^2 \leq 0 \quad \forall t$$

Therefore V decreases as t increases.
Also V decreases as the distance from the origin decreases.
Therefore the distance from the origin must decrease as t increases.
$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = 0$. All orbits terminate at the origin.
Again, the origin is an asymptotically stable point.

Energy considerations and Stability:

For a system of differential equations that arises from the description of a physical system, if the total energy of the system is constant or decreasing and a critical point corresponds to a point of minimum potential energy of the system, then the critical point should be stable.

If the critical point corresponds to a maximum of potential energy (such as the upside-down position of the pendulum in section 4.01), then the critical point should be unstable.

If (0, 0) is an asymptotically stable critical point of the system

$$\frac{dx}{dt} = f(x, y) \quad , \quad \frac{dy}{dt} = g(x, y) \qquad \textbf{(3)}$$

then there must exist some domain D, containing (0, 0), such that all solutions in D must approach (0, 0) as $t \to \infty$.

Suppose that an energy function $V(x, y)$ exists such that $V(0, 0) = 0$ and $V(x, y) > 0$ everywhere else in D. Then, following any open orbit in D, V must decrease to zero as $t \to \infty$. The converse of these statements is more useful:
If V decreases to zero as $t \to \infty$ on every trajectory in D, then every trajectory in D must approach the origin as $t \to \infty$ and the origin is therefore asymptotically stable.

Definitions:

Let $V(x, y)$ be defined on some domain D that contains the origin.

V is **positive definite** on D if $V(0, 0) = 0$ and $V(x, y) > 0$ for all other points in D.
V is **negative definite** on D if $V(0, 0) = 0$ and $V(x, y) < 0$ for all other points in D.

V is **positive semi-definite** on D if $V(0, 0) = 0$ and $V(x, y) \geq 0$ for all other points in D.
V is **negative semi-definite** on D if $V(0, 0) = 0$ and $V(x, y) \leq 0$ for all other points in D.

A function $V(x, y)$ is a **Lyapunov function** for the system

$$\frac{dx}{dt} = f(x,y) \quad , \quad \frac{dy}{dt} = g(x,y) \tag{3}$$

if there exists some neighbourhood of the origin in which

- V is a differentiable function of x and y;
- $V > 0$ except at the origin, where $V = 0$; and
- For any solution $(x(t), y(t))$ of **(3)** there exists a t_o such that $\frac{dV}{dt} \leq 0$ for all $t \geq t_o$.

Theorem:

If $V(x, y)$ is a Lyapunov function for the system **(3)**, then:

If $\frac{dV}{dt}$ is negative semidefinite, then (0, 0) is stable.
If $\frac{dV}{dt}$ is negative definite, then (0, 0) is asymptotically stable.
If $\frac{dV}{dt}$ is positive definite, then (0, 0) is unstable.

Also note that, by the chain rule and **(3)**,

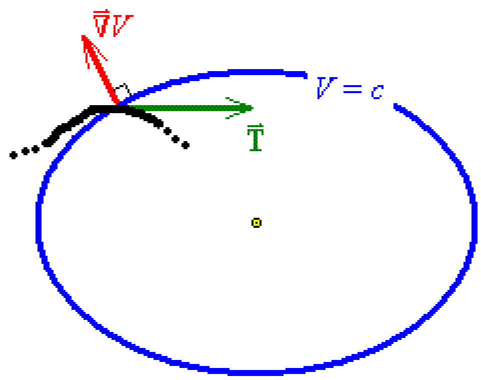
$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = \frac{\partial V}{\partial x}\cdot f(x,y) + \frac{\partial V}{\partial y}\cdot g(x,y)$$

and that

$$\frac{dV}{dt} = \vec{\nabla} V \bullet \vec{\mathbf{T}}$$

where $\vec{\nabla} V = \frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}}$ is the gradient vector of the scalar function $V(x, y)$ and $\vec{\mathbf{T}} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} = f(x,y)\hat{\mathbf{i}} + g(x,y)\hat{\mathbf{j}}$ is the **tangent vector** to the trajectory $(x(t), y(t))$.

If $\frac{dV}{dt}$ is negative definite, then the two vectors must point in directions more than 90° apart, everywhere in the region (except possibly at the origin). But the gradient vector points in the direction of increasing V, at right angles to the contours $V =$ constant.



V is positive definite, so its gradient vector points outward, away from the origin.
Therefore the trajectories must point inward, everywhere in the region where V is positive definite and $\frac{dV}{dt}$ is negative definite.

The general quadratic function

$$V(x,y) = ax^2 + bxy + cy^2$$

is positive definite if and only if $a > 0$ and $b^2 - 4ac < 0$
and
is negative definite if and only if $a < 0$ and $b^2 - 4ac < 0$

Example 4.10.1

The populations of a pair of competing species are modelled by the system

$$\frac{dx}{dt} = x(1 - x - y)$$

$$\frac{dy}{dt} = y(0.75 - y - 0.5x)$$

Investigate the stability of the critical point at (0.5, 0.5).

Transform the critical point to the origin with the change of coordinates

$$w = x - 0.5 ; \qquad z = y - 0.5$$

The system becomes

$$\frac{dw}{dt} = (w+0.5)\big(1 - (w+0.5) - (z+0.5)\big) = -0.5w - 0.5z - w^2 - wz$$

$$\frac{dz}{dt} = (z+0.5)\big(0.75 - (z+0.5) - 0.5(w+0.5)\big) = -0.25w - 0.5z - 0.5wz - z^2$$

There are many possible choices for a Lyapunov function, among the simplest of which is

$$V(w, z) = w^2 + z^2$$

V is clearly positive definite: $V(0, 0) = 0$ and $V(w, z) > 0$ everywhere else.

$$\frac{dV}{dt} = \frac{\partial V}{\partial w}\cdot\frac{dw}{dt} + \frac{\partial V}{\partial z}\cdot\frac{dz}{dt}$$

$$= 2w\left(-0.5w - 0.5z - w^2 - wz\right) + 2z\left(-0.25w - 0.5z - 0.5wz - z^2\right)$$

$$= -\left(w^2 + 1.5wz + z^2\right) - \left(2w^3 + 2w^2z + wz^2 + 2z^3\right)$$

In the quadratic expression $-\left(w^2 + 1.5wz + z^2\right)$, $a = c = -1$ and $b = -1.5$.

$a < 0$ and $b^2 - 4ac < 0$, so that $-\left(w^2 + 1.5wz + z^2\right)$ is negative definite.

The cubic terms can be of either sign, but sufficiently close to $(w, z) = (0, 0)$ they will be negligible compared to the quadratic terms. Therefore a region does exist around (0, 0) such that V is positive definite and $\frac{dV}{dt}$ is negative definite. The critical point must therefore be asymptotically stable.

By using a more complicated Lyapunov function and obtaining bounds on where its derivative is negative definite, one can estimate how far the region of asymptotic stability extends around the critical point.

Example 4.10.1 (continued)

Note that we can also investigate stability by finding the eigenvalues of the linear system that approximates the non-linear system near the critical point:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -0.5 & -0.5 \\ -0.25 & -0.5 \end{pmatrix} \begin{pmatrix} x-0.5 \\ y-0.5 \end{pmatrix}$$

The characteristic equation is

$$\det\left(A-\lambda I\right) = 0 \quad \Rightarrow \quad \left(-0.5-\lambda\right)^2 - \left(-0.5\right)\left(-0.25\right) = 0$$

$$\Rightarrow \quad \left(\lambda+0.5\right)^2 = 0.125 \quad \Rightarrow \quad \lambda+0.5 = \pm\sqrt{0.125} \quad \Rightarrow \quad \lambda = -0.5\pm\sqrt{0.125}$$

which is a real distinct negative pair.
The critical point is therefore an asymptotically stable node.