6.03 Examples

Example 6.03.1

(a) Find extremals $y(x)$ for $I = \int_{x_0}^{x_1} \frac{(y')^2}{x^3}\, dx$.

(b) Find the extremal that passes through the points (0, 1) and (1, 4).

(c) Prove that the extremal in part (b) minimizes the integral I.

(a) $F = \frac{(y')^2}{x^3} \quad \Rightarrow \quad \frac{\partial F}{\partial y} = 0$

Euler's equation simplifies to

$$\frac{d}{dx}\left(\frac{\partial}{\partial y'}\frac{(y')^2}{x^3}\right) = 0 \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{2y'}{x^3}\right) = 0$$

$$\Rightarrow \quad \frac{2y'}{x^3} = c_1 \quad \Rightarrow \quad y' = \tfrac{1}{2}c_1 x^3 \quad \Rightarrow \quad y = \tfrac{1}{8}c_1 x^4 + c_2$$

Redefining the arbitrary constants, this leads to the two-parameter family of extremals

$$y(x) = A x^4 + B$$

(b) The curve $y(x) = A x^4 + B$ must pass through both (0, 1) and (1, 4).

$1 = 0 + B \quad \Rightarrow \quad B = 1$

$4 = A(1)^4 + 1 \quad \Rightarrow \quad A = 3$

Therefore the extremal through (0, 1) and (1, 4) is $\Gamma_o : \quad y = 3x^4 + 1$.

Example 6.03.1 (continued)

(c) To prove that $y = 3x^4 + 1$ really is the path between (0, 1) and (1, 4) that minimizes the value of the integral $I = \int_{x_0}^{x_1} \frac{(y')^2}{x^3}\, dx$, consider the related family of functions

$\Gamma: \quad y = 3x^4 + 1 + g(x), \quad$ where $\quad g(0) = g(1) = 0 \quad$ and $g(x)$ is otherwise arbitrary.

$$I(\Gamma) = \int_0^1 \frac{(y')^2}{x^3}\, dx = \int_0^1 \frac{\left(12x^3 + g'(x)\right)^2}{x^3}\, dx$$

$$= \int_0^1 \frac{\left(12x^3\right)^2 + 24x^3 g'(x) + \left(g'(x)\right)^2}{x^3}\, dx$$

$$= \int_0^1 \frac{\left(12x^3\right)^2}{x^3}\, dx + 24\int_0^1 g'(x)\, dx + \int_0^1 \frac{\left(g'(x)\right)^2}{x^3}\, dx$$

$$\Rightarrow \quad I(\Gamma) = I(\Gamma_0) + 24\left(\cancel{g(1)} - \cancel{g(0)}\right) + \int_0^1 \frac{\left(g'(x)\right)^2}{x^3}\, dx > I(\Gamma_0) \quad \text{for } 0 < x < 1.$$

Note that the integral $\int_0^1 \frac{\left(g'(x)\right)^2}{x^3}\, dx$ is necessarily positive, because $g'(x)$ cannot be identically zero on [0, 1] and the integrand is non-negative on [0, 1].
Also $g(0) = g(1) = 0$. Therefore $I(\Gamma) > I(\Gamma_o)$ and Γ_o minimizes I.

If F is explicitly independent of x and y, so that the integral to be minimized is of the form $I = \int_{x_0}^{x_1} F(y')\,dx$, then Euler's equation simplifies to

$$\frac{d}{dx}\left(F_{y'}\right) - F_y = 0 \quad \Rightarrow \quad \underbrace{F_{y'x}}_{=0} + y'\underbrace{F_{y'y}}_{=0} + y''F_{y'y'} - \underbrace{F_y}_{=0} \equiv 0$$

$$\Rightarrow \quad \boxed{y''F_{y'y'} \equiv 0}$$

If $F_{y'y'} \not\equiv 0$ then $y'' \equiv 0 \quad \Rightarrow \quad \boxed{y(x) = Ax + B}$ (a line).

Example 6.03.2 (Example 6.01.1 revisited)

Show that the geodesic on $\mathbb{R}^2$ between any two points $x = a$ and $x = b$ is the straight line between the two points.

The arc length s between any two points $x = a$ and $x = b$ along any path C in $\mathbb{R}^2$ is the line integral

$$s = \int_C ds = \int_C \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

This integral is of the form $I = \int_{x_0}^{x_1} F(y')\,dx$, where $F(y') = \sqrt{1 + (y')^2}$.

Clearly

$$F_{y'y'} \not\equiv 0 \quad \Rightarrow \quad y''(x) = 0 \ \forall x \quad \Rightarrow \quad y(x) = Ax + B$$

which is a straight line.
But the extremal must pass through both points.
Only one straight line can pass through a pair of distinct points on $\mathbb{R}^2$.
Therefore the geodesic is the straight line between the two points.

If F is explicitly independent of y, so that the integral to be minimized is of the form $I = \int_{x_0}^{x_1} F(x, y')\, dx$, then Euler's equation simplifies to

$$\frac{d}{dx}\left(F_{y'}\right) - 0 = 0 \quad \Rightarrow \quad \boxed{F_{y'} = c_1}$$

If F is explicitly independent of x, so that the integral to be minimized is of the form $I = \int_{x_0}^{x_1} F(y, y')\, dx$, then multiply Euler's equation

$$\frac{d}{dx}\left(F_{y'}\right) - F_y = 0$$

by y' to obtain

$$y'\frac{d}{dx}\left(F_{y'}\right) - y'F_y = 0 \quad \Rightarrow \quad \left(\frac{d}{dx}\left(y'F_{y'}\right) - y''F_{y'}\right) - y'F_y = 0$$

But $\dfrac{dF}{dx} = \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y}\dfrac{dy}{dx} + \dfrac{\partial F}{\partial y'}\dfrac{dy'}{dx} = 0 + y'\dfrac{\partial F}{\partial y} + y''\dfrac{\partial F}{\partial y'}$

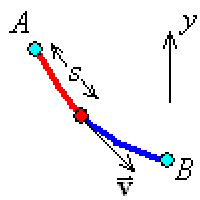
$$\Rightarrow \quad \frac{d}{dx}\left(y'F_{y'}\right) - \frac{dF}{dx} = 0 \quad \Rightarrow \quad \boxed{y'F_{y'} - F = c_1}$$

Example 6.03.3 The Brachistochrone Problem of Bernoulli (1696)

Find the curve $y = f(x)$ such that a particle sliding under gravity but without friction on the curve from the point $A(x_0, y_0)$ to the point $B(x_1, y_1)$ reaches B in the least time.

The sum of kinetic and potential energy of the particle is constant along the curve:

$E = \tfrac{1}{2}mv^2 + mgy = \text{const.}$



$$v = \frac{ds}{dt} \quad \Rightarrow \quad E = \tfrac{1}{2}m\left(\frac{ds}{dt}\right)^2 + mgy = \text{const.}$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{\frac{2E}{m} - 2gy} \quad \Rightarrow \quad dt = \frac{ds}{\sqrt{\frac{2E}{m} - 2gy}} = \frac{\sqrt{1 + (y')^2}\, dx}{\sqrt{\frac{2E}{m} - 2gy}}$$

Example 6.03.3 (continued)

Therefore the time taken to slide down the curve $y(x)$ is

$$t\left[y(x)\right] = \int_{x_A}^{x_B} \sqrt{\frac{1 + (y')^2}{\frac{2E}{m} - 2gy}}\, dx$$

If the point A is at the origin, y is measured *downwards* and the particle is released from rest, then the total energy is $E = \tfrac{1}{2}m(0)^2 + mg(0) = 0$ and the integral for travel time simplifies to

$$t\left[y(x)\right] = \int_0^{x_B} \sqrt{\frac{1 + (y')^2}{2gy}}\, dx$$

The integrand is an explicit function of y and y' only, not x.

When F is explicitly independent of x, $y'F_{y'} - F = c_1$

$$\Rightarrow \quad y'\frac{y'}{\sqrt{2gy}\sqrt{1 + (y')^2}} - \sqrt{\frac{1 + (y')^2}{2gy}} = c_1$$

$$\Rightarrow \quad (y')^2 - \left(1 + (y')^2\right) = c_1\sqrt{2gy\left(1 + (y')^2\right)}$$

$$\Rightarrow \quad 1 = c_1^{\,2}\, 2gy\left(1 + (y')^2\right) \qquad \Rightarrow \quad y\left(1 + (y')^2\right) = c_2$$

Use the substitution $y' = \dfrac{dy}{dx} = \tan\phi$. Then

$$y\left(1 + \tan^2\phi\right) = c_2 \quad \Rightarrow \quad y\sec^2\phi = c_2 \quad \Rightarrow \quad y = c_2\cos^2\phi = c_2\frac{(1+\cos 2\phi)}{2}$$

$$dx = \frac{dy}{\tan\phi} = c_2\frac{(2\cos\phi)(-\sin\phi)}{\tan\phi}\, d\phi = -c_2\frac{2\sin\phi\cos\phi}{\left(\dfrac{\sin\phi}{\cos\phi}\right)}\, d\phi$$

$$\Rightarrow \quad dx = -c_2\left(2\cos^2\phi\right) d\phi = -c_2\left(1+\cos 2\phi\right) d\phi$$

$$\Rightarrow \quad x = -c_2\int(1+\cos 2\phi)\, d\phi = -c_2\left(\phi + \frac{\sin 2\phi}{2}\right) + c_3$$

Therefore the solution can be expressed in parametric form by

$$\left(x(\phi), y(\phi)\right) = \left(c_3 + r\left(2\phi + \sin 2\phi\right), -r\left(1 + \cos 2\phi\right)\right)$$

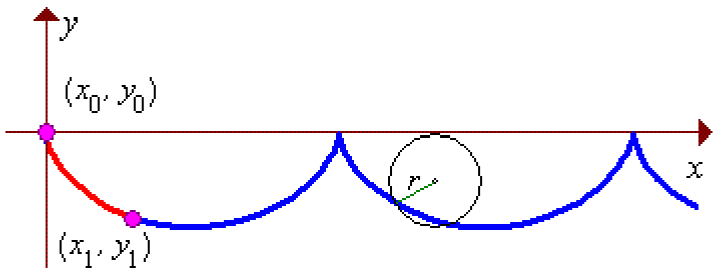
where $r = -\dfrac{c_2}{2}$.

Example 6.03.3 (continued)

Replacing 2ϕ by $\theta+\pi$ and defining $a = c_3 + r\pi$,

$$\boxed{\left(x(\theta), y(\theta)\right) = \left(a + r(\theta - \sin\theta), -r(1 - \cos\theta)\right)}$$

which is the parametric equation of a two-parameter family of **cycloids**.



Parameter a shifts the curve horizontally, while r changes the magnitude of the radius of the generating circle. [A cycloid is the path generated by a point on the circumference of a circle that rolls without slipping along an axis. θ is the angle through which the rolling circle of radius r has rotated.]

Example 6.03.4 (The Catenary)

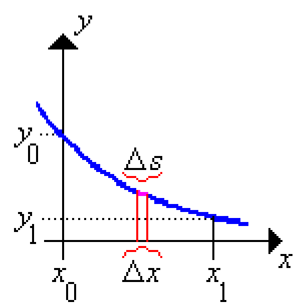
Find the equation $y = f(x)$ of the curve between points $A(x_0, y_0)$ and $B(x_1, y_1)$ which is such that the curved surface of the surface of revolution swept out by the curve around the x-axis has the least possible area.

The element of curved surface area is $2\pi y\,\Delta s$, where Δs is the element of arc length.

The total curved surface area is therefore

$$A = 2\pi \int_{x=x_0}^{x=x_1} y\,ds = 2\pi \int_{x_0}^{x_1} y\,\frac{ds}{dx}\,dx$$

$$= 2\pi \int_{x_0}^{x_1} y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$



The integrand is of the form $F(y, y')$, with no explicit dependence on x.

Therefore the extremal is the solution of $y'\dfrac{\partial F}{\partial y'} - F = c_1$, where

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<u>Example 6.03.4</u> (continued)

$$F(y, y') = y\sqrt{1 + (y')^2}$$

$$\Rightarrow \quad y'y\frac{y'}{\sqrt{1+(y')^2}} - y\sqrt{1+(y')^2} = c_1$$

$$\Rightarrow \quad y\left((y')^2 - 1 - (y')^2\right) = c_1\sqrt{1+(y')^2} \qquad \Rightarrow \quad y = -c_1\sqrt{1+(y')^2}$$

$$\Rightarrow \quad y^2 = c_1^{\,2}\left(1+(y')^2\right) \qquad \Rightarrow \quad \frac{dy}{dx} = \pm\sqrt{\frac{y^2}{c_1^{\,2}} - 1}$$

Let $y = c_1 \cosh t$

then $\dfrac{dy}{dx} = \pm\sqrt{\cosh^2 t - 1} = \pm\sqrt{\sinh^2 t} = \pm\sinh t$

But $\dfrac{dy}{dx}\cdot\dfrac{dx}{dt} = \dfrac{dy}{dt} = c_1 \sinh t = \pm c_1\dfrac{dy}{dx} \quad \Rightarrow \quad \dfrac{dx}{dt} = \pm c_1 \quad \Rightarrow \quad x = \pm c_1 t + c_2$

$$\Rightarrow \quad t = \pm\frac{x}{c_1} - \frac{c_2}{c_1}$$

Let $A = c_1$ and $B = -\dfrac{c_2}{c_1}$ and note that $\cosh(t)$ is an even function.

Then the two-parameter family of extremals is

$$y(x) = A\cosh\left(\frac{x}{A} + B\right)$$

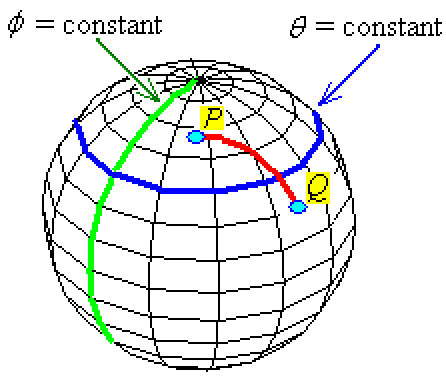
which is the catenary curve.

Example 6.03.5

Find the geodesic (shortest path) between two points P and Q on the surface of a sphere.

Let the radius of the sphere be a and choose the coordinate system such that the origin is at the centre of the sphere. The relationship between the Cartesian coordinates (x, y, z) of any point on the sphere and its spherical polar coordinates (θ, ϕ) is

$x = a \sin\theta \cos\phi$
$y = a \sin\theta \sin\phi$
$z = a \cos\theta$



Note that the radial coordinate r is constant ($r = a$) everywhere on the sphere.
The element of arc length in the spherical polar coordinate system is

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2$$

But, on the sphere, $dr = 0$

$$\Rightarrow \quad ds^2 = a^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$\Rightarrow \quad \left(\frac{ds}{d\theta}\right)^2 = a^2\left(1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

The distance along a path on the sphere between points P and Q is therefore

$$s = \int_P^Q ds = \int_P^Q \frac{ds}{d\theta}\,d\theta = \int_P^Q a\sqrt{1 + \sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta$$

The geodesic between P and Q on the surface of the sphere is the function $\phi(\theta)$ that minimizes the integral for s.

For x read θ, for y read ϕ, for y' read $\dfrac{d\phi}{d\theta}$.

The integrand is $F(\theta, \phi, \phi') = a\sqrt{1 + \sin^2\theta\left(\dfrac{d\phi}{d\theta}\right)^2}$.

The integrand is an explicit function of θ and $\dfrac{d\phi}{d\theta}$, but not of ϕ.

Example 6.03.5 (continued)

Therefore the Euler equation for the extremal, $\frac{d}{d\theta}\left(F_{\phi'}\right) - F_{\phi} = 0$, simplifies to

$$\frac{\partial F}{\partial \phi'} = c \qquad \Rightarrow \quad \frac{a \sin^2\theta \left(\frac{d\phi}{d\theta}\right)}{\sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}} = c$$

$$\Rightarrow \quad \left(a \sin^2\theta \left(\frac{d\phi}{d\theta}\right)\right)^2 = c^2 \left(1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2\right)$$

$$\Rightarrow \quad \sin^2\theta\left(a^2 \sin^2\theta - c^2\right)\left(\frac{d\phi}{d\theta}\right)^2 = c^2 \qquad \Rightarrow \quad \left(\frac{d\phi}{d\theta}\right)^2 = \frac{c^2}{\sin^2\theta\left(a^2\sin^2\theta - c^2\right)}$$

$$\Rightarrow \quad \frac{d\phi}{d\theta} = \frac{c}{\sin\theta\sqrt{a^2\sin^2\theta - c^2}}$$

After substitutions, this can be integrated to a function $\phi(\theta)$, which, upon conversion back into Cartesian coordinates, can be found to lie entirely on a plane through the origin. But the intersection of any plane through the origin with the sphere is just a great circle on the sphere.

Alternatively, reorient the coordinate system (or rotate the sphere) so that one of the two points is at the north pole ($\theta = 0$). Then $\frac{\partial F}{\partial \phi'} = c$ becomes $\frac{\partial F}{\partial \phi'} = 0$ (because $\sin\theta = 0$ at the pole and c must have the same value everywhere on the path).

$$\Rightarrow \quad \frac{a \sin^2\theta \left(\frac{d\phi}{d\theta}\right)}{\sqrt{1 + \sin^2\theta \left(\frac{d\phi}{d\theta}\right)^2}} = 0 \qquad \Rightarrow \quad \sin^2\theta\left(\frac{d\phi}{d\theta}\right) = 0$$

$\sin\theta \neq 0$ along the path between the points, so

$$\frac{d\phi}{d\theta} = 0 \qquad \Rightarrow \quad \phi = \text{constant}$$

which, again, is an arc of a great circle (a line of longitude from the north pole to the other point).

Therefore the geodesic between any two points on the sphere is the shorter arc of the great circle that passes through both points.

Example 6.03.6

Find the path $y = f(x)$ between the points (0, 0) and (π/2, 0) that provides an extremum for the value of the integral

$$I = \int_0^{\pi/2} \left((y')^2 - y^2 - 4y \sin x \right) dx$$

$$F = (y')^2 - y^2 - 4y\sin x$$

$$\Rightarrow \quad \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = \frac{d}{dx}(2y') + 2y + 4\sin x = 0$$

$$\Rightarrow \quad y'' + y = -2\sin x$$

This is a second order linear ODE with constant coefficients and a pair of boundary conditions (solution curve passes through (0, 0) and (π/2, 0)).

A.E.: $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm j$

C.F.: $y_C = A\cos x + B\sin x$

P.S.: Method of undetermined coefficients:
$R(x) = -2 \sin x$, but $\sin x$ is part of the complementary function.

Therefore try $y_P = cx\cos x + dx\sin x \quad \Rightarrow \quad y_P'' = -2c\sin x + 2d\cos x - y_P$

Substitute y_P into the ODE:

$$y_P'' + y_P = -2c\sin x + 2d\cos x = -2\sin x$$

$$\Rightarrow \quad c = 1, \quad d = 0$$

$$\therefore \quad y_P = x\cos x$$

G.S.: $y = (x + A)\cos x + B\sin x$

Impose the boundary conditions:

$(0,0)$: $\quad 0 = A + 0$

$\left(\frac{\pi}{2}, 0\right)$: $\quad 0 = 0 + B$

Therefore the complete solution is

$$\boxed{y = f(x) = x\cos x}$$

It can be shown that this sole extremal solution leads to $I = -\frac{\pi}{4} < 0$

The trivial path $y \equiv 0$ leads to $I = 0$

Therefore the extremum must be an absolute minimum.