7. Fourier Series and Fourier Transforms

Fourier series have multiple purposes, including the provision of series solutions to some linear partial differential equations with boundary conditions (as will be reviewed in Chapter 8). Fourier transforms are often used to extract frequency information from time series data. For lack of time in this course, only a brief introduction is provided here.

Sections in this Chapter:

Sections for reference only, ***not*** examinable in this course:

7.01 Orthogonal Functions

The inner product (or scalar product or dot product) of two vectors **u** and **v** is defined in Cartesian coordinates in $\mathbb{R}^3$ by

$$\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \sum_{k=1}^{3} u_k v_k = u_1 v_1 + u_2 v_2 + u_3 v_3$$

The inner product possesses the four properties:

Commutative: $\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = \vec{\mathbf{v}} \bullet \vec{\mathbf{u}}$

Scalar multiplication: $(k\vec{\mathbf{u}}) \bullet \vec{\mathbf{v}} = k(\vec{\mathbf{u}} \bullet \vec{\mathbf{v}}), \quad k \in \mathbb{R}$

Positive definite: $\vec{\mathbf{u}} \bullet \vec{\mathbf{u}} \begin{cases} = 0 & (\text{if } \vec{\mathbf{u}} = \vec{\mathbf{0}}) \\ > 0 & (\text{if } \vec{\mathbf{u}} \neq \vec{\mathbf{0}}) \end{cases}$

Associative: $\vec{\mathbf{u}} \bullet (\vec{\mathbf{v}} + \vec{\mathbf{w}}) = \vec{\mathbf{u}} \bullet \vec{\mathbf{v}} + \vec{\mathbf{u}} \bullet \vec{\mathbf{w}}$

Vectors $\vec{\mathbf{u}}, \vec{\mathbf{v}}$ are orthogonal if and only iff $\vec{\mathbf{u}} \bullet \vec{\mathbf{v}} = 0$.
A pair of non-zero orthogonal vectors intersects at right angles.

The inner product of two real-valued functions f_1 and f_2 on an interval $[a, b]$ may be defined in a way that also possesses these four properties:

$$\boxed{(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx}$$

Two functions f_1 and f_2 are said to be **orthogonal** on an interval [a, b] if their inner product is zero:

$$(f_1, f_2) = \int_a^b f_1(x)\, f_2(x)\, dx = 0$$

A set of real-valued functions $\{ \phi_0(x), \phi_1(x), \phi_2(x), \cdots, \phi_n(x) \}$ is orthogonal on the interval [a, b] if the inner product of any two of them is zero:

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x) \phi_n(x)\, dx = 0 \quad (m \neq n)$$

If, in addition, the inner product of any function in the set with itself is unity, then the set is **orthonormal**:

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x) \phi_n(x)\, dx = \delta_{mn} = \begin{cases} 0 & (m \neq n) \\ 1 & (m = n) \end{cases}$$

where δ_{mn} is the "Kronecker delta" symbol.

Just as any vector in $\mathbb{R}^3$ may be represented by a linear combination of the three Cartesian basis vectors, (which form the orthonormal set {**i**, **j**, **k**}), so a real valued function $f(x)$ defined on $[a, b]$ may be written as a linear combination of the elements of an infinite orthonormal set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ on $[a, b]$:

$$f(x) = c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots$$

To find the coefficients c_n, multiply $f(x)$ by $\phi_n(x)$ and integrate over $[a, b]$:

$$\int_a^b f(x)\phi_n(x)\,dx = c_0\int_a^b \phi_0(x)\phi_n(x)\,dx + c_1\int_a^b \phi_1(x)\phi_n(x)\,dx + \ldots$$

$$= \sum_{m=0}^{\infty} c_m \int_a^b \phi_m(x)\phi_n(x)\,dx$$

But the $\{\phi_n(x)\}$ are an orthonormal set. Therefore all but one of the terms in the infinite series are zero. The exception is the term for which $m = n$, where the integral is unity. Therefore

$$c_n = \int_a^b f(x)\phi_n(x)\,dx$$

and

$$\boxed{f(x) = \sum_{n=0}^{\infty}\left(\left(\int_a^b f(x)\phi_n(x)\,dx\right)\phi_n(x)\right)}$$

If the set is orthogonal but not orthonormal, then the form for $f(x)$ changes to

$$\boxed{f(x) = \sum_{n=0}^{\infty}\left(\left(\frac{\int_a^b f(x)\phi_n(x)\,dx}{\int_a^b \phi_n^2(x)\,dx}\right)\phi_n(x)\right)}$$

The orthogonal set $\{\phi_n(x)\}$ is **complete** if the only function that is orthogonal to all members of the set is the zero function $f(x) \equiv 0$. An expansion of every function $f(x)$ in terms of an orthogonal or orthonormal set $\{\phi_n(x)\}$ is not possible if $\{\phi_n(x)\}$ is not complete.

Also note that a generalised form of an inner product can be defined using a weighting function $w(x)$, so that, in terms of a complete orthogonal set $\{\phi_n(x)\}$,

$$f(x) = \sum_{n=0}^{\infty}\left(\left(\frac{\int_a^b w(x)f(x)\phi_n(x)\,dx}{\int_a^b w(x)\phi_n^2(x)\,dx}\right)\phi_n(x)\right)$$

We shall usually be concerned with the case $w(x) \equiv 1$ only.

Example 7.01.1

Show that the set { sin nx } $\left(n \in \mathbb{N}\right)$ is orthogonal but not orthonormal and not complete on $[-\pi, +\pi]$.

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(\cos\left(m-n\right)x - \cos\left(m+n\right)x\right) dx$$

If $m \neq n$, then

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2}\left[\frac{\sin\left(m-n\right)x}{m-n} - \frac{\sin\left(m+n\right)x}{m+n}\right]_{-\pi}^{\pi} = 0$$

The set { sin nx } is therefore orthogonal on $[-\pi, +\pi]$.

If $m = n$, then

$$\int_{-\pi}^{\pi} \sin nx \sin nx \, dx = \frac{1}{2}\int_{-\pi}^{\pi} \left(1 - \cos 2nx\right) dx = \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi \neq 1$$

so the set is **not** orthonormal, (although the set $\left\{\dfrac{\sin nx}{\sqrt{\pi}}\right\}$ **is** orthonormal).

To show that the set is not complete on $[-\pi, +\pi]$, we need to find a non-trivial function that is orthogonal to sin nx for all positive integer values of n.

Note that sin nx is an odd function of x and that the range of integration is symmetric about $x = 0$. The product of any odd function with any even function is another odd function. The integral of any odd function over a range of integration that is symmetric about $x = 0$ is zero. This leads us to try any even function. The simplest non-trivial even function is $f(x) \equiv 1$.

$$\int_{-\pi}^{\pi} 1 \sin nx \, dx = \left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi} = 0 \quad \forall n \in \mathbb{N}$$

The function $f(x) \equiv 1$ is therefore orthogonal to all members of the set { sin nx }.
The set { sin nx } is therefore not complete.

7.02 Definitions of Fourier Series

Example 7.02.1

Show that the set $\left\{1, \left\{\cos\left(\frac{n\pi x}{L}\right)\right\}, \left\{\sin\left(\frac{n\pi x}{L}\right)\right\}\right\}$, $(n \in \mathbb{N})$ is orthogonal but not orthonormal on [–*L*, *L*].

Inner product of any two distinct sine functions ($m \neq n$):

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} \left(\cos\frac{(m-n)\pi x}{L} - \cos\frac{(m+n)\pi x}{L} \right) dx$$

$$= \frac{1}{2}\left[\frac{L}{(m-n)\pi} \sin\frac{(m-n)\pi x}{L} - \frac{L}{(m+n)\pi} \sin\frac{(m+n)\pi x}{L} \right]_{-L}^{L} = 0$$

Inner product of any two distinct cosine functions ($m \neq n$):

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} \left(\cos\frac{(m+n)\pi x}{L} + \cos\frac{(m-n)\pi x}{L} \right) dx$$

$$= \frac{1}{2}\left[\frac{L}{(m+n)\pi} \sin\frac{(m+n)\pi x}{L} + \frac{L}{(m-n)\pi} \sin\frac{(m-n)\pi x}{L} \right]_{-L}^{L} = 0$$

This result holds also for $m = 0$, for which $\cos\left(\frac{m\pi x}{L}\right) \equiv 1$.

Inner product of any sine function with any cosine function:

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L} \left(\sin\frac{(m+n)\pi x}{L} + \sin\frac{(m-n)\pi x}{L} \right) dx$$

$$= \left\{ \begin{array}{ll} \frac{1}{2}\left[\frac{-L}{(m+n)\pi} \cos\frac{(m+n)\pi x}{L} - \frac{L}{(m-n)\pi} \cos\frac{(m-n)\pi x}{L} \right]_{-L}^{L} & (m \neq n) \\ \frac{1}{2}\left[\frac{-L}{2n\pi} \cos\frac{2n\pi x}{L} - 0 \right]_{-L}^{L} & (m = n) \end{array} \right\} = 0$$

This result holds also for $m \neq n = 0$, for which $\cos\left(\frac{n\pi x}{L}\right) \equiv 1$.

Example 7.02.1 (continued)

Therefore the set $\left\{1, \left\{\cos\left(\frac{n\pi x}{L}\right)\right\}, \left\{\sin\left(\frac{n\pi x}{L}\right)\right\}\right\}, \quad \left(n \in \mathbb{N}\right)$ is orthogonal on $[-L, L]$.

Inner product of any sine function with itself:

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L}\left(1 - \cos\frac{2n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\left[x - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right]_{-L}^{L} = L$$

Inner product of any cosine function with itself ($n > 0$):

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{1}{2}\int_{-L}^{L}\left(1 + \cos\frac{2n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\left[x + \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\right]_{-L}^{L} = L$$

Inner product of the function 1 with itself:

$$\int_{-L}^{L} 1 \times 1 \, dx = \left[x\right]_{-L}^{L} = 2L \neq L$$

Therefore the set is not orthonormal for any choice of L, although the related set
$\left\{\frac{1}{\sqrt{2L}}, \left\{\frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right)\right\}, \left\{\frac{1}{\sqrt{L}}\sin\left(\frac{n\pi x}{L}\right)\right\}\right\}, \quad \left(n \in \mathbb{N}\right)$ **is** orthonormal on $[-L, L]$.

Using the results from Example 7.02.1, we can express most real-valued functions $f(x)$ defined on $(-L, L)$, in terms of an infinite series of trigonometric functions:

The **Fourier series** of $f(x)$ on the interval $(-L, L)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx\,, \quad (n = 0, 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\,, \quad (n = 1, 2, 3, \ldots)$$

The $\{a_n, b_n\}$ are the Fourier coefficients of $f(x)$.

Note that the cosine functions (and the function 1) are even, while the sine functions are odd.

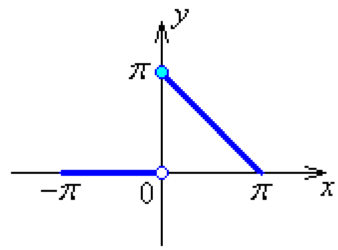
If $f(x)$ is even ($f(-x) = +f(x)$ for all x), then $b_n = 0$ for all n, leaving a Fourier cosine series (and perhaps a constant term) only for $f(x)$.

If $f(x)$ is odd ($f(-x) = -f(x)$ for all x), then $a_n = 0$ for all n, leaving a Fourier sine series only for $f(x)$.

Example 7.02.2

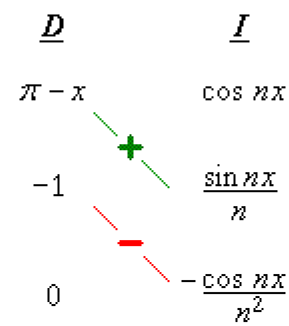
Expand $f(x) = \begin{cases} 0 & (-\pi < x < 0) \\ \pi - x & (0 \le x < +\pi) \end{cases}$ in a Fourier series.

$L = \pi.$



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 + \frac{1}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^2}{-2}\right]_0^{\pi} = \frac{\pi}{2}$$

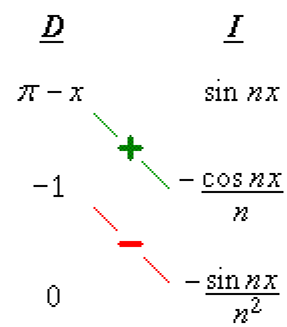


$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 + \frac{1}{\pi}\int_0^{\pi} (\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\sin nx \; - \; \cos nx}{n^2}\right]_0^{\pi} = \frac{1 - (-1)^n}{n^2 \pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 + \frac{1}{\pi}\int_0^{\pi} (\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{n(\pi - x)\cos nx \; + \; \sin nx}{-n^2}\right]_0^{\pi} = \frac{1}{n}$$



Therefore the Fourier series for $f(x)$ is

$$\boxed{f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \qquad (-\pi < x < +\pi)}$$

Example 7.02.2 (Additional Notes – also see "www.engr.mun.ca/~ggeorge/9420/demos/")

The first few partial sums in the Fourier series

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right) \qquad (-\pi < x < +\pi)$$

are

$$S_0 = \frac{\pi}{4}$$

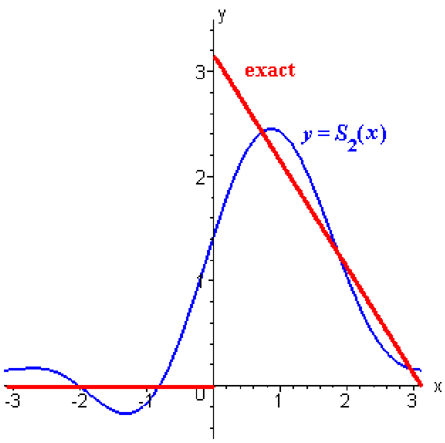
$$S_1 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x$$

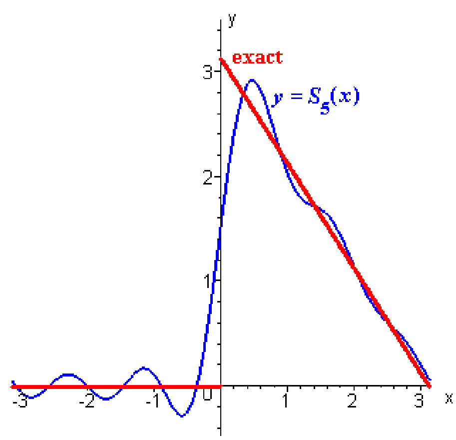
$$S_2 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x$$

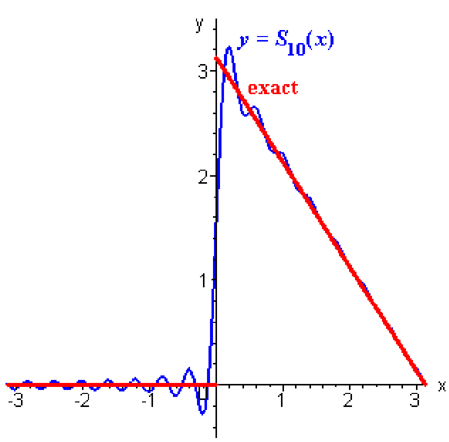
$$S_3 = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x + \frac{2}{9\pi}\cos 3x + \frac{1}{3}\sin 3x$$

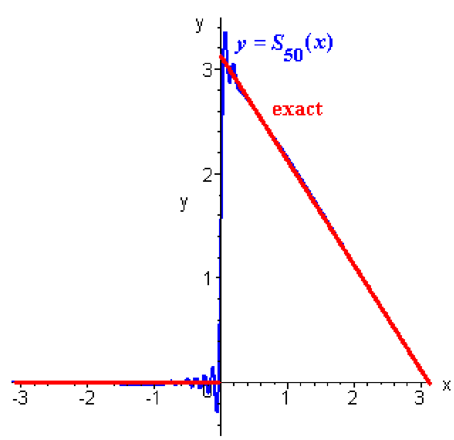
and so on.

The graphs of successive partial sums approach $f(x)$ more closely, except in the vicinity of any discontinuities, (where a systematic overshoot occurs, the **Gibbs phenomenon**).









Example 7.02.3

Find the Fourier series expansion for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

$L = 1$.

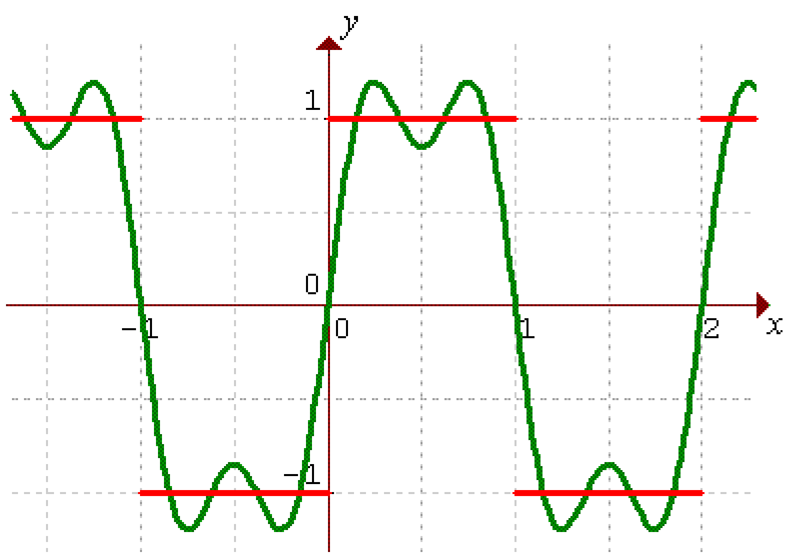
The function is odd ($f(-x) = -f(x)$ for all x).
Therefore $a_n = 0$ for all n. We will have a Fourier sine series only.

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0} -\sin n\pi x\,dx + \int_{0}^{1} \sin n\pi x\,dx$$

$$= \left[\frac{\cos n\pi x}{n\pi}\right]_{-1}^{0} + \left[\frac{-\cos n\pi x}{n\pi}\right]_{0}^{1} = \frac{2\left(1-(-1)^n\right)}{n\pi}$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\sin n\pi x\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\sin(2k-1)\pi x\right)$$

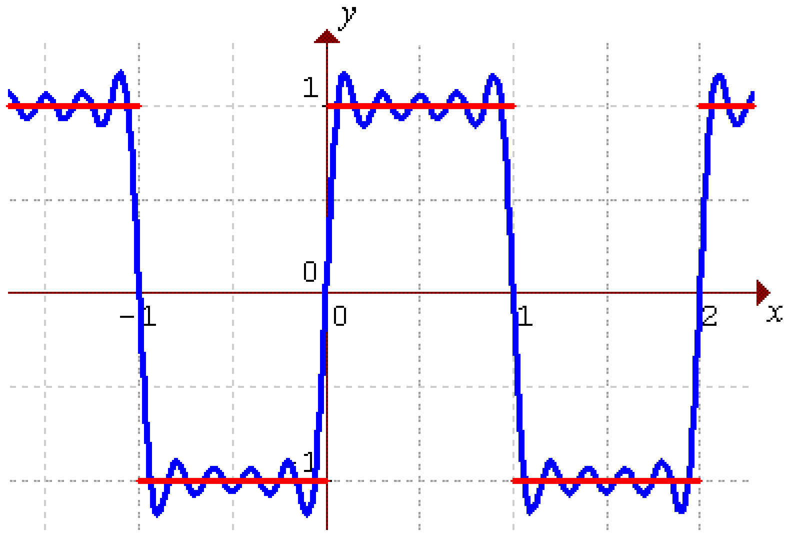
The graphs of the third and ninth partial sums (containing two and five non-zero terms respectively) are displayed here, together with the exact form for $f(x)$, with a **periodic extension** beyond the interval (–1, +1) that is appropriate for the square wave.



$$y = S_3(x)$$

Example 7.02.3 (continued)

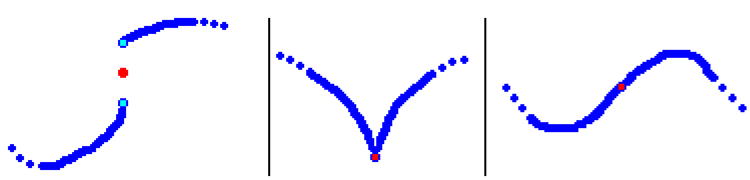
$y = S_9(x)$



Convergence

At all points $x = x_o$ in $(-L, L)$ where $f(x)$ is continuous and is either differentiable or the limits $\lim_{x \to x_o^-} f'(x)$ and $\lim_{x \to x_o^+} f'(x)$ both exist, the Fourier series converges to $f(x)$.

At finite discontinuities, (where the limits $\lim_{x \to x_o^-} f'(x)$ and $\lim_{x \to x_o^+} f'(x)$ both exist), the Fourier series converges to $\dfrac{f(x_o -) + f(x_o +)}{2}$,
(using the abbreviations $f(x_o -) = \lim_{x \to x_o^-} f(x)$ and $f(x_o +) = \lim_{x \to x_o^+} f(x)$).



$f(x)$ not continuous at $x = x_o$ | continuous but not differentiable | continuous and differentiable

In all cases, the Fourier series at $x = x_o$ converges to $\dfrac{f(x_o -) + f(x_o +)}{2}$ (the red dot).