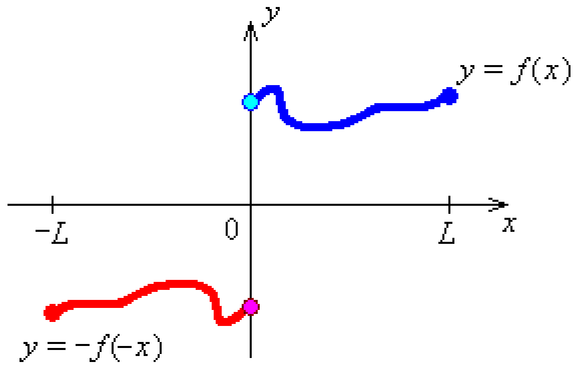
7.03 Half-Range Fourier Series

A Fourier series for $f(x)$, valid on $[0, L]$, may be constructed by extension of the domain to $[-L, L]$.

An odd extension leads to a **Fourier sine series**:

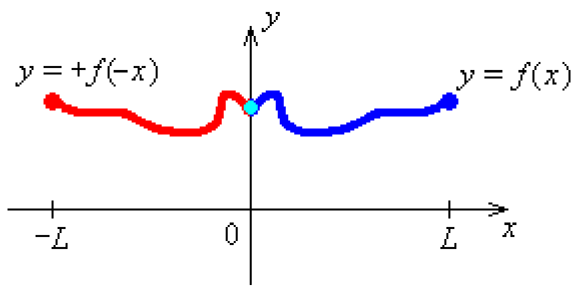


$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad (n = 1, 2, 3, \ldots)$$

An even extension leads to a **Fourier cosine series**:



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \quad (n = 0, 1, 2, 3, \ldots)$$

and there is automatic continuity of the Fourier cosine series at $x = 0$ and at $x = \pm L$.

Example 7.03.1

Find the Fourier sine series and the Fourier cosine series for $f(x) = x$ on $[0, 1]$.

$f(x) = x$ happens to be an odd function of x for any domain centred on $x = 0$. The odd extension of $f(x)$ to the interval $[-1, 1]$ is $f(x)$ itself.

Evaluating the Fourier sine coefficients,

$$b_n = \frac{2}{1}\int_0^1 x \sin\left(\frac{n\pi x}{1}\right) dx\,, \quad (n = 1, 2, 3, \ldots) \qquad \Rightarrow$$

$$b_n = 2\left[-\frac{x}{n\pi}\cos(n\pi x) + \frac{1}{(n\pi)^2}\sin(n\pi x)\right]_0^1 = \frac{2}{n\pi}\times(-1)^{n+1}$$

D		I
x		$\sin n\pi x$
	$+$	
1		$-\frac{\cos n\pi x}{n\pi}$
	$-$	
0		$-\frac{\sin n\pi x}{(n\pi)^2}$

Therefore the Fourier sine series for $f(x) = x$ on $[0, 1]$ (which is also the Fourier series for $f(x) = x$ on $[-1, 1]$) is

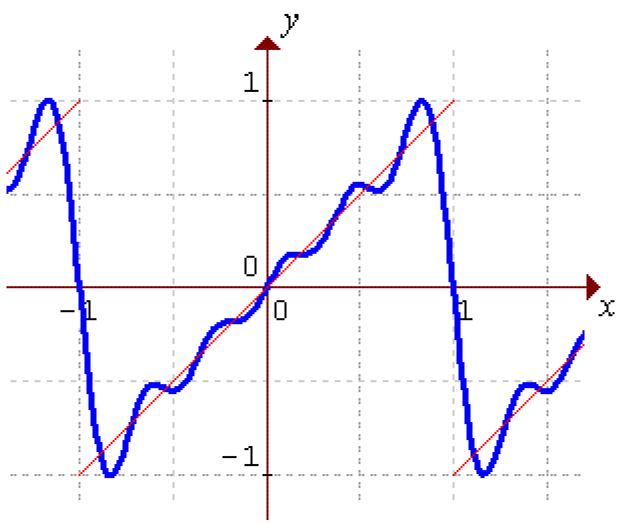
$$f(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(n\pi x)}{n\pi}$$

or

$$f(x) = \frac{2}{\pi}\left(\sin \pi x - \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} - \frac{\sin 4\pi x}{4} + \ldots\right)$$

This function happens to be continuous and differentiable at $x = 0$, but is clearly discontinuous at the endpoints of the interval ($x = \pm 1$).

Fifth order partial sum of the Fourier sine series for $f(x) = x$ on $[0, 1]$



Example 7.03.1 (continued)

The even extension of $f(x)$ to the interval $[-1, 1]$ is $f(x) = |x|$.

Evaluating the Fourier cosine coefficients,

$$a_n = \frac{2}{1}\int_0^1 x\cos\left(\frac{n\pi x}{1}\right)dx\,, \quad (n = 1, 2, 3, \ldots)$$

$$\Rightarrow \quad a_n = 2\left[\frac{x}{n\pi}\sin(n\pi x) + \frac{1}{(n\pi)^2}\cos(n\pi x)\right]_0^1$$

$$= \frac{2\left((-1)^n - 1\right)}{(n\pi)^2}$$

D		I
x		$\cos n\pi x$
	$+$	
1		$\frac{\sin n\pi x}{n\pi}$
	$-$	
0		$-\frac{\cos n\pi x}{(n\pi)^2}$

and $\quad a_0 = \frac{2}{1}\int_0^1 x\,dx = \left[x^2\right]_0^1 = 1$

Evaluating the first few terms,

$$a_0 = 1\,,\quad a_1 = \frac{-4}{\pi^2}\,,\quad a_2 = 0\,,\quad a_3 = \frac{-4}{9\pi^2}\,,\quad a_4 = 0\,,\quad a_5 = \frac{-4}{25\pi^2}\,,\quad a_6 = 0\,,\ldots$$

or $\quad a_n = \begin{cases} 1 & (n = 0) \\ \dfrac{-4}{(n\pi)^2} & (n = 1, 3, 5, \ldots) \\ 0 & (n = 2, 4, 6, \ldots) \end{cases}$

Therefore the Fourier cosine series for $f(x) = x$ on $[0, 1]$ (which is also the Fourier series for $f(x) = |x|$ on $[-1, 1]$) is

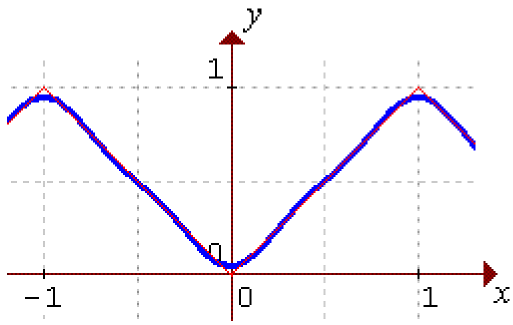
$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{\cos\left((2k-1)\pi x\right)}{(2k-1)^2}$$

or

$$f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\cos \pi x + \frac{\cos 3\pi x}{9} + \frac{\cos 5\pi x}{25} + \frac{\cos 7\pi x}{49} + \ldots\right)$$

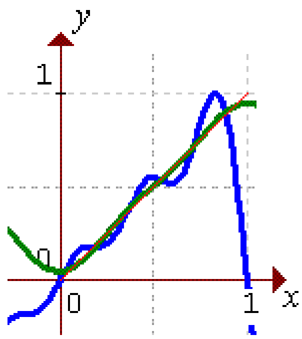
Example 7.03.1 (continued)

Third order partial sum of the Fourier cosine series for $f(x) = x$ on [0, 1]



Note how rapid the convergence is for the cosine series compared to the sine series.

$y = S_3(x)$ for cosine series and $y = S_5(x)$ for sine series for $f(x) = x$ on [0, 1]



7.04 Frequency Spectrum

The Fourier series may be combined into a single cosine series.
Let p be the fundamental period. If the function $f(x)$ is not periodic at all on $[-L, L]$, then the fundamental period of the extension of $f(x)$ to the entire real line is $p = 2L$.

Define the fundamental frequency $\omega = \dfrac{2\pi}{p} = \dfrac{\pi}{L}$.
The Fourier series for $f(x)$ on $[-L, L]$ is, from page 7.07,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega x) + b_n \sin(n\omega x)\right)$$

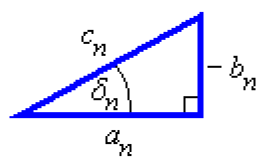
where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\omega x)\,dx, \quad (n = 0, 1, 2, 3, \ldots)$$

and

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\omega x)\,dx, \quad (n = 1, 2, 3, \ldots)$$

Let the **phase** angle δ_n be such that $\tan\delta_n = -\dfrac{b_n}{a_n}$,



so that $\sin\delta_n = -\dfrac{b_n}{c_n}$ and $\cos\delta_n = +\dfrac{a_n}{c_n}$

where the **amplitude** is $c_n = \sqrt{a_n^{\,2} + b_n^{\,2}}$.
Also, in the trigonometric identity $\cos A\cos B - \sin A\sin B \equiv \cos(A+B)$,
replace A by $n\omega x$ and B by δ_n. Then

$$a_n \cos(n\omega x) + b_n \sin(n\omega x) = (c_n\cos\delta_n)\cos(n\omega x) - (c_n \sin\delta_n)\sin(n\omega x)$$

$$= c_n\cos(n\omega x + \delta_n), \quad \text{where} \quad \boxed{\omega = \frac{2\pi}{p} = \frac{\pi}{L}}, \quad \boxed{c_n = \sqrt{a_n^{\,2}+b_n^{\,2}}} \quad \text{and} \quad \boxed{\tan\delta_n = -\frac{b_n}{a_n}}$$

Therefore the phase angle or **harmonic** form of the Fourier series is

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n\cos(n\omega x + \delta_n)}$$

Example 7.04.1

Plot the frequency spectrum for the standard square wave,

$$f(x) = \begin{cases} -1 & (-1 < x < 0) \\ +1 & (0 \le x < +1) \end{cases}$$

From Example 7.02.3, the Fourier series for the standard square wave is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\sin n\pi x\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\sin(2k-1)\pi x\right)$$

The fundamental frequency is $\omega = \pi$.

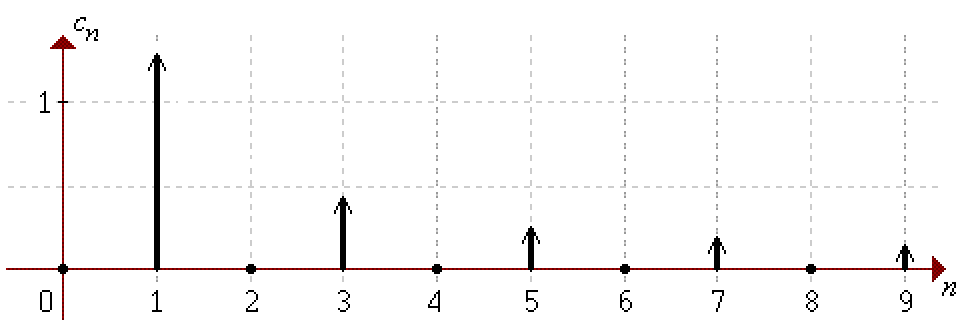
The absence of cosine terms $\Rightarrow \quad a_n = 0 \quad \forall n \qquad \Rightarrow \quad c_n = b_n$ and $\delta_n = -\frac{\pi}{2} \quad \forall n$.

The harmonic form of the Fourier series is therefore

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\cos\left(n\pi x - \tfrac{\pi}{2}\right)\right) = \frac{4}{\pi}\sum_{k=1}^{\infty}\left(\frac{1}{2k-1}\cos\left((2k-1)\pi x - \tfrac{\pi}{2}\right)\right)$$

The amplitudes are therefore

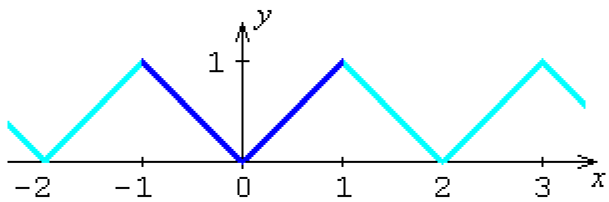
$$\boxed{c_n = \begin{cases} \dfrac{4}{n\pi} & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases}}$$



Example 7.04.2

Plot the frequency spectrum for the periodic extension of

$$f(x) = |x|, \quad -1 < x < 1$$



$f(x)$ is even $\Rightarrow$ $b_n = 0 \ \forall n$ $\Rightarrow$ $\delta_n = 0 \ \forall n$ and $c_n = |a_n| \ \forall n > 0$.

$$c_0 = |a_0| = \frac{1}{1}\int_{-1}^{1} |x|\, dx = 2\int_0^1 x\, dx = \left[x^2\right]_0^1 = 1$$

For $n > 0$,

$$a_n = \frac{1}{1}\int_{-1}^{1} |x| \cos(n\pi x)\, dx = 2\int_0^1 x\cos(n\pi x)\, dx$$

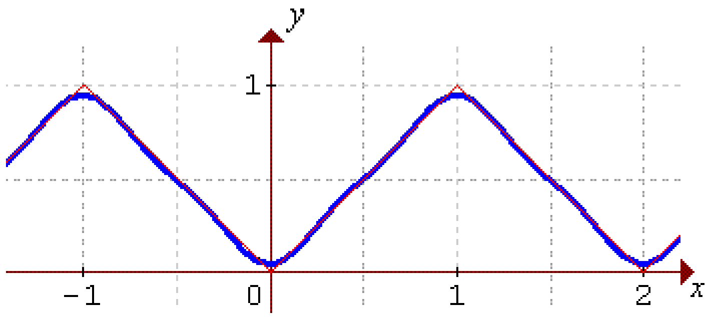
$$= 2\left[x\frac{\sin(n\pi x)}{n\pi} + \frac{\cos(n\pi x)}{(n\pi)^2}\right]_0^1 = \frac{2\left((-1)^n - 1\right)}{(n\pi)^2}$$

Therefore

$$f(x) = \frac{1}{2} - 2\sum_{n=1}^{\infty} \frac{1-(-1)^n}{(n\pi)^2}\cos(n\pi x)$$

$$= \frac{1}{2} - \frac{4}{\pi^2}\cos \pi x - \frac{4}{9\pi^2}\cos 3\pi x - \frac{4}{25\pi^2}\cos 5\pi x - \ldots$$

(which converges very rapidly, as this third partial sum demonstrates)



Example 7.04.2 (continued)

The harmonic amplitudes are

$$c_n = \left\{ \begin{array}{cc} \dfrac{1}{2} & (n=0) \\ \dfrac{2\left(1-(-1)^n\right)}{(n\pi)^2} & (n \in \mathbb{N}) \end{array} \right\} = \left\{ \begin{array}{cc} \dfrac{1}{2} & (n=0) \\ 0 & (n \text{ even}, n \geq 2) \\ \dfrac{4}{(n\pi)^2} & (n \text{ odd}) \end{array} \right.$$

The frequencies therefore diminish rapidly: