8. Partial Differential Equations

Partial differential equations (PDEs) are equations involving functions of more than one variable and their partial derivatives with respect to those variables.

Most (but not all) physical models in engineering that result in partial differential equations are of at most second order and are often linear. (Some problems such as elastic stresses and bending moments of a beam can be of fourth order). In this course we shall have time to look at only a small subset of second order linear partial differential equations.

Sections in this Chapter:

8.01 Major Classifications of Common PDEs

A general second order linear partial differential equation in two Cartesian variables can be written as

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x\,\partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)$$

Three main types arise, based on the value of $D = B^2 - 4AC$ (a discriminant):
Hyperbolic, wherever (x, y) is such that $D > 0$;
Parabolic, wherever (x, y) is such that $D = 0$;
Elliptic, wherever (x, y) is such that $D < 0$.

Among the most important partial differential equations in engineering are:

The **wave equation**: $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\nabla^2 u$

or its one-dimensional special case $\dfrac{\partial^2 u}{\partial t^2} = c^2\,\dfrac{\partial^2 u}{\partial x^2}$ [which is hyperbolic everywhere]

(where u is the displacement and c is the speed of the wave);

The **heat** (or **diffusion**) **equation**: $\mu\rho\dfrac{\partial u}{\partial t} = K\,\nabla^2 u + \vec{\nabla}K\bullet\vec{\nabla}u$

a one-dimensional special case of which is

$$\frac{\partial u}{\partial t} = \frac{K}{\mu\rho}\frac{\partial^2 u}{\partial x^2} \qquad \text{[which is parabolic everywhere]}$$

(where u is the temperature, μ is the specific heat of the medium, ρ is the density and K is the thermal conductivity);

The potential (or **Laplace's**) **equation**: $\nabla^2 u = 0$

a special case of which is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ [which is elliptic everywhere]

The complete solution of a PDE requires additional information, in the form of initial conditions (values of the dependent variable and its first partial derivatives at $t = 0$), boundary conditions (values of the dependent variable on the boundary of the domain) or some combination of these conditions.

8.02 The Wave Equation – Vibrating Finite String

The wave equation is

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u}$$

If $u(x, t)$ is the vertical displacement of a point at location x on a vibrating string at time t, then the governing PDE is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

If $u(x, y, t)$ is the vertical displacement of a point at location (x, y) on a vibrating membrane at time t, then the governing PDE is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

or, in plane polar coordinates (r, θ), (appropriate for a circular drum),

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)$$

For spherically symmetric waves in $\mathbb{R}^3$, the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \frac{c^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r} \right) = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} \right)$$

The latter two cases lead to solutions involving Bessel functions.

<u>Example 8.02.1</u>

An elastic string of length L is fixed at both ends ($x = 0$ and $x = L$). The string is displaced into the form $y = f(x)$ and is released from rest. Find the displacement $y(x, t)$ at all locations on the string ($0 < x < L$) and at all subsequent times ($t > 0$).

The boundary value problem for the displacement function $y(x, t)$ is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 < x < L \quad \text{and} \quad t > 0$$

Both ends fixed for all time: $y(0, t) = y(L, t) = 0 \quad \text{for} \quad t \geq 0$

Initial configuration of string: $y(x, 0) = f(x) \quad \text{for} \quad 0 \leq x \leq L$

String released from rest: $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = 0 \qquad \text{for } 0 \leq x \leq L$

Example 8.02.1 (continued)

Separation of Variables (or Fourier Method)

Attempt a solution of the form $y(x, t) = X(x)\,T(t)$

Substitute $y(x, t) = X(x)\,T(t)$ into the PDE:

$$\frac{\partial^2}{\partial t^2}\big(X(x)T(t)\big) = c^2 \frac{\partial^2}{\partial x^2}\big(X(x)T(t)\big) \quad \Rightarrow \quad X\frac{d^2T}{dt^2} = c^2 \frac{d^2X}{dx^2}T$$

$$\Rightarrow \quad \frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$

The left hand side of this equation is a function of t only. At any instant t it must have the same value at all values of x. Therefore the right hand side, which is a function of x only, must at any one instant have that same value at all values of x.

By a similar argument, the right hand side of this equation is a function of x only. At any location x it must have the same value at all times t. Therefore the left hand side, which is a function of t only, must at any one location have that same value at all times t.

Thus both sides of this differential equation must be the same absolute constant, which we shall represent for now by $-k$.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k \quad \Rightarrow \quad \frac{d^2X}{dx^2} + kX = 0$$

The general solution of this simple second order ODE is a combination of two exponential functions of x if $k < 0$, is a linear function of x if $k = 0$ and is a combination of sine and cosine functions of x if $k > 0$.

It is not possible for both ends of the string to be fixed for all time in the first two cases (unless we admit the trivial solution $y(x, t) \equiv 0$, a string that never moves from its equilibrium position). Therefore $k > 0$. Replace k by λ^2 (guaranteed positive for all real λ except $\lambda = 0$).

We now have the pair of ODEs

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + \lambda^2 c^2 T = 0$$

The general solutions are

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad \text{and} \quad T(t) = C\cos(\lambda ct) + D\sin(\lambda ct)$$

respectively, where A, B, C and D are arbitrary constants.

Example 8.02.1 (continued)

Consider the boundary conditions:
$y(0,t) = X(0)T(t) = 0 \quad \forall t \geq 0$
For a non-trivial solution, this requires $X(0)=0 \quad \Rightarrow \quad A=0$.

$$y(L,t) = X(L)T(t) = 0 \quad \forall t \geq 0 \qquad \Rightarrow \quad X(L)=0$$

$$\Rightarrow \quad B\sin(\lambda L)=0 \quad \Rightarrow \quad \lambda_n = \frac{n\pi}{L}, \quad (n \in \mathbb{Z})$$

We now have a solution only for a discrete set of eigenvalues λ_n, with corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

Consider the initial condition:

$$\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = X(x)T'(0) = 0 \quad \forall x \quad \Rightarrow \quad T'(0) = 0$$

$$T'(t) = -C\lambda c\sin(\lambda ct) + D\lambda c\cos(\lambda ct) \quad \Rightarrow \quad T'(0) = D\lambda c = 0 \quad \Rightarrow \quad D = 0$$

Therefore our complete solution for $y(x, t)$ is now some linear combination of

$$y_n(x,t) = X_n(x)T_n(t) = C_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

There is one condition remaining to be satisfied.
The initial configuration of the string is: $y(x, 0) = f(x)$ for $0 \leq x \leq L$.

$$\Rightarrow \quad y(x,0) = \sum_{n=1}^{\infty} C_n\sin\left(\frac{n\pi x}{L}\right) = f(x)$$

This is precisely the Fourier sine series expansion of $f(x)$ on $[0, L]$!
From Fourier series theory (Chapter 7), the coefficients C_n are

$$C_n = \frac{2}{L}\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du$$

Therefore our complete solution is

$$\boxed{y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)}$$

Example 8.02.1 (continued)

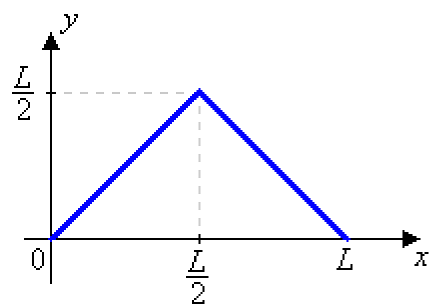
This solution is valid for any initial displacement function $f(x)$ that is continuous with a piece-wise continuous derivative on $[0, L]$ with $f(0) = f(L) = 0$.

If the initial displacement is itself sinusoidal $\left(f(x) = a \sin\left(\frac{n\pi x}{L}\right) \text{ for some } n \in \mathbb{N} \right)$, then the complete solution is a single term from the infinite series,

$$y(x,t) = a \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi c t}{L}\right).$$

Example 8.02.2

Suppose that the initial configuration is triangular:



$$y(x,0) = f(x) = \begin{cases} x & \left(0 \le x \le \frac{1}{2}L\right) \\ L - x & \left(\frac{1}{2}L < x \le L\right) \end{cases}$$

Then the Fourier sine coefficients are

$$C_n = \frac{2}{L}\int_0^L f(u) \sin\left(\frac{n\pi u}{L}\right) du$$

$$= \frac{2}{L}\int_0^{L/2} u \sin\left(\frac{n\pi u}{L}\right) du + \frac{2}{L}\int_{L/2}^{L} (L-u) \sin\left(\frac{n\pi u}{L}\right) du$$

D		I	D		I
u		$\sin\frac{n\pi u}{L}$	$L-u$		$\sin\frac{n\pi u}{L}$
	$+$			$+$	
1		$-\frac{L}{n\pi}\cos\frac{n\pi u}{L}$	-1		$-\frac{L}{n\pi}\cos\frac{n\pi u}{L}$
	$-$			$-$	
0		$-\left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi u}{L}$	0		$-\left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi u}{L}$

$$= \frac{2}{L}\cdot\left(\frac{L}{n\pi}\right)^2 \left\{ \left[-\left(\frac{n\pi u}{L}\right)\cos\left(\frac{n\pi u}{L}\right) + \sin\left(\frac{n\pi u}{L}\right) \right]_0^{L/2} \right.$$

$$\left. + \left[-\left(\frac{n\pi(L-u)}{L}\right)\cos\left(\frac{n\pi u}{L}\right) - \sin\left(\frac{n\pi u}{L}\right) \right]_{L/2}^{L} \right\}$$

Example 8.02.2 (continued)

$$= \frac{2L}{(n\pi)^2}\left\{\left(-\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right)\right) - (0-0)\right.$$

$$\left. + (-0+0) - \left(-\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right)\right\}$$

$$= \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \qquad \text{But} \quad \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & (n \text{ even}) \\ \pm 1 & (n \text{ odd}) \end{cases}$$

$$\sin\left(\frac{(2k-1)\pi}{2}\right) = (-1)^{k+1}, \quad (k \in \mathbb{N})$$

Therefore sum over the odd integer values of n only ($n = 2k - 1$).

$$C_k = \frac{4L}{\left((2k-1)\pi\right)^2}(-1)^{k+1}$$

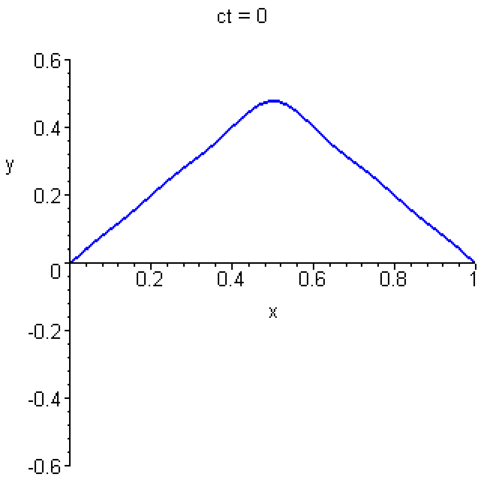
and

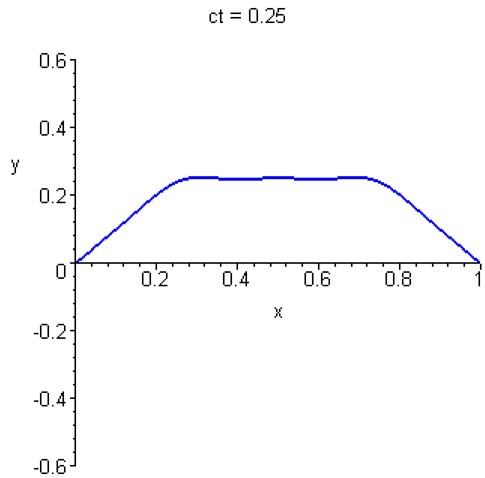
$$\boxed{y(x,t) = \frac{4L}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin\left(\frac{(2k-1)\pi x}{L}\right)\cos\left(\frac{(2k-1)\pi c t}{L}\right)}$$

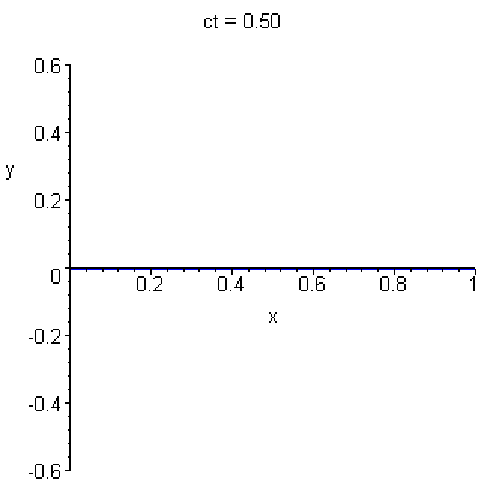
See the web page "www.engr.mun.ca/~ggeorge/9420/demos/ex8022.html" for an animation of this solution.

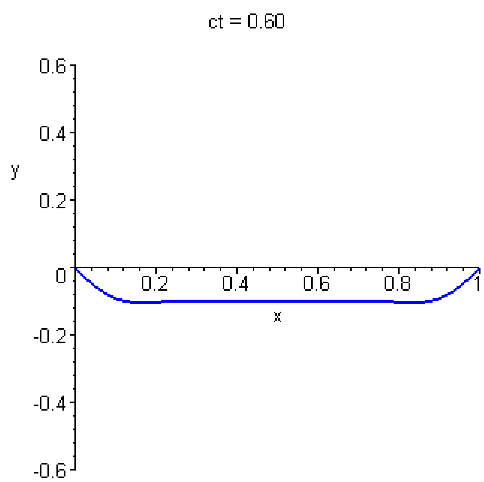
Example 8.02.2 (continued)

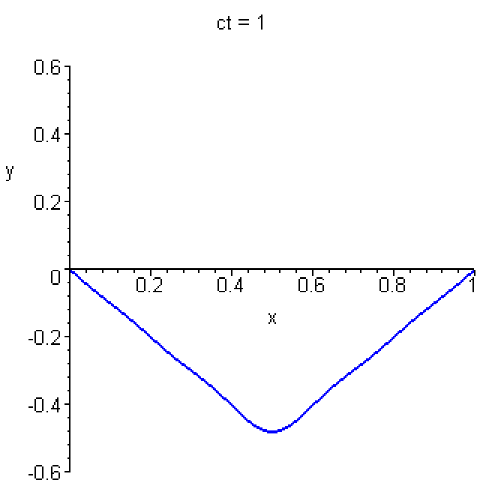
Some snapshots of the solution are shown here:











These graphs were generated from the Fourier series, truncated after the fifth non-zero term.