Example 8.02.3

An elastic string of length L is fixed at both ends ($x = 0$ and $x = L$). The string is initially in its equilibrium state [$y(x, 0) = 0$ for all x] and is released with the initial velocity $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$. Find the displacement $y(x, t)$ at all locations on the string ($0 < x < L$) and at all subsequent times ($t > 0$).

The boundary value problem for the displacement function $y(x, t)$ is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } 0 < x < L \quad \text{and} \quad t > 0$$

Both ends fixed for all time: $y(0, t) = y(L, t) = 0$ for $t \geq 0$

Initial configuration of string: $y(x, 0) = 0$ for $0 \leq x \leq L$

String released with initial velocity: $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = g(x) \qquad \text{for } 0 \leq x \leq L$

As before, attempt a solution by the method of the **separation of variables**.

Substitute $y(x, t) = X(x)\, T(t)$ into the PDE:

$$\frac{\partial^2}{\partial t^2}\big(X(x)T(t)\big) = c^2 \frac{\partial^2}{\partial x^2}\big(X(x)T(t)\big) \qquad \Rightarrow \quad X \frac{d^2T}{dt^2} = c^2 \frac{d^2X}{dx^2} T$$

Again, each side must be a negative constant.

$$\Rightarrow \quad \frac{1}{c^2 T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

We now have the pair of ODEs

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + \lambda^2 c^2 T = 0$$

The general solutions are

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x) \quad \text{and} \quad T(t) = C\cos(\lambda ct) + D\sin(\lambda ct)$$

respectively, where A, B, C and D are arbitrary constants.

Example 8.02.3 (continued)

Consider the boundary conditions:
$y(0,t) = X(0)T(t) = 0 \quad \forall t \geq 0$
For a non-trivial solution, this requires $X(0)=0 \quad \Rightarrow \quad A=0$.

$y(L,t) = X(L)T(t) = 0 \quad \forall t \geq 0 \qquad \Rightarrow \quad X(L)=0$
$\Rightarrow \quad B\sin(\lambda L)=0 \quad \Rightarrow \quad \lambda_n = \dfrac{n\pi}{L}, \quad (n \in \mathbb{Z})$

We now have a solution only for a discrete set of eigenvalues λ_n, with corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad (n = 1, 2, 3, \ldots)$$

and

$$y_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{L}\right)T_n(t), \quad (n = 1, 2, 3, \ldots)$$

So far, the solution has been identical to Example 8.02.1.

Consider the initial condition $y(x, 0) = 0$:

$$y(x,0) = 0 \quad \Rightarrow \quad X(x)T(0) = 0 \quad \forall x \quad \Rightarrow \quad T(0) = 0$$

The initial value problem for $T(t)$ is now

$$T'' + \lambda^2c^2T = 0, \quad T(0)=0, \qquad \text{where} \quad \lambda = \frac{n\pi}{L}$$

the solution to which is

$$T_n(t) = C_n\sin\left(\frac{n\pi c t}{L}\right), \qquad (n \in \mathbb{N})$$

Our eigenfunctions for y are now

$$y_n(x,t) = X_n(x)T_n(t) = C_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right), \qquad (n \in \mathbb{N})$$

Example 8.02.3 (continued)

Differentiate term by term and impose the initial velocity condition:

$$\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = \sum_{n=1}^{\infty} C_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = g(x)$$

which is just the Fourier sine series expansion for the function $g(x)$.
The coefficients of the expansion are

$$C_n \frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(u) \sin\left(\frac{n\pi u}{L}\right) du$$

which leads to the complete solution

$$y(x,t) = \frac{2}{\pi c}\sum_{n=1}^{\infty} \frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right) du\right) \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi c t}{L}\right)$$

This solution is valid for any initial velocity function $g(x)$ that is continuous with a piece-wise continuous derivative on $[0, L]$ with $g(0) = g(L) = 0$.

The solutions for Examples 8.02.1 and 8.02.3 may be superposed.

Let $y_1(x,t)$ be the solution for initial displacement $f(x)$ and zero initial velocity.
Let $y_2(x,t)$ be the solution for zero initial displacement and initial velocity $g(x)$.
Then $y(x,t) = y_1(x,t) + y_2(x,t)$ satisfies the wave equation
(the sum of any two solutions of a linear homogeneous PDE is also a solution),
and satisfies the boundary conditions $y(0, t) = y(L, t) = 0$:

$y(x,0) = y_1(x,0) + y_2(x,0) = f(x) + 0$,
which satisfies the condition for initial displacement $f(x)$.
$y_t(x,0) = y_{1t}(x,0) + y_{2t}(x,0) = 0 + g(x)$,
which satisfies the condition for initial velocity $g(x)$.

Therefore the sum of the two solutions is the complete solution for initial displacement $f(x)$ **and** initial velocity $g(x)$:

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right) du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$
$$+ \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right) du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

Example 8.02.4

An elastic string of length 1 m is fixed at both ends (x = 0 and x = 1). The string is initially in the shape of an arc of a parabola [$y(x, 0) = x - x^2$ for $0 \le x \le 1$] and is released with the initial velocity $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = x - x^2 \quad (0 \le x \le 1)$. It is known that the wave speed is $c = 5\ \text{m s}^{-1}$. Find the displacement $y(x, t)$ at all locations on the string ($0 < x < 1$) and at all subsequent times ($t > 0$).

In the formula for the complete solution of the wave equation,

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right)$$
$$+ \frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\left(\int_0^L g(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c t}{L}\right)$$

we know that $L = 1$, $c = 5$ and $f(x) = g(x) = x - x^2$ for $0 \le x \le 1$.

Both integrals inside the summations are the same:

$$\int_0^1 \left(u - u^2\right)\sin\left(\frac{n\pi u}{1}\right)du =$$

D		I
$u - u^2$		$\sin n\pi u$
	+	
$1 - 2u$		$-\frac{\cos n\pi u}{n\pi}$
	−	
-2		$-\frac{\sin n\pi u}{(n\pi)^2}$
	+	
0		$\frac{\cos n\pi u}{(n\pi)^3}$

$$\left[\left(\frac{(u-1)u}{n\pi} - \frac{2}{(n\pi)^3}\right)\cos n\pi u + \frac{1-2u}{(n\pi)^2}\sin n\pi u\right]_0^1$$

$$= \left(\left(0 - \frac{2}{(n\pi)^3}\right)(-1)^n + 0\right) - \left(\left(0 - \frac{2}{(n\pi)^3}\right) + 0\right)$$

$$= \frac{2}{(n\pi)^3}\left(1 - (-1)^n\right) = \begin{cases} 0 & (n \text{ even}) \\ \dfrac{4}{(n\pi)^3} & (n \text{ odd}) \end{cases}$$

Let (odd n) = $2k - 1$

The complete solution is

$$y(x,t) = \frac{8}{\pi^3}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^3}\sin(2k-1)\pi x\left(\cos 5(2k-1)\pi t + \frac{\sin 5(2k-1)\pi t}{5(2k-1)\pi}\right)$$

A Maple file for this solution is available at "www.engr.mun.ca/~ggeorge/9420/demos/ex8024.mws".

8.03 The Wave Equation – Vibrating Infinite String

Example 8.03.1

An elastic string of infinite length is displaced into the form $y = f(x)$ and is released from rest. Find the displacement $y(x, t)$ at all locations on the string $x \in \mathbb{R}$ and at all subsequent times $(t > 0)$.

The boundary value problem for the displacement function $y(x, t)$ is:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } -\infty < x < \infty \quad \text{and} \quad t > 0$$

Initial configuration of string: $\qquad y(x, 0) = f(x) \quad \text{for} \quad x \in \mathbb{R}$

String released from rest: $\qquad \left.\frac{\partial y}{\partial t}\right|_{(x,0)} = 0 \qquad \text{for} \quad x \in \mathbb{R}$

We no longer have the additional boundary conditions of fixed endpoints.
However, it is reasonable to insist upon a bounded solution.

Separation of Variables (or Fourier Method)

Attempt a solution of the form $y(x, t) = X(x)\,T(t)$
Again we find the linked pair of ordinary differential equations

$$X'' + \omega^2 X = 0 \quad \text{and} \quad T'' + \omega^2 c^2 T = 0$$

If $\omega = 0$ then $X(x) = ax + b$. However, for a bounded solution, we require $a = 0$.
For other values of ω, $X(x) = a \cos \omega x + b \sin \omega x$, which is bounded for all x and all ω.
The $\omega = 0$ case is a special case of this solution.

We have a continuum of eigenvalues ω with corresponding eigenfunctions

$$X_\omega(x) = a_\omega \cos(\omega x) + b_\omega \sin(\omega x)$$

It then follows that $\qquad T_\omega(t) = c_\omega \cos \omega ct + d_\omega \sin \omega ct$

Imposing the initial condition of zero velocity,

$$\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = X(x)T'(0) = X(x)d_\omega \omega c = 0 \qquad \forall x \in \mathbb{R} \qquad \Rightarrow \quad d_\omega = 0$$

Example 8.03.1 (continued)

Therefore we have, for any real ω, a solution of the wave equation and the initial velocity condition,

$$y_\omega(x,t) = X_\omega(x)T_\omega(t) = \left(a_\omega \cos(\omega x) + b_\omega \sin(\omega x)\right)\cos(\omega c t)$$

[where c_ω has been absorbed into the other arbitrary constants a_ω and b_ω.]

The superposition of solutions now leads to an integral, not a discrete sum.

$$y(x,t) = \int_0^\infty y_\omega(x,t)\,d\omega = \int_0^\infty \left(a_\omega \cos(\omega x) + b_\omega \sin(\omega x)\right)\cos(\omega c t)\,d\omega$$

Imposing the remaining condition,

$$y(x,0) = \int_0^\infty \left(a_\omega \cos(\omega x) + b_\omega \sin(\omega x)\right)d\omega = f(x)$$

But this is just the Fourier integral representation of $f(x)$ on $(-\infty, \infty)$.
Therefore a_ω and b_ω are just the Fourier integral coefficients

$$a_\omega = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos(\omega u)\,du \quad \text{and} \quad b_\omega = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin(\omega u)\,du$$

The complete solution is

$$y(x,t) = \int_0^\infty \left[\left(\frac{1}{\pi}\int_{-\infty}^\infty f(u)\cos(\omega u)\,du\right)\cos(\omega x) + \left(\frac{1}{\pi}\int_{-\infty}^\infty f(u)\sin(\omega u)\,du\right)\sin(\omega x)\right]\cos(\omega c t)\,d\omega$$

which, after re-iteration (interchanging the order of integration) is

$$y(x,t) = \frac{1}{\pi}\int_{-\infty}^\infty \int_0^\infty \left(\cos(\omega u)\cos(\omega x) + \sin(\omega u)\sin(\omega x)\right) f(u)\cos(\omega c t)\,d\omega\,du$$

$$\Rightarrow \quad y(x,t) = \frac{1}{\pi}\int_{-\infty}^\infty f(u)\int_0^\infty \cos\left(\omega(u-x)\right)\cos(\omega c t)\,d\omega\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(u)\int_0^\infty \left(\cos\left(\omega(u-x+ct)\right) + \cos\left(\omega(u-x-ct)\right)\right)d\omega\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^\infty f(u)\left[\frac{\sin\left(\omega(u-x+ct)\right)}{u-x+ct} + \frac{\sin\left(\omega(u-x-ct)\right)}{u-x-ct}\right]_{\omega=0}^{\omega=\infty} du$$

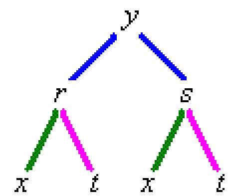
8.04 d'Alembert Solution

One form of the solution to Example 8.03.1,

$$y(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(u)\left[\frac{\sin\big(\omega(u-x+ct)\big)}{u-x+ct} + \frac{\sin\big(\omega(u-x-ct)\big)}{u-x-ct}\right]_{\omega=0}^{\omega=\infty} du$$

suggests that, in general, one might seek solutions to the wave equation of the form

$$y(x,t) = \frac{f(x+ct) + f(x-ct)}{2}$$



Let $r = x + ct$ and $s = x - ct$, then $y(r,s) = \dfrac{f(r) + f(s)}{2}$ and

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial x} = \frac{1}{2}\Big(\big(f'(r)+0\big)\times 1 + \big(0+f'(s)\big)\times 1\Big),$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial r}\left(\frac{\partial y}{\partial x}\right)\frac{\partial r}{\partial x} + \frac{\partial}{\partial s}\left(\frac{\partial y}{\partial x}\right)\frac{\partial s}{\partial x} = \frac{1}{2}\big(f''(r)\times 1 + f''(s)\times 1\big),$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial y}{\partial s}\frac{\partial s}{\partial t} = \frac{1}{2}\Big(\big(f'(r)+0\big)\times c + \big(0+f'(s)\big)\times(-c)\Big),$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial r}\left(\frac{\partial y}{\partial t}\right)\frac{\partial r}{\partial t} + \frac{\partial}{\partial s}\left(\frac{\partial y}{\partial t}\right)\frac{\partial s}{\partial t} = \frac{1}{2}\big(c\,f''(r)\times c - c\,f''(s)\times(-c)\big),$$

$$\Rightarrow \quad \frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{2}\big(f''(r) + f''(s)\big) - \frac{1}{2c^2}\big(c^2 f''(r) + c^2 f''(s)\big) = 0,$$

Therefore $y(x,t) = \dfrac{f(x+ct) + f(x-ct)}{2}$ is a solution to the wave equation for all twice differentiable functions $f(u)$. This is part of the d'Alembert solution.

This d'Alembert solution satisfies the initial displacement condition:

$$y(x,0) = \frac{f(x+0) + f(x-0)}{2} = f(x)$$

Also $\left.\dfrac{\partial}{\partial t}y(x,t)\right|_{t=0} = \left.\dfrac{c\,f'(x+ct) - c\,f'(x-ct)}{2}\right|_{t=0} = \dfrac{c\,f'(x) - c\,f'(x)}{2} = 0$

The d'Alembert solution therefore satisfies both initial conditions.

A more general d'Alembert solution to the wave equation for an infinitely long string is

$$y(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$$

This satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad \text{for } -\infty < x < \infty \quad \text{and} \quad t > 0$$

and

Initial configuration of string: $y(x, 0) = f(x)$ for $x \in \mathbb{R}$

and

Initial speed of string: $\left.\frac{\partial y}{\partial t}\right|_{(x,0)} = g(x)$ for $x \in \mathbb{R}$

for any twice differentiable functions $f(x)$ and $g(x)$.

Physically, this represents two identical waves, moving with speed c in opposite directions along the string.

<u>Proof</u> that $y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du$ satisfies both initial conditions:

$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du \quad \Rightarrow \quad y(x,0) = \frac{1}{2c}\int_{x}^{x} g(u)\,du = 0$$

Using a Leibnitz differentiation of the integral:

$$\frac{\partial y}{\partial t} = \frac{1}{2c}\left(g(x+ct)\cdot\frac{\partial}{\partial t}(x+ct) - g(x-ct)\cdot\frac{\partial}{\partial t}(x-ct) + \int_{x-ct}^{x+ct} \frac{\partial}{\partial t} g(u)\,du \right)$$

$$= \frac{1}{2c}\left(c\,g(x+ct) + c\,g(x-ct) + 0\right) = \frac{g(x+ct) + g(x-ct)}{2}$$

$$\Rightarrow \quad \left.\frac{\partial y}{\partial t}\right|_{t=0} = \frac{g(x+0) + g(x-0)}{2} = g(x)$$

Example 8.04.1

An elastic string of infinite length is displaced into the form $y = \cos \pi x/2$ on [–1, 1] only (and $y = 0$ elsewhere) and is released from rest. Find the displacement $y(x, t)$ at all locations on the string $x \in \mathbb{R}$ and at all subsequent times ($t > 0$).

For this solution to the wave equation we have initial conditions

$$y(x,0) = f(x) = \begin{cases} \cos\left(\dfrac{\pi x}{2}\right) & (-1 \le x \le 1) \\ 0 & (\text{otherwise}) \end{cases}$$

and

$$\frac{\partial y}{\partial t}(x,0) = g(x) = 0$$

The d'Alembert solution is

$$y(x,t) = \frac{f(x+ct) + f(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\,du = \frac{f(x+ct) + f(x-ct)}{2} + 0$$

where $f(x+ct) = \begin{cases} \cos\left(\dfrac{\pi(x+ct)}{2}\right) & (-1-ct \le x \le 1-ct) \\ 0 & (\text{otherwise}) \end{cases}$

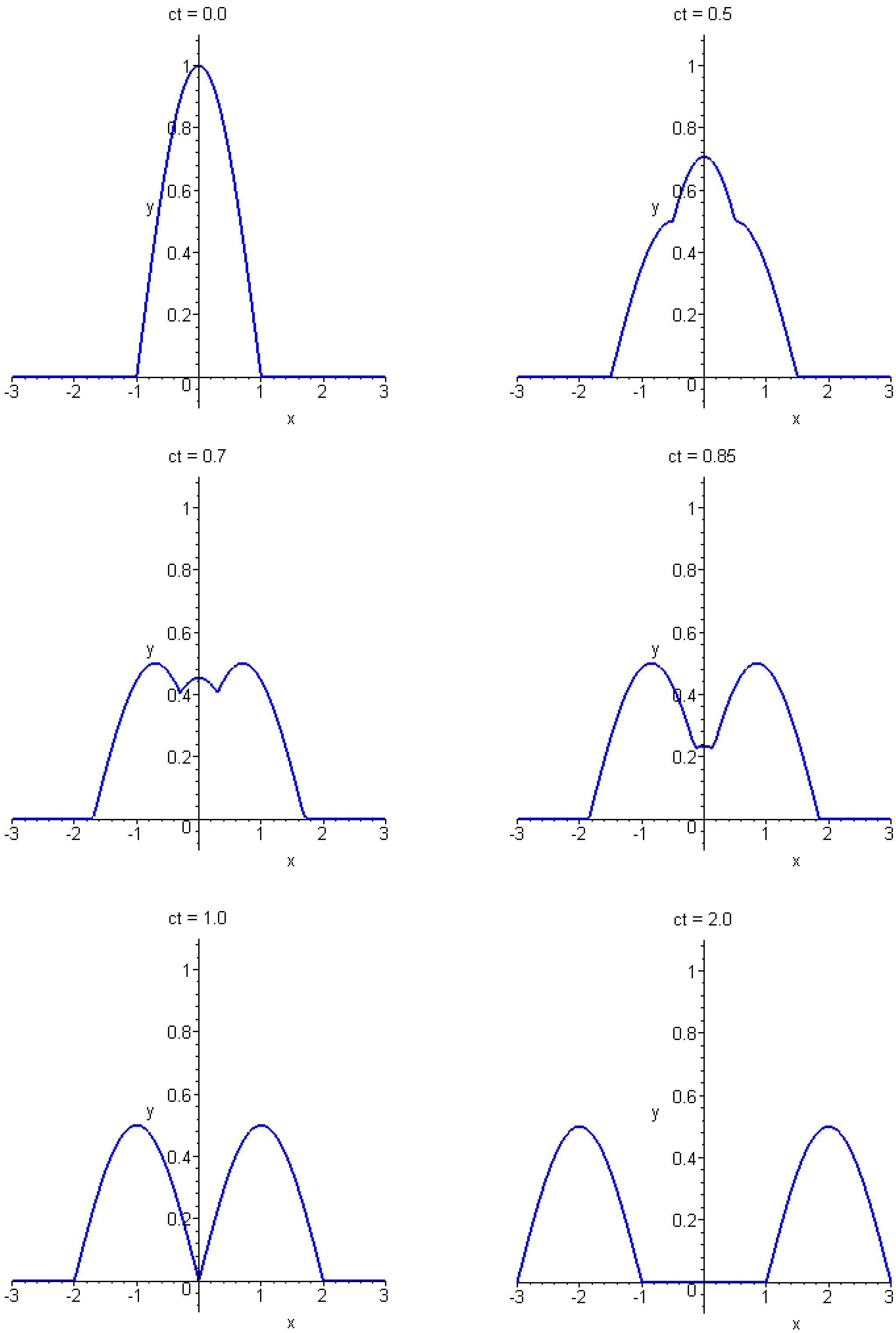
and $f(x-ct) = \begin{cases} \cos\left(\dfrac{\pi(x-ct)}{2}\right) & (-1+ct \le x \le 1+ct) \\ 0 & (\text{otherwise}) \end{cases}$

We therefore obtain two waves, each of the form of a single half-period of a cosine function, moving apart from a superposed state at $x = 0$ at speed c in opposite directions.

See the web page "www.engr.mun.ca/~ggeorge/9420/demos/ex8041.html" for an animation of this solution.

Example 8.04.1 (continued)

Some snapshots of the solution are shown here:



A more general case of a d'Alembert solution arises for the homogeneous PDE with constant coefficients

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

The characteristic (or auxiliary) equation for this PDE is

$$A\lambda^2 + B\lambda + C = 0$$

This leads to the complementary function (which is also the general solution for this homogeneous PDE)

$$\boxed{u(x, y) = f_1(y+\lambda_1 x) + f_2(y+\lambda_2 x),}$$

where

$$\boxed{\lambda_1 = \frac{-B-\sqrt{D}}{2A} \quad \text{and} \quad \lambda_2 = \frac{-B+\sqrt{D}}{2A}}$$

and $D = B^2 - 4AC$

and f_1, f_2 are **arbitrary** twice-differentiable functions of their arguments.

λ_1 and λ_2 are the roots (or eigenvalues) of the characteristic equation.

In the event of equal roots, the solution changes to

$$u(x,y) = f_1(y+\lambda x) + h(x,y)\,f_2(y+\lambda x)$$

where $h(x, y)$ is any non-trivial linear function of x and/or y (except $y + \lambda x$).

The wave equation is a special case with $y = t$, $A = 1$, $B = 0$, $C = -1/c^2$ and $\lambda = \pm 1/c$.

Example 8.04.2

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x, 0) = -x^2$$

$$u_y(x, 0) = 0$$

(a) Classify the partial differential equation.

(b) Find the value of u at $(x, y) = (0, 1)$.

(a) Compare this PDE to the standard form

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

$$A = 1\,, \quad B = -3\,, \quad C = 2 \quad \Rightarrow \quad D = 9 - 4\times 2 = 1 > 0$$

Therefore the PDE is **hyperbolic** everywhere.

(b) $$\lambda = \frac{+3 \pm \sqrt{1}}{2} = 1 \text{ or } 2$$

$$\Rightarrow \quad u(x, y) = f\,(y + x) + g(y + 2x)$$