Example 8.04.2

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$
$$u(x, 0) = -x^2$$
$$u_y(x, 0) = 0$$

(a) Classify the partial differential equation.
(b) Find the value of $u$ at $(x, y) = (0, 1)$.

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(a) Compare this PDE to the standard form

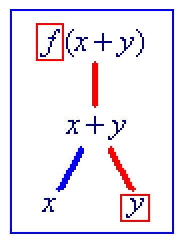
$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} = 0$$

$A = 1$, $B = -3$, $C = 2$ $\Rightarrow$ $D = 9 - 4\times 2 = 1 > 0$

Therefore the PDE is **hyperbolic** everywhere.

(b) $\lambda = \dfrac{+3 \pm \sqrt{1}}{2} = 1 \text{ or } 2$

Chain rule diff'n




$\Rightarrow \quad u(x, y) = f(y + x) + g(y + 2x)$

$\Rightarrow \quad u_y(x, y) = f'(y + x) + g'(y + 2x)$

Boundary conditions:
$u(x, 0) = f(x) + g(2x) = -x^2$ **(1)**
and
$u_y(x, 0) = f'(x) + g'(2x) = 0$ **(2)**

$\dfrac{\partial}{\partial x}(1) = f'(x) + 2g'(2x) = -2x$ **(3)**

**(3) – (2)** $\Rightarrow \quad g'(2x) = -2x \quad \Rightarrow \quad g'(x) = -x$

$\Rightarrow \quad g(x) = -\tfrac{1}{2}x^2 + k \qquad \Rightarrow \quad g(y+2x) = -\tfrac{1}{2}(y+2x)^2 + k$

Example 8.04.2 (continued)

Also **(1)** $\Rightarrow$ $f(x) = -x^2 - g(2x) = -x^2 + \frac{1}{2}(2x)^2 - k = x^2 - k$

$\Rightarrow$ $f(y + x) = (y + x)^2 - k$

Therefore $u(x, y) = f(y + x) + g(y + 2x)$
$= (y + x)^2 - k - (y + 2x)^2 / 2 + k$

$$= \frac{1}{2}\left(2y^2 + 4xy + 2x^2 - y^2 - 4xy - 4x^2\right) = \frac{1}{2}\left(y^2 - 2x^2\right)$$

The complete solution is therefore $u(x, y) = \frac{1}{2}\left(y^2 - 2x^2\right)$

$$\Rightarrow \quad u(0,1) = \frac{1}{2}\left(1^2 - 0^2\right) = \mathbf{\frac{1}{2}}$$

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[It is easy (though tedious) to confirm that $u(x, y) = \frac{1}{2}\left(y^2 - 2x^2\right)$ satisfies the partial differential equation $\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x\,\partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$ together with both initial conditions $u(x, 0) = -x^2$ and $u_y(x, 0) = 0$.]

[Also note that the arbitrary constants of integration for $f$ and $g$ cancelled each other out. This cancellation happens generally for this d'Alembert method of solution.]

Example 8.04.3

Find the complete solution to

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 14,$$

$$u(x, 0) = 2x + 1,$$

$$u_y(x, 0) = 4 - 6x.$$

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This PDE is non-homogeneous.

For the particular solution, we require a function such that the combination of second partial derivatives resolves to the constant 14. It is reasonable to try a quadratic function of $x$ and $y$ as our particular solution.

Try $\quad u_{\text{P}} = ax^2 + bxy + cy^2$

$$\Rightarrow \quad \frac{\partial u_{\text{P}}}{\partial x} = 2ax + by \quad \text{and} \quad \frac{\partial u_{\text{P}}}{\partial y} = bx + 2cy$$

$$\Rightarrow \quad \frac{\partial^2 u_{\text{P}}}{\partial x^2} = 2a, \quad \frac{\partial^2 u_{\text{P}}}{\partial x\,\partial y} = b \quad \text{and} \quad \frac{\partial^2 u_{\text{P}}}{\partial y^2} = 2c$$

$$\Rightarrow \quad 6\frac{\partial^2 u_{\text{P}}}{\partial x^2} - 5\frac{\partial^2 u_{\text{P}}}{\partial x\,\partial y} + \frac{\partial^2 u_{\text{P}}}{\partial y^2} = 12a - 5b + 2c = 14$$

We have one condition on three constants, two of which are therefore a free choice.
Choose $b = 0$ and $c = a$, then $14a = 14 \Rightarrow c = a = 1$
Therefore a particular solution is $u_{\text{P}} = x^2 + y^2$
[But we could have chosen, for example, $a = b = 0$ and $c = 7$ instead $\rightarrow u_{\text{P}} = 7y^2$ ]

Complementary function:

$A = 6, \quad B = -5, \quad C = 1 \quad \Rightarrow \quad D = 25 - 4\times 6 = 1 > 0$

Therefore the PDE is **hyperbolic** everywhere.

$$\lambda = \frac{+5 \pm \sqrt{1}}{12} = \frac{1}{3} \text{ or } \frac{1}{2}$$

The complementary function is

$$u_{\text{C}}(x, y) = f\left(y + \tfrac{1}{3}x\right) + g\left(y + \tfrac{1}{2}x\right)$$

and the general solution is

$$u(x, y) = f\left(y + \tfrac{1}{3}x\right) + g\left(y + \tfrac{1}{2}x\right) + x^2 + y^2$$

Example 8.04.3 (continued)

$$u(x,y) = f\left(y+\tfrac{1}{3}x\right) + g\left(y+\tfrac{1}{2}x\right) + x^2+y^2$$

$$\Rightarrow \frac{\partial u}{\partial y} = f'\left(y+\tfrac{1}{3}x\right) + g'\left(y+\tfrac{1}{2}x\right) + 2y$$

Imposing the two boundary conditions:

$$u(x,0) = f\left(\tfrac{1}{3}x\right) + g\left(\tfrac{1}{2}x\right) + x^2 = 2x+1 \qquad \textbf{(1)}$$

and

$$u_y(x,0) = f'\left(\tfrac{1}{3}x\right) + g'\left(\tfrac{1}{2}x\right) + 0 = 4-6x \qquad \textbf{(2)}$$

$$\frac{\partial}{\partial x}\textbf{(1)} = \frac{1}{3}f'\left(\tfrac{1}{3}x\right) + \frac{1}{2}g'\left(\tfrac{1}{2}x\right) + 2x = 2 \qquad \textbf{(3)}$$

**(2) – 2×(3)** $\Rightarrow \ \dfrac{1}{3}f'\left(\tfrac{1}{3}x\right) - 4x = 4 - 6x - 4$

$$\Rightarrow f'\left(\tfrac{1}{3}x\right) = -6x = -18\left(\tfrac{1}{3}x\right) \qquad \Rightarrow f'(x) = -18x$$

$$\Rightarrow f(x) = -9x^2+k$$

**(1)** $\Rightarrow \ g\left(\tfrac{1}{2}x\right) = 2x+1-x^2 - f\left(\tfrac{1}{3}x\right) = 2x+1-x^2 + 9\left(\dfrac{x^2}{9}\right) - k$

$$\Rightarrow g\left(\tfrac{1}{2}x\right) = 4\left(\tfrac{1}{2}x\right) + 1 - k$$

$$\Rightarrow g(x) = 4x + 1 - k$$

But

$$u(x,y) = f\left(y+\tfrac{1}{3}x\right) + g\left(y+\tfrac{1}{2}x\right) + x^2+y^2$$

$$\Rightarrow u(x,y) = -9\left(y+\tfrac{1}{3}x\right)^2 + k + 4\left(y+\tfrac{1}{2}x\right) + 1 - k + x^2+y^2$$

[again the arbitrary constants cancel - they can be omitted safely.]

$$= -9y^2-6xy-x^2 + 4y + 2x + 1 + x^2+y^2$$

Therefore the complete solution is

$$\boxed{u(x,y) = 1 + 2x + 4y - 6xy - 8y^2}$$

Example 8.04.3 - Alternative Treatment of the Particular Solution

Find the complete solution to

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x\,\partial y} + \frac{\partial^2 u}{\partial y^2} = 14,$$

$$u(x, 0) = 2x + 1,$$

$$u_y(x, 0) = 4 - 6x.$$

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This PDE is non-homogeneous.

For the particular solution, we require a function such that the combination of second partial derivatives resolves to the constant 14. It is reasonable to try a quadratic function of $x$ and $y$ as our particular solution.

Try $\quad u_{\text{P}} = ax^2 + bxy + cy^2$

$$\Rightarrow \quad \frac{\partial u_{\text{P}}}{\partial x} = 2ax + by \quad \text{and} \quad \frac{\partial u_{\text{P}}}{\partial y} = bx + 2cy$$

$$\Rightarrow \quad \frac{\partial^2 u_{\text{P}}}{\partial x^2} = 2a, \quad \frac{\partial^2 u_{\text{P}}}{\partial x\,\partial y} = b \quad \text{and} \quad \frac{\partial^2 u_{\text{P}}}{\partial y^2} = 2c$$

$$\Rightarrow \quad 6\frac{\partial^2 u_{\text{P}}}{\partial x^2} - 5\frac{\partial^2 u_{\text{P}}}{\partial x\,\partial y} + \frac{\partial^2 u_{\text{P}}}{\partial y^2} = 12a - 5b + 2c = 14$$

We have one condition on three constants, two of which are therefore a free choice.

Let us leave the free choice unresolved for now.

Complementary function:

$$\lambda = \frac{+5 \pm \sqrt{1}}{12} = \frac{1}{3} \text{ or } \frac{1}{2}$$

The complementary function is

$$u_{\text{C}}(x, y) = f\left(y + \tfrac{1}{3}x\right) + g\left(y + \tfrac{1}{2}x\right)$$

and the general solution is

$$u(x, y) = f\left(y + \tfrac{1}{3}x\right) + g\left(y + \tfrac{1}{2}x\right) + ax^2 + bxy + cy^2$$

Example 8.04.3 (continued)

$$u(x,y) = f\left(y+\tfrac{1}{3}x\right) + g\left(y+\tfrac{1}{2}x\right) + ax^2 + bxy + cy^2$$
$$\Rightarrow \frac{\partial u}{\partial y} = f'\left(y+\tfrac{1}{3}x\right) + g'\left(y+\tfrac{1}{2}x\right) + bx + 2cy$$

Imposing the two boundary conditions:

$$u(x,0) = f\left(\tfrac{1}{3}x\right) + g\left(\tfrac{1}{2}x\right) + ax^2 + 0 + 0 = 2x+1 \qquad \textbf{(A)}$$

and

$$u_y(x,0) = f'\left(\tfrac{1}{3}x\right) + g'\left(\tfrac{1}{2}x\right) + bx + 0 = 4-6x \qquad \textbf{(B)}$$

$$\frac{d}{dx}(\mathbf{A}) = \frac{1}{3}f'\left(\tfrac{1}{3}x\right) + \frac{1}{2}g'\left(\tfrac{1}{2}x\right) + 2ax = 2 \qquad \textbf{(C)}$$

**(B) – 2×(C)** $\Rightarrow \quad \frac{1}{3}f'\left(\tfrac{1}{3}x\right) + (b-4a)x = 4 - 6x - 4$

$$\Rightarrow f'\left(\tfrac{1}{3}x\right) = 3(4a-b-6)x = 9(4a-b-6)\left(\tfrac{1}{3}x\right) \qquad \Rightarrow f'(x) = 9(4a-b-6)x$$

$$\Rightarrow f(x) = \frac{9(4a-b-6)}{2}x^2 + k$$

**(A)** $\Rightarrow \quad g\left(\tfrac{1}{2}x\right) = 2x+1-ax^2 - f\left(\tfrac{1}{3}x\right) = 2x+1-ax^2 - \frac{\not{9}(4a-b-6)}{2}\left(\frac{x^2}{\not{9}}\right) - k$

$$\Rightarrow g(x) = 2(2x) + 1 - \left(\frac{2a+4a-b-6}{2}\right)(2x)^2 - k = 2(b+6-6a)x^2 + 4x + 1 - k$$

But

$$u(x,y) = f\left(y+\tfrac{1}{3}x\right) + g\left(y+\tfrac{1}{2}x\right) + ax^2 + bxy + cy^2$$

$$\Rightarrow u(x,y) = \frac{9(4a-b-6)}{2}\left(y+\tfrac{1}{3}x\right)^2 + k$$
$$+ 2(b+6-6a)\left(y+\tfrac{1}{2}x\right)^2 + 4\left(y+\tfrac{1}{2}x\right) + 1 - k + ax^2 + bxy + cy^2$$

[again the arbitrary constants cancel - they can be omitted safely.]

$$= \frac{9(4a-b-6)}{2}\left(y^2 + \tfrac{2}{3}xy + \tfrac{1}{9}x^2\right) + 2(b+6-6a)\left(y^2 + xy + \tfrac{1}{4}x^2\right)$$
$$+ 4y + 2x + 1 + ax^2 + bxy + cy^2$$

Example 8.04.3 (continued)

$$= \left(\frac{9(4a-b-6)+4(b+6-6a)+2c}{2}\right)y^2 + \left(3(4a-b-6)+2(b+6-6a)+b\right)xy$$

$$+ \left(\frac{(4a-b-6)+(b+6-6a)+2a}{2}\right)x^2 + 4y+2x+1$$

$$= \left(\frac{12a-5b+2c-30}{2}\right)y^2 - 6xy + 0x^2 + 4y+2x+1$$

But $12a-5b+2c = 14 \;\Rightarrow\; \dfrac{12a-5b+2c-30}{2} = \dfrac{14-30}{2} = -8$

Therefore the complete solution is

$$\boxed{u(x,y) = 1 + 2x + 4y - 6xy - 8y^2}$$

and note how the values of the two free parameters have no effect whatsoever on the solution.

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In practice, assign values to the free parameters in the particular solution only after one of the two arbitrary functions from the complementary function has been determined - if possible, make that function zero. In the example above, when we found

$f'(x) = 9(4a-b-6)x$,

choose $a=0$ and $b=-6 \;\Rightarrow\; f'(x)=0$

and, from $12a-5b+2c = 14$, we also have $c=-8$.

**(A)** $\Rightarrow\; g\left(\tfrac{1}{2}x\right) = 2x+1-ax^2 - f\left(\tfrac{1}{3}x\right) = 2x+1-0-0 \;\Rightarrow\; g(x) = 4x+1$

The general solution then becomes

$$u(x,y) = f\left(y+\tfrac{1}{3}x\right) + g\left(y+\tfrac{1}{2}x\right) + ax^2+bxy+cy^2$$

$$u(x,y) = 0 + 4\left(y+\tfrac{1}{2}x\right)+1 + 6xy-8y^2 \;\Rightarrow$$

$$\boxed{u(x,y) = 1 + 2x + 4y - 6xy - 8y^2}$$

as before, but much faster!