# ENGI 9420 Engineering Analysis
# Assignment 2 Solutions

2012 Fall
[Second order ODEs, Laplace transforms; Sections 1.01-1.09]

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1. Use Laplace transforms to solve the initial value problem [10]

$$\frac{dy}{dx} + 2y = 2\,, \qquad y(0) = 4$$

[This was Question 1 on Assignment 1]

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Let $Y(s) = \mathcal{L}\{y(x)\}$, then the Laplace transform of the initial value problem is

$$(sY - 4) + 2Y = \frac{2}{s} \quad \Rightarrow \quad (s+2)Y = \frac{2}{s} + 4 = \frac{2+4s}{s}$$

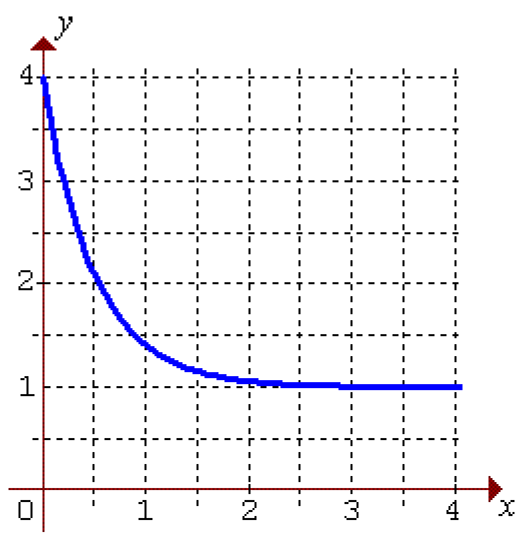
$$\Rightarrow \quad Y = \frac{4s+2}{s(s+2)} = \frac{a}{s} + \frac{b}{s+2}$$

By the cover-up rule,

$$a = \frac{0+2}{\cancel{0}(0+2)} = 1 \quad \text{and} \quad b = \frac{-8+2}{-2\cancel{(-2+2)}} = 3$$

$$\Rightarrow \quad Y = \frac{1}{s} + \frac{3}{s+2} = \mathcal{L}\{1\} + 3\mathcal{L}\left\{e^{-2x}\right\} \quad \Rightarrow$$

$$\boxed{y(x) = 1 + 3e^{-2x}}$$

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2. Find the complete solution of the initial value problem

$$\frac{d^2 y}{dx^2} + 7\frac{dy}{dx} + 10y = 6e^{-2x}, \qquad y(0)=1,\ y'(0)=-3$$

(a) without using Laplace transforms; and [8]
(b) using Laplace transforms. [8]

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(a) A.E.: $\lambda^2 + 7\lambda + 10 = 0 \;\Rightarrow\; (\lambda+5)(\lambda+2)=0 \;\Rightarrow\; \lambda = -5 \text{ or } -2$

C.F.: $y_C = Ae^{-5x} + Be^{-2x}$

Particular solution by the method of <u>undetermined coefficients</u>:

The right side function $r = 6e^{-2x}$ is a simple multiple of one of the basis functions of the complementary function. Therefore try

$$y_P = cxe^{-2x} \quad\Rightarrow\quad y_P' = c(1-2x)e^{-2x} \quad\Rightarrow\quad y_P'' = c(4x-4)e^{-2x}$$

Substitute this trial particular solution into the inhomogeneous ODE:

$$y_P'' + 7y_P' + 10y_P = c(4x-4 + 7-14x + 10x)e^{-2x} = 3ce^{-2x} = 6e^{-2x}$$

$$\Rightarrow \quad c = 2 \quad \Rightarrow \quad y_P = 2xe^{-2x}$$

**OR**

Particular solution by the method of <u>variation of parameters</u>:

$$y_1 = e^{-5x}, \quad y_2 = e^{-2x}, \quad r = 6e^{-2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-5x} & e^{-2x} \\ -5e^{-5x} & -2e^{-2x} \end{vmatrix} = 3e^{-7x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix} = -y_2 r = -e^{-2x}\left(6e^{-2x}\right) = -6e^{-4x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix} = +y_1 r = e^{-5x}\left(6e^{-2x}\right) = 6e^{-7x}$$

$$u' = \frac{W_1}{W} = \frac{-6e^{-4x}}{3e^{-7x}} = -2e^{3x} \quad\Rightarrow\quad u = -\frac{2}{3}e^{3x}$$

$$v' = \frac{W_2}{W} = \frac{6e^{-7x}}{3e^{-7x}} = 2 \quad\Rightarrow\quad v = 2x$$

$$\Rightarrow \quad y_P = uy_1 + vy_2 = -\frac{2}{3}e^{3x}e^{-5x} + 2xe^{-2x} = 2xe^{-2x} - \frac{2}{3}e^{-2x}$$

But any constant multiple of $e^{-2x}$ is part of the complementary function and can be absorbed into it. Therefore the particular solution is $y_P = 2xe^{-2x}$.

2 (a) (continued)

The <u>general solution</u> is

$$y = y_C + y_P = Ae^{-5x} + (2x+B)e^{-2x}$$

$$\Rightarrow \quad y' = -5Ae^{-5x} + (2-4x-2B)e^{-2x}$$

Applying the initial conditions,

$$y(0)=1 \quad \Rightarrow \quad 1 = A + B \quad \Rightarrow \quad B = 1-A \qquad \textbf{(1)}$$

$$y'(0)=-3 \quad \Rightarrow \quad -3 = -5A + (2-2B) \qquad \textbf{(2)}$$

Substitute equation **(1)** into equation **(2)**:

$$-3 = -5A + (2-2+2A) \quad \Rightarrow \quad -3A=-3 \quad \Rightarrow \quad A=1$$

$$A=1 \quad \Rightarrow \quad B = 1-1 = 0$$

The complete solution is therefore

$$\boxed{y(x) = e^{-5x} + 2xe^{-2x}}$$

[It is tedious but straightforward to verify that this complete solution does satisfy the initial value problem.]

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(b) Let $Y(s)=\mathscr{L}\{y(x)\}$, then the Laplace transform of the initial value problem

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 10y = 6e^{-2x}, \qquad y(0)=1,\ y'(0)=-3$$

is $\left(s^2Y - 1s + 3\right) + 7(sY-1) + 10Y = \dfrac{6}{s+2}$

$$\Rightarrow \quad \left(s^2+7s+10\right)Y = \frac{6}{s+2} + s - 3 + 7$$

$$\Rightarrow \quad (s+2)(s+5)Y = \frac{6}{s+2} + s + 4 = \frac{6+s^2+6s+8}{s+2}$$

$$\Rightarrow \quad Y = \frac{s^2+6s+14}{(s+2)^2(s+5)} = \frac{a}{s+2} + \frac{b}{(s+2)^2} + \frac{c}{s+5}$$

By the cover-up rule, $c = \dfrac{25-30+14}{(-5+2)^2\,\cancel{(-5+5)}} = \dfrac{9}{9} = 1$

$$s^2+6s+14 = a(s+2)(s+5) + b(s+5) + 1(s+2)^2$$

$$s=-2 \quad \Rightarrow \quad 4-12+14 = 0+3b+0 \quad \Rightarrow \quad b = \frac{6}{3} = 2$$

Matching coefficients of $s^2$: $\quad 1 = a+0+1 \quad \Rightarrow \quad a=0$

2 (b) (continued)

$$\Rightarrow \quad Y = \frac{3}{(s+2)^2} + \frac{1}{s+5} = 3\mathscr{L}\left\{ x e^{-2x} \right\} + \mathscr{L}\left\{ e^{-5x} \right\} \quad \Rightarrow$$

$$\boxed{y(x) = e^{-5x} + 2x e^{-2x}}$$

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3. An underdamped mass-spring system, with an oscillating force applied, is modelled by the ordinary differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 10\sin t$$

Find the general solution $x(t)$

(a) without using Laplace transforms; and [6]

(b) using Laplace transforms. [10]

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(a) A.E.: $\lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j$

C.F.: $x_C = e^{-t}\left(A\cos t + B\sin t\right)$

Particular solution by the method of <u>undetermined coefficients</u>:

The right side function $r = 10\sin t$ is independent of the complementary function.
[10 sin $t$ is not a multiple of either $e^{-t}\sin t$ or $e^{-t}\cos t$ .]
Therefore try

$x_P = c\cos t + d\sin t \quad \Rightarrow \quad x_P' = -c\sin t + d\cos t$

$\Rightarrow \quad x_P'' = -c\cos t - d\sin t$

Substitute this trial particular solution into the inhomogeneous ODE:

$x_P'' + 2x_P' + 2x_P = (-c + 2d + 2c)\cos t + (-d - 2c + 2d)\sin t = 10\sin t$

$\Rightarrow \quad 2d + c = 0 \quad$ and $\quad d - 2c = 10$

$\Rightarrow \quad 5d = 10 \quad \Rightarrow \quad d = 2 \quad$ and $\quad c = -4$

$\Rightarrow \quad x_P = 2\sin t - 4\cos t$

**OR**

3 (a) (continued)

Particular solution by the method of variation of parameters:

$$x_1 = e^{-t}\cos t\,, \quad x_2 = e^{-t}\sin t\,, \quad r = 10\sin t$$

$$W = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-t}\cos t & e^{-t}\sin t \\ e^{-t}\left(-\cos t - \sin t\right) & e^{-t}\left(\cos t - \sin t\right) \end{vmatrix}$$

$$= e^{-2t}\left(\cos^2 t - \cos t\sin t + \cos t\sin t + \sin^2 t\right) = e^{-2t}$$

$$W_1 = \begin{vmatrix} 0 & x_2 \\ r & x_2' \end{vmatrix} = -x_2 r = -e^{-t}\sin t\left(10\sin t\right) = -10e^{-t}\sin^2 t$$

$$W_2 = \begin{vmatrix} x_1 & 0 \\ x_1' & r \end{vmatrix} = +x_1 r = e^{-t}\cos t\left(10\sin t\right) = 10e^{-t}\cos t\sin t$$

We shall need the double angle formulae:

$$\sin 2t \equiv 2\sin t\cos t \quad \text{and} \quad \cos 2t \equiv 2\cos^2 t - 1 \equiv 1 - 2\sin^2 t$$

$$u' = \frac{W_1}{W} = \frac{-10e^{-t}\sin^2 t}{e^{-2t}} = -10e^t\sin^2 t = -5e^t\left(1 - \cos 2t\right)$$

After a tedious integration by parts, we find

$$u = e^t\left(2\sin 2t + \cos 2t - 5\right)$$

$$v' = \frac{W_2}{W} = \frac{10e^{-t}\cos t\sin t}{e^{-2t}} = 10e^t\cos t\sin t = 5e^t\sin 2t$$

After another tedious integration by parts, we find

$$v = e^t\left(\sin 2t - 2\cos 2t\right)$$

$$\Rightarrow \quad x_P = u\,x_1 + v\,x_2$$

$$= e^t\left(2\sin 2t + \cos 2t - 5\right)e^{-t}\cos t \; + \; e^t\left(\sin 2t - 2\cos 2t\right)e^{-t}\sin t$$

$$= \cancel{4\sin t\cos^2 t} + \cos t - \cancel{2\sin^2 t\cos t} - 5\cos t$$

$$+ \; \cancel{2\sin^2 t\cos t} + 2\sin t - \cancel{4\sin t\cos^2 t}$$

Clearly the method of undetermined coefficients is much faster in this case!

The general solution is

$$\boxed{x(t) = e^{-t}\left(A\cos t + B\sin t\right) - 4\cos t + 2\sin t}$$

[It is tedious but straightforward to verify that this general solution does satisfy the ordinary differential equation.]

3 (b) Let $X(s) = \mathscr{L}\{x(t)\}$, then the Laplace transform of the ordinary differential equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 10\sin t$$

is $\left(s^2 X - as - b\right) + 2\left(sX - a\right) + 2X = \dfrac{10}{s^2+1}$, where $a = x(0)$ and $b = x'(0)$.

$$\Rightarrow \left(s^2+2s+2\right)X = \frac{10}{s^2+1} + as + b + 2a = \frac{as^3 + (b+2a)s^2 + as + (b+2a+10)}{s^2+1}$$

$$\Rightarrow X = \frac{as^3 + (b+2a)s^2 + as + (b+2a+10)}{\left(s^2+1\right)\left((s+1)^2+1\right)} = \frac{cs+d}{s^2+1} + \frac{e(s+1)+f}{(s+1)^2+1}$$

$$\Rightarrow as^3 + (b+2a)s^2 + as + (b+2a+10) = (cs+d)\left((s+1)^2+1\right) + \left(e(s+1)+f\right)\left(s^2+1\right)$$

Matching coefficients:

$s^3: \quad a+0+0+0 = c + e$ **(A)**

$s^2: \quad 0+(b+2a)+0+0 = 2c+d + e+f$ **(B)**

$s^1: \quad 0+0+a+0 = 2c+2d + e$ **(C)**

$s^0: \quad 0+0+0+(b+2a+10) = 2d + e+f$ **(D)**

**(C) – (A)** $\Rightarrow c+2d=0 \Rightarrow c=-2d$

**(D) – (B)** $\Rightarrow d-2c = 10 \Rightarrow d+4d=10 \Rightarrow d=\frac{10}{5}=2 \Rightarrow c=-4$

**(A)** $\Rightarrow e = a+4$

**(B)** $\Rightarrow f = b+2a+8-2-a-4 = a+b+2$

$e$ and $f$ are expressed in terms of the unknown constants $a$, $b$.
$e$ and $f$ are therefore arbitrary constants - relabel them as $A$, $B$.

$$\Rightarrow X = \frac{-4s+2}{s^2+1} + \frac{A(s+1)+B}{(s+1)^2+1} = \mathscr{L}\left\{-4\cos t + 2\sin t + Ae^{-t}\cos t + Be^{-t}\sin t\right\}$$

Therefore the general solution is

$$\boxed{x(t) = e^{-t}\left(A\cos t + B\sin t\right) - 4\cos t + 2\sin t}$$

4. Find the general solution of the ordinary differential equation [10]

$$\frac{d^2 y}{dx^2} + y = \sec x , \qquad \left(0 \le x < \frac{\pi}{2}\right)$$

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A.E.: $\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = \pm j$

C.F.: $y_C = A\cos x + B\sin x$

Particular solution:

The right side (sec $x$) is not of a form for which the particular solution can be found by the method of undetermined coefficients. There is no choice but to find the particular solution by the method of variation of parameters.

$$y_1 = \cos x , \quad y_2 = \sin x , \quad r = \sec x = \frac{1}{\cos x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix} = -y_2 r = -\sin x \cdot \frac{1}{\cos x} = -\tan x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix} = +y_1 r = \cos x \cdot \frac{1}{\cos x} = 1$$

$$u' = \frac{W_1}{W} = \frac{-\tan x}{1} = \frac{-\sin x}{\cos x} \quad \Rightarrow \quad u = \int \frac{-\sin x}{\cos x}\,dx = \ln\left|\cos x\right|$$

$$v' = \frac{W_2}{W} = \frac{1}{1} = 1 \quad \Rightarrow \quad v = x$$

$$\Rightarrow \quad y_P = u\,y_1 + v\,y_2 = \left(\ln\left|\cos x\right|\right)\cos x + x\sin x$$

The general solution is therefore

$$\boxed{y(x) = \left(A + \ln\left|\cos x\right|\right)\cos x + (x+B)\sin x}$$

Laplace transforms cannot be used in this question, because $\mathcal{L}\{\sec x\}$ does not exist in terms of elementary functions.

4 (continued)

Verification of this solution (not required):

$$y(x) = \left(A + \ln|\cos x|\right)\cos x + (x+B)\sin x$$

$$\Rightarrow \quad y'(x) = \left(\frac{-\sin x}{\cos x}\right)\cos x - \left(A + \ln|\cos x|\right)\sin x + 1\sin x + (x+B)\cos x$$

$$\Rightarrow \quad y''(x) = -\left(-\frac{\sin x}{\cos x}\right)\sin x - \left(A + \ln|\cos x|\right)\cos x + 1\cos x - (x+B)\sin x$$

$$\Rightarrow \quad y''(x) = +\frac{\sin^2 x}{\cos x} + \cos x - y(x)$$

$$\Rightarrow \quad y'' + y = \frac{\sin^2 x + \cos^2 x}{\cos x} = \frac{1}{\cos x} = \sec x \quad ✓$$

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5. A mass-spring system is at rest until it is struck by a hammer at time $t = 4$ (seconds). [10]
The response $x(t)$ is modelled by the initial value problem

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 53x = 21\,\delta(t-4), \qquad x(0) = x'(0) = 0$$

where $\delta(t-a)$ is the Dirac delta function.
Use Laplace transforms to find the complete solution of this initial value problem.

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Let $X(s) = \mathscr{L}\{x(t)\}$, then the Laplace transform of the initial value problem is

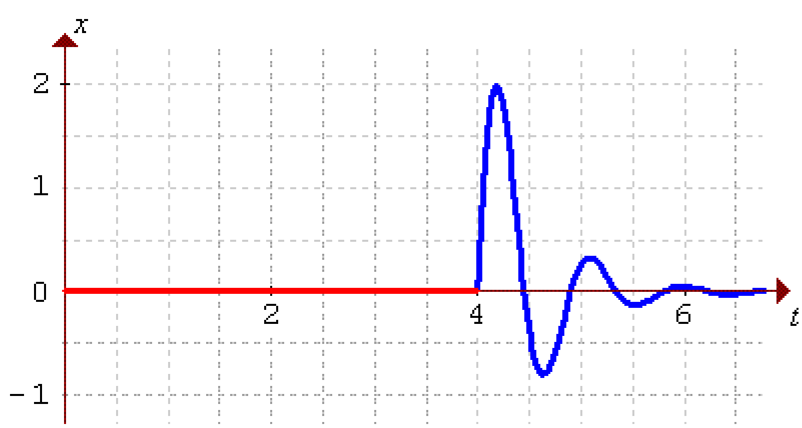
$$\left(s^2X - 0 - 0\right) + 4(sX - 0) + 53X = 21e^{-4s} \quad \Rightarrow \quad \left((s+2)^2 + 49\right)X = 21e^{-4s}$$

$$\Rightarrow \quad X(s) = \frac{21e^{-4s}}{(s+2)^2 + 7^2} = 3\,\mathscr{L}\left\{e^{-2t}\sin 7t\right\}e^{-4s}$$

Using the second shift theorem, the complete solution is

$$\boxed{x(t) = 3\,e^{-2(t-4)}\sin 7(t-4)\,H(t-4)}$$

This method for the solution is *much* faster than the method of variation of parameters!

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6. Find the Laplace transform $F(s)$ of [10]

$$f(t) = t\,e^{-2t}\cos 3t$$

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Quoting the two standard identities

$$\mathscr{L}\left\{e^{-at}\cos\omega t\right\} = \frac{s+a}{(s+a)^2+\omega^2} \quad \text{and} \quad \mathscr{L}\left\{t\,f(t)\right\} = -\frac{d}{ds}\mathscr{L}\left\{f(t)\right\}:$$

$$F(s) = \mathscr{L}\left\{f(t)\right\} = \mathscr{L}\left\{t\,e^{-2t}\cos 3t\right\} = -\frac{d}{ds}\mathscr{L}\left\{e^{-2t}\cos 3t\right\}$$

$$= -\frac{d}{ds}\left(\frac{s+2}{(s+2)^2+3^2}\right) = -\left(\frac{1\left((s+2)^2+3^2\right) - (s+2)\left(2(s+2)+0\right)}{\left((s+2)^2+3^2\right)^2}\right)$$

$$= \frac{(2-1)(s+2)^2-9}{\left((s+2)^2+9\right)^2} = \frac{s^2+4s-5}{\left(s^2+4s+13\right)^2}$$

Therefore

$$\boxed{F(s) = \frac{s^2+4s-5}{\left(s^2+4s+13\right)^2}}$$

**OR**

Applying the first shift theorem to $\mathscr{L}\left\{t\cos\omega t\right\} = \dfrac{s^2-\omega^2}{\left(s^2+\omega^2\right)^2}$ :

$$F(s) = \mathscr{L}\left\{t\,e^{-2t}\cos 3t\right\} = \mathscr{L}\left\{t\cos 3t\right\}\Big|_{s\to s+2} = \frac{(s+2)^2-3^2}{\left((s+2)^2+3^2\right)^2}$$

Therefore

$$\boxed{F(s) = \frac{s^2+4s-5}{\left(s^2+4s+13\right)^2}}$$

7. Find the function $f(t)$ whose Laplace transform is [12]

$$F(s) = \frac{48}{(s+2)(s^2+4s+20)}$$

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Solution by partial fractions:

$$F(s) = \frac{48}{(s+2)(s^2+4s+20)} = \frac{a}{s+2} + \frac{b(s+2) + 4c}{(s+2)^2+4^2}$$

$$\Rightarrow \quad 48 = a(s^2+4s+20) + b(s+2)^2 + 4c(s+2)$$

$$s = -2 \quad \Rightarrow \quad 48 = a(4-8+20) + 0 + 0 \quad \Rightarrow \quad a = \tfrac{48}{16} = 3$$

[or use the cover-up rule to find $a$ ]

Coefficients of $s^2$: $\Rightarrow \quad 0 = 3 + b + 0 \quad \Rightarrow \quad b = -3$

$$s=0 \quad \Rightarrow \quad 48 = 3(20) + -3(2)^2 + 4c(2) \quad \Rightarrow \quad 48-60+12 = 8c \quad \Rightarrow \quad c = 0$$

$$\Rightarrow \quad F(s) = \frac{3}{s+2} - \frac{3(s+2)}{(s+2)^2+4^2} = 3\mathscr{L}\left\{e^{-2t}\right\} - 3\mathscr{L}\left\{e^{-2t}\cos 4t\right\}$$

Therefore

$$\boxed{f(t) = 3(1 - \cos 4t)e^{-2t}}$$

or

$$f(t) = 6e^{-2t}\sin^2 2t$$

Solution by first shift theorem and integration property:

$$F(s) = \frac{48}{(s+2)(s^2+4s+20)} = \frac{48}{(s+2)\left((s+2)^2+16\right)} = \left.\frac{48}{s(s^2+16)}\right|_{s \to s+2}$$

By the first shift theorem, the inverse Laplace transform of $F(s)$ is

$$f(t) = 12\,\mathscr{L}^{-1}\left\{\frac{4}{s(s^2+4^2)}\right\}e^{-2t}$$

Using the integration property of Laplace transforms:

$$f(t) = \left(\int_0^t 12\,\mathscr{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\}d\tau\right)e^{-2t} = 3\left(\int_0^t 4\sin 4\tau \, d\tau\right)e^{-2t}$$

$$\Rightarrow \quad f(t) = 3\left[-\cos 4\tau\right]_0^t e^{-2t} = \boxed{f(t) = 3(1 - \cos 4t)e^{-2t}}$$

7. (continued)

Solution by convolution:

$$F(s) = \frac{48}{(s+2)(s^2+4s+20)} = 12\cdot\frac{1}{s+2}\cdot\frac{4}{(s+2)^2+4^2}$$

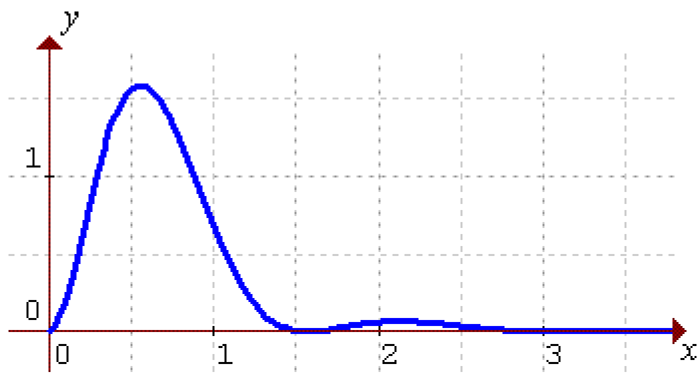
$$= 12\,\mathscr{L}\left\{e^{-2t}\right\}\mathscr{L}\left\{e^{-2t}\sin 4t\right\}$$

$$\Rightarrow \quad f(t) = 12\,e^{-2t} * \left(e^{-2t}\sin 4t\right) = 12\int_0^t e^{-2(t-\tau)}e^{-2\tau}\sin 4\tau\, d\tau$$

$$= 3e^{-2t}\int_0^t 4\sin 4\tau\, d\tau = 3e^{-2t}\left[-\cos 4\tau\right]_{\tau=0}^{\tau=t} = 3e^{-2t}\left(-\cos 4t + 1\right)$$

Therefore

$$\boxed{f(t) = 3(1 - \cos 4t)e^{-2t}}$$



8. Use the integration property of Laplace transforms, $\mathcal{L}^{-1}\left\{\dfrac{F(s)}{s}\right\} = \int_0^t \mathcal{L}^{-1}\{F(s)\}\,d\tau$

twice, in order to establish $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2\left(s^2+\omega^2\right)}\right\} = \dfrac{\omega t - \sin\omega t}{\omega^3}$ [8]

***and*** confirm this result using partial fractions. [8]

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Using the integration property:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{\sin\omega t}{\omega}$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+\omega^2}\right\} = \int_0^t \frac{\sin\omega\tau}{\omega}\,d\tau = \left[\frac{-\cos\omega\tau}{\omega^2}\right]_0^t = \frac{-\cos\omega t + 1}{\omega^2}$$

$$\Rightarrow \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{1}{s^2+\omega^2}\right\} = \int_0^t \frac{1-\cos\omega\tau}{\omega^2}\,d\tau = \left[\frac{\omega\tau-\sin\omega\tau}{\omega^3}\right]_0^t = \frac{\omega t-\sin\omega t - 0}{\omega^3}$$

Therefore

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s^2+\omega^2\right)}\right\} = \frac{\omega t-\sin\omega t}{\omega^3}}$$

Using partial fractions:

$$\frac{1}{s^2\left(s^2+\omega^2\right)} = \frac{a}{s} + \frac{b}{s^2} + \frac{cs+d\omega}{s^2+\omega^2}$$

$$\Rightarrow \quad 1 = as\left(s^2+\omega^2\right) + b\left(s^2+\omega^2\right) + cs^3 + d\omega s^2$$

Matching coefficients of powers of $s$:

$$s^0: \quad \Rightarrow \quad 1 = 0 + b\omega^2 + 0 + 0 \quad \Rightarrow \quad b = \frac{1}{\omega^2}$$

$$s^1: \quad \Rightarrow \quad 0 = a\omega^2 + 0 + 0 + 0 \quad \Rightarrow \quad a = 0$$

$$s^2: \quad \Rightarrow \quad 0 = 0 + \frac{1}{\omega^2} + 0 + d\omega \quad \Rightarrow \quad d = -\frac{1}{\omega^3}$$

$$s^3: \quad \Rightarrow \quad 0 = 0 + 0 + c + 0 \quad \Rightarrow \quad c = 0$$

$$\Rightarrow \quad \frac{1}{s^2\left(s^2+\omega^2\right)} = \frac{1}{\omega^3}\left(\frac{\omega}{s^2} - \frac{\omega}{s^2+\omega^2}\right) = \frac{1}{\omega^3}\left(\omega\mathcal{L}\{t\} - \mathcal{L}\{\sin\omega t\}\right)$$

Therefore

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2\left(s^2+\omega^2\right)}\right\} = \frac{\omega t-\sin\omega t}{\omega^3}}$$

8. (continued)

Also note that this inverse can be found via convolution:

$$\frac{1}{s^2\left(s^2+\omega^2\right)} = \mathscr{L}\{t\}\cdot\mathscr{L}\left\{\frac{\sin\omega t}{\omega}\right\} = \mathscr{L}\left\{t*\frac{\sin\omega t}{\omega}\right\}$$

$$\Rightarrow \quad \mathscr{L}^{-1}\left\{\frac{1}{s^2\left(s^2+\omega^2\right)}\right\} = t*\frac{\sin\omega t}{\omega} = \frac{1}{\omega}\int_0^t (t-\tau)\sin\omega\tau\,d\tau$$

After a two-step integration by parts, this integral evaluates to

$$-\frac{1}{\omega^3}\Big[\omega(t-\tau)\cos\omega\tau + \sin\omega\tau\Big]_{\tau=0}^{\tau=t} = -\frac{1}{\omega^3}\Big(\left(0+\sin\omega t\right) - \left(\omega t-0\right)\Big)$$

Therefore

$$\boxed{\mathscr{L}^{-1}\left\{\frac{1}{s^2\left(s^2+\omega^2\right)}\right\} = \frac{\omega t-\sin\omega t}{\omega^3}}$$

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