ENGI 9420 Engineering Analysis
Assignment 3 Solutions

2012 Fall

[Series solution of ODEs, matrix algebra; numerical methods; Chapters 1, 2 and 3]

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1. Find a power series solution about $x = 0$, as far as the term in x^7, to the ordinary differential equation

$$\frac{d^2 y}{dx^2} + y = \sec x, \qquad \left(0 \le x < \frac{\pi}{2}\right)$$

[This ODE is also in Assignment 2 Question 4.]

You may quote the Maclaurin series expansion for $\sec x$:

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \ldots \qquad \left(|x| < \tfrac{\pi}{2}\right)$$

This ODE is analytic for all x in $\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)$.

Therefore $x = 0$ is not a singular point.

Attempt a simple power series solution:

$$y = \sum_{n=0}^{\infty} a_n x^n \quad \Rightarrow \quad y'' = \sum_{n=0}^{\infty} a_n n(n-1) x^{n-2}$$

$$\Rightarrow \quad y'' = 0 + 0 + \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n$$

Substitute the series for y, y'' and $\sec x$ into the ODE:

$$\Rightarrow \quad \sum_{n=0}^{\infty} \left(a_{n+2}(n+2)(n+1) + a_n\right) x^n = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \frac{277x^8}{8064} + \ldots$$

Matching coefficients:

$$x^0: \quad 2\times 1 a_2 + a_0 = 1 \quad \Rightarrow \quad a_2 = \tfrac{1}{2}\left(1 - a_0\right)$$

$$x^1: \quad 3\times 2 a_3 + a_1 = 0 \quad \Rightarrow \quad a_3 = -\tfrac{1}{6} a_1$$

$$x^2: \quad 4\times 3 a_4 + a_2 = \tfrac{1}{2} \quad \Rightarrow \quad a_4 = \tfrac{1}{12}\left(\tfrac{1}{2} - a_2\right) = -\tfrac{1}{12}\left(\tfrac{1}{2} - \tfrac{1}{2}\left(1 - a_0\right)\right) = +\tfrac{1}{24} a_0$$

$$x^3: \quad 5\times 4 a_5 + a_3 = 0 \quad \Rightarrow \quad a_5 = -\tfrac{1}{20} a_3 = -\tfrac{1}{20}\left(-\tfrac{1}{6} a_1\right) = \tfrac{1}{120} a_1$$

1. (continued)

$$x^4: \quad 6\times5a_6 + a_4 = \tfrac{5}{24} \quad\Rightarrow\quad a_6 = \tfrac{1}{30}\left(\tfrac{5}{24}-a_4\right) = \tfrac{1}{30}\left(\tfrac{5}{24}-\tfrac{1}{24}a_0\right) = \tfrac{1}{720}\left(5-a_0\right)$$

$$x^5: \quad 7\times6a_7 + a_5 = 0 \quad\Rightarrow\quad a_7 = -\tfrac{1}{42}a_5 = -\tfrac{1}{42}\left(\tfrac{1}{120}a_1\right) = -\tfrac{1}{5040}a_1$$

$$x^6: \quad 8\times7a_8 + a_6 = \tfrac{61}{720}$$

$$\Rightarrow\quad a_8 = \tfrac{1}{56}\left(\tfrac{61}{720}-a_6\right) = \tfrac{1}{56}\left(\tfrac{61}{720}-\tfrac{1}{720}\left(5-a_0\right)\right) = \tfrac{1}{40320}\left(56+a_0\right)$$

The leading coefficients a_0 and a_1 are arbitrary constants, which are also the initial values $y(0)$ and $y'(0)$ respectively. Rewrite them as A and B respectively.

The general solution is

$$\boxed{\begin{aligned} y(x) = {} & A\left(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \tfrac{1}{720}x^6 + \ldots\right) \\ & + B\left(x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \tfrac{1}{5040}x^7 + \ldots\right) \\ & + \left(\tfrac{1}{2}x^2 + \tfrac{1}{144}x^6 + \tfrac{1}{720}x^8 + \ldots\right) \end{aligned}}$$

Upon recognizing the presence of factorial functions, we can write the general solution as

$$\boxed{\begin{aligned} y(x) = {} & A\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) \\ & + B\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right) \\ & + \left(\frac{x^2}{2!} + \frac{5x^6}{6!} + \frac{56x^8}{8!} + \ldots\right) \end{aligned}}$$

One can recover the complementary function quickly, upon recognizing the Maclaurin series for the sine and cosine functions:

$$\boxed{y(x) = A\cos x + B\sin x + \left(\frac{x^2}{2!} + \frac{5x^6}{6!} + \frac{56x^8}{8!} + \ldots\right)}$$

The particular solution turns out to be the Maclaurin series expansion for

$$x\sin x + (\cos x)\ln\left|\cos x\right|$$

A brief Maple worksheet provides this Maclaurin series expansion.

2. Use the method of Frobenius to find the general solution of the ordinary differential equation

$$x^2 \frac{d^2y}{dx^2} + \tfrac{1}{2} x(1+4x)\frac{dy}{dx} + \left(x-\tfrac{1}{2}\right)y = 0$$

as a series about $x = 0$, as far as the fourth non-zero term.

$$\underbrace{x^2}_{P(x)} \frac{d^2y}{dx^2} + \underbrace{\tfrac{1}{2} x(1+4x)}_{Q(x)}\frac{dy}{dx} + \underbrace{\left(x-\tfrac{1}{2}\right)}_{R(x)}y = \underbrace{0}_{F(x)}$$

$x\dfrac{Q}{P} = \tfrac{1}{2}(1+4x), \quad x^2\dfrac{R}{P} = x-\tfrac{1}{2}$ and $\dfrac{F}{P} = 0$ are all analytic.

Therefore $x = 0$ is a regular singular point of this ODE.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \Rightarrow \quad y' = \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1}$$

$$\Rightarrow \quad y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2}$$

Substitute the series into the ODE:

$$\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2+2} + \frac{1}{2}\sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1+1}$$

$$+ 2\sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1+2} + \sum_{n=0}^{\infty} a_n x^{n+r+1} - \frac{1}{2}\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

Adjust the index of summation, such that the exponent of x is $n+r$ in all cases.
The ODE becomes

$$\sum_{n=0}^{\infty} a_n \left((n+r)(n+r-1) + \frac{1}{2}(n+r) - \frac{1}{2}\right)x^{n+r}$$

$$+ \sum_{n=1}^{\infty} a_{n-1}\left(2(n+r-1) + 1\right)x^{n+r} = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} (n+r-1)\left(n+r+\tfrac{1}{2}\right)a_n x^{n+r} + \sum_{n=1}^{\infty}\left(2(n+r) - 1\right)a_{n-1}x^{n+r} = 0$$

$$\Rightarrow \quad (r-1)\left(r+\tfrac{1}{2}\right)a_0 x^r$$

$$+ \sum_{n=1}^{\infty}\left[(n+r-1)\left(n+r+\tfrac{1}{2}\right)a_n + \left(2(n+r)-1\right)a_{n-1}\right]x^{n+r}$$

This equation must be true for all values of x.
Therefore the coefficient of every power of x must be zero.
But a_0 cannot be identically zero.

$$\Rightarrow \quad (r-1)\left(r+\tfrac{1}{2}\right) = 0 \quad \Rightarrow \quad r = 1 \text{ or } -\tfrac{1}{2}$$

2. (continued)

Case $r = 1$:

This value of r is a positive integer, so that the series becomes an ordinary power series.

$$\Rightarrow \quad (n+1-1)\left(n+1+\tfrac{1}{2}\right)a_n \;+\; \left(2(n+1)\;-\;1\right)a_{n-1} \;=\; 0 \qquad \forall n > 0$$

$$\Rightarrow \quad a_n \;=\; \frac{-2(2n+1)}{n(2n+3)}a_{n-1}$$

Employing this recurrence relation to find a_1 to a_4 ,

$$\Rightarrow \quad a_1 \;=\; \frac{-2(2+1)}{1(2+3)}a_0 \;=\; -\frac{6}{5}a_0$$

$$\Rightarrow \quad a_2 \;=\; \frac{-2(4+1)}{2(4+3)}a_1 \;=\; -\frac{5}{7}\times-\frac{6}{5}a_0 \;=\; +\frac{6}{7}a_0$$

$$\Rightarrow \quad a_3 \;=\; \frac{-2(6+1)}{3(6+3)}a_2 \;=\; -\frac{14}{27}\times\frac{6}{7}a_0 \;=\; -\frac{4}{9}a_0$$

$$\Rightarrow \quad a_4 \;=\; \frac{-2(8+1)}{4(8+3)}a_3 \;=\; -\frac{9}{22}\times-\frac{4}{9}a_0 \;=\; +\frac{2}{11}a_0$$

$$\Rightarrow \quad y_1(x) \;=\; \sum_{n=0}^{\infty} a_n x^{n+1} \;=\; a_0\left(x - \frac{6}{5}x^2 + \frac{6}{7}x^3 - \frac{4}{9}x^4 + \frac{2}{11}x^5 \;-\; \dots\right)$$

Case $r = -1/2$:

$$\left(n-\tfrac{1}{2}-1\right)\left(n-\tfrac{1}{2}+\tfrac{1}{2}\right)a_n \;+\; \left(2\left(n-\tfrac{1}{2}\right)\;-\;1\right)a_{n-1} \;=\; 0 \qquad \forall n > 0$$

$$\Rightarrow \quad a_n \;=\; -\frac{2(n-1)}{n(n-3)}a_{n-1} \qquad \forall n > 0$$

$$\Rightarrow \quad a_1 \;=\; -\frac{2(1-1)}{1(1-3)}a_0 \;=\; 0 \qquad \Rightarrow \quad a_n = 0 \quad \forall n > 0$$

This series is therefore finite, with only one term:

$$y_2(x) \;=\; a_0 x^{-1/2}$$

The function $y_2(x)$ is clearly independent of the function $y_1(x)$.
Therefore the general solution is

$$\boxed{y(x) \;=\; \frac{A}{\sqrt{x}} \;+\; B\left(x \;-\; \tfrac{6}{5}x^2 \;+\; \tfrac{6}{7}x^3 \;-\; \tfrac{4}{9}x^4 \;+\; \tfrac{2}{11}x^5 \;-\; \dots\right)}$$

(where A and B are arbitrary constants).

3. Find the entire series solution (using the method of Frobenius, adapted to a first order ODE) about $x = 0$ of the ordinary differential equation

$$\frac{dy}{dx} + \frac{4y}{x} = 2$$

$P = 1, \quad Q = \frac{4}{x}, \quad F = 2$

$x\frac{Q}{P} = 4$ and $\frac{F}{P} = 2$ are both analytic

Therefore $x = 0$ is a regular singular point of this ODE.

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad \Rightarrow \quad y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$

Substitute the series into the ODE:

$$y' + \frac{4}{x}y = \sum_{n=0}^{\infty} a_n (n+r+4) x^{n+r-1} = 2$$

The leading term $(n = 0)$ is $a_0 (r+4) x^{r-1}$

If $r = 1$ then $a_0 (1+4) x^0 = 2 \quad \Rightarrow \quad a_0 = \tfrac{2}{5}$

The ODE then becomes

$$\frac{2}{5} \times 5x^0 + \sum_{n=1}^{\infty} a_n (n+5) x^n = 2 \quad \Rightarrow \quad a_n = 0 \quad \forall n > 0$$

$$\Rightarrow \quad y(x) = \sum_{n=0}^{\infty} a_n x^{n+1} = \frac{2}{5}x \quad - \text{ this is a particular solution.}$$

If $r \neq 1$ then $a_0 (r+4) = 0 \quad \Rightarrow \quad r = -4 \quad (a_0 \text{ cannot be zero})$

$$\Rightarrow \quad \sum_{n=1}^{\infty} a_n n x^{n-5} = 2 \quad \Rightarrow \quad a_n = 0$$

except for $n = 0$, where a_0 is arbitrary, (relabel it as A)

and for $n = 5$, where $5a_5 = 2 \quad \Rightarrow \quad a_5 = \tfrac{2}{5}$

$$\Rightarrow \quad y(x) = \sum_{n=0}^{\infty} a_n x^{n-4} = Ax^{-4} + 0 + 0 + 0 + 0 + \frac{2}{5}x^1 + 0 + \ldots$$

which includes the case $r = 1$.

Therefore the general solution is

$$\boxed{y(x) = \frac{2}{5}x + \frac{A}{x^4}}$$

[Note that this ODE is linear. It is easy to verify that the solution above is correct.]

4. In Chapter 2 of the lecture notes, dimensional analysis is used to derive the functional form of the Planck length L_P in terms of the universal constants G, h and c.

In the study of the steady flow of incompressible fluid through a pipe, it is known that the volume of liquid issuing per second from a pipe, Q, depends on the coefficient of viscosity η (measured in the units kg m^{-1}s^{-1}), the radius a of the pipe (measured in metres) and the pressure gradient p/l set up along the pipe (measured in the units N m^{-2}m^{-1} or, equivalently, kg m^{-2}s^{-2}).

Use a similar dimensional analysis to derive the functional form for Q in terms of η, a and p/l.

$Q = k\eta^x a^y \left(\frac{p}{l}\right)^z$, where k is a constant of proportionality that cannot be determined by dimensional analysis alone. Q is a volume per unit time (m^{3}s^{-1}).
Conducting the dimensional analysis,

$$\left[L^3T^{-1}\right] = \left[M\,L^{-1}T^{-1}\right]^x \cdot \left[L\right]^y \cdot \left[M\,L^{-2}T^{-2}\right]^z = \left[M^{x+z}\,L^{-x+y-2z}\,T^{-x-2z}\right]$$

$$\Rightarrow \begin{array}{c c} & \begin{array}{ccc} \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} \end{array} \\ \begin{array}{c} \boldsymbol{M} \\ \boldsymbol{L} \\ \boldsymbol{T} \end{array} & \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 1 & -2 & 3 \\ -1 & 0 & -2 & -1 \end{array}\right] \end{array} \xrightarrow[R_3+R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{R_3\times(-1)} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2+R_3]{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\Rightarrow \quad x=-1, \quad y=4, \quad z=1$

Therefore

$$\boxed{Q = \frac{k\,a^4}{\eta}\left(\frac{p}{l}\right)}$$

which is Poiseuille’s formula.

By methods beyond the scope of this course, it can be shown that $k = \frac{\pi}{8}$.

Plain text output from the linear system reduction program is available at this link.

5. Find the intersection of the three planes

$$x + 2y + 3z = 0$$
$$2x + 5y + 2z + 1 = 0$$
$$x + 2y + 4z - 1 = 0$$

Where the three planes all meet, all three equations must be true simultaneously.
Form and reduce the augmented matrix that is equivalent to this triple of simultaneous linear equations.

$$\left[\mathrm{A} \mid \vec{\mathbf{b}}\right] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 5 & 2 & -1 \\ 1 & 2 & 4 & +1 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Upon reaching this triangular form, one can see that rank A = rank [A | b] = n so that there is a unique solution.

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 11 & 2 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2 + 4R_3]{R_1 - 11R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Geometrically, the three planes intersect each other in a single point.
From this reduced echelon form, we can read the unique solution

$$\boxed{(x, y, z) = (-9,\ 3,\ 1)}$$

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6. By reducing the appropriate linear system to echelon form, show that the intersection of the three planes

$$x + y + z + 1 = 0$$
$$x + 2y + 3z + 4 = 0$$
$$4x + 3y + 2z + 1 = 0$$

is the line $\dfrac{x-2}{1} = \dfrac{y-(-3)}{-2} = \dfrac{z-0}{1}$.

The linear system for (x, y, z) corresponding to the simultaneous equations of the three planes is

6 (continued)

$$\begin{array}{c} \\ \text{Plane 1} \\ \text{Plane 2} \\ \text{Plane 3} \end{array} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 1 & 2 & 3 & -4 \\ 4 & 3 & 2 & -1 \end{array}\right] \end{array}$$

$$\xrightarrow[R_3 - 4R_1]{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & -1 \\ 0 & 1 & 2 & -3 \\ 0 & -1 & -2 & 3 \end{array}\right]$$

At this point, one can see that (row 3) = – (row 2), so that
row 3 will cancel down to a row of all zeroes and
rank A = rank [A | b] = 2 < n. There are therefore infinitely many solutions.
More precisely, one can detect a one-parameter family of solutions.
Completing the row reduction,

$$\xrightarrow[R_3 + R_2]{R_1 - R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 2 \\ 0 & \boxed{1} & 2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is the reduced row echelon form for [A | b].
There are leading one's for x and y only.
z is therefore a free parameter (call it t). Reading the solutions from this echelon form,

$$z = t\,;$$
$$y + 2t = -3 \quad \Rightarrow \quad y = -2t - 3\,;$$
$$x - t = 2 \quad \Rightarrow \quad x = t + 2$$
$$\Rightarrow \quad (x, y, z) = (t+2, -2t-3, t) = (2, -3, 0) + (1, -2, 1)t$$

This one vector equation is also a set of three simultaneous scalar equations for x, y, z in terms of t. Make t the subject of all three equations, then

$$t = \frac{x-2}{1} = \frac{y-(-3)}{-2} = \frac{z-0}{1}$$

which is the equation of a line. Geometrically, a one parameter family of solutions should generate a one-dimensional object (the line).

7. Find the value of the determinant and, if it exists, the inverse matrix, for the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ -1 & 0 & 2 & 1 \end{bmatrix}$$

Form the augmented matrix

$$[A \,|\, I] = \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ -1 & 0 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

Transform the matrix to reduced echelon form:

$$\xrightarrow[R_4 + R_1]{R_2 - \frac{1}{2}R_1} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_4} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right]$$

The left matrix is now in triangular form.
The only row operation so far that has changed the value of the determinant is the row interchange. The value of the determinant of the original matrix A is therefore
$-1 \times (1 \times 2 \times 1 \times ½) = -1$.
Non-zero determinant $\Rightarrow$ **A^{-1} exists**.

Resuming the row reduction:

$$\xrightarrow{R_2 \div 2} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right]$$

7. (continued)

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{cccc|cccc} 1 & 0 & -2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 - R_3]{R_1 + 2R_3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_4 \times 2]{} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 - \frac{1}{2}R_4]{R_2 + \frac{1}{2}R_4} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 \end{array}\right] = \left[\,\mathrm{I} \,|\, \mathrm{A}^{-1}\right]$$

The only row operations that changed the value of the determinant were one row interchange and two cancelling row multiplications (the first by a factor of ½ and the second by a factor of 2). All other row operations were of the type (from one row subtract or add a multiple of another row), which do not affect the value of the determinant. Therefore the determinant of the original matrix A is –1 × the determinant of the echelon form I. But det I = 1. Therefore det A = –1.

Also, the fact that the reduced echelon form of A is the identity matrix allows us to conclude that A is invertible and that the inverse is the right-hand matrix in the reduced echelon form of the augmented matrix. Therefore

$$\boxed{\det \mathrm{A} = -1 \quad \text{and} \quad \mathrm{A}^{-1} = \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 1 & -1 & \frac{1}{2} \\ \frac{1}{2} & -1 & 1 & 0 \\ -1 & 2 & 0 & 0 \end{bmatrix}}$$

[One may check the answer, using A A^{-1} = A^{-1}A = I.]

7. (continued)

<u>Additional Note</u>:

It is possible to use a cofactor expansion to evaluate the determinant and inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ -1 & 0 & 2 & 1 \end{bmatrix}$$

However, this is a *much* slower method than row reduction (Gaussian elimination) to echelon form. The details of this alternative method are on a separate web page.

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8. In Assignment 1, Question 5, find, correct to the nearest millimetre, the initial head H of water needed in the conical tank in order for the tank to drain completely in exactly one minute, using

(a) the method of bisection (***or*** a graphical zoom-in, in which case provide sketches or screenshots) and explain your choice of initial values.

The time T required to drain the conical tank of head H completely was found in assignment 1 to be

$$T = \frac{1}{21}\sqrt{\frac{2H}{3g}}\left(2\sqrt{3}H^2 + 20H + 30\sqrt{3}\right)$$

A tank of head 30 cm was found to drain in approximately 26 s.
The functional form for $T(H)$ involves a higher power than the square.

It is therefore certain that a tank of head 60 cm will require more than 60 seconds to drain. In the bisection method, take as starting values $H = 30$ and $H = 60$.
[next page]

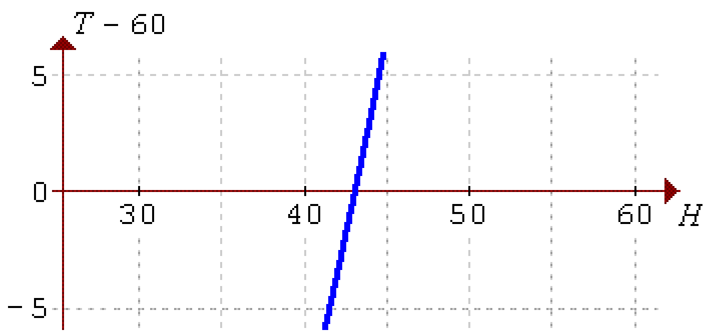
8 (a) (continued)

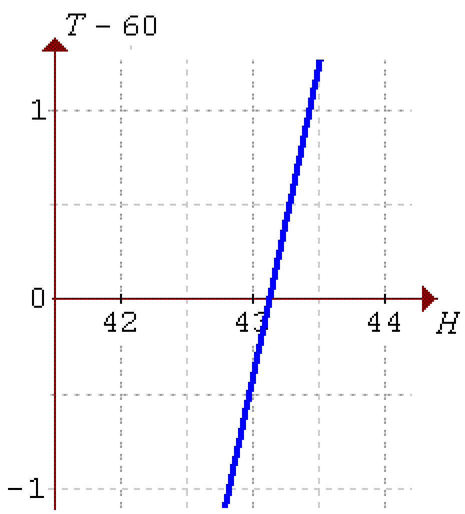
H	***T***
30.000 000	25.6308
60.000 000	131.9522
45.000 000	66.3421
37.500 000	43.1277
41.250 000	53.9868
43.125 000	59.9734
44.062 500	63.1095
43.593 750	61.5294
43.359 375	60.7484
43.242 188	60.3602
43.183 594	60.1666
43.154 297	60.0700
43.139 648	**60.0217**
43.132 324	**59.9975**

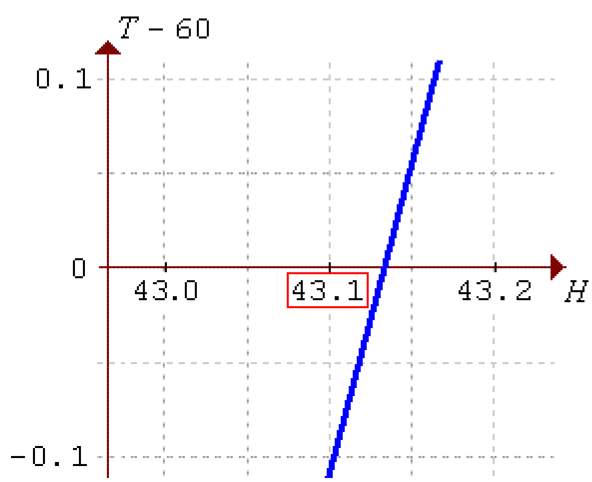
At this point one can deduce that, correct to the nearest millimetre, a drainage time of exactly one minute will occur when

$$\boxed{H = \mathbf{43.1}\text{ cm}}$$

An alternative to tabular bisection is a sequence of graphical zooms, as illustrated here.

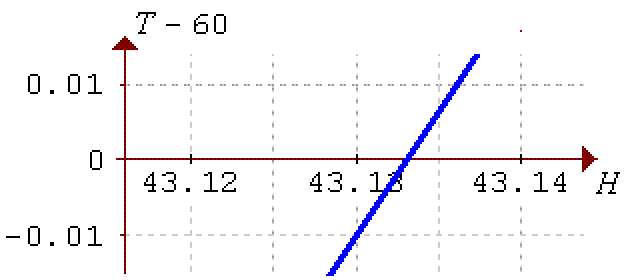


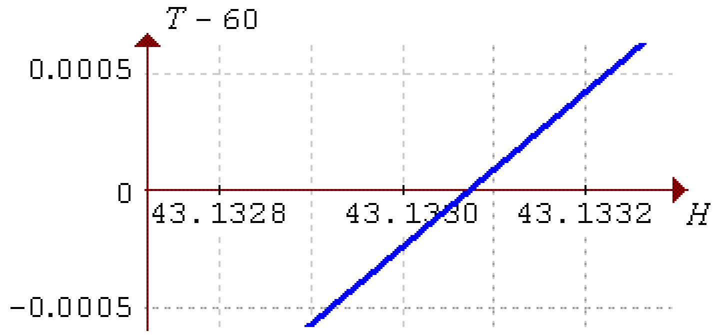




8 (a) (continued)

This process of graphical zooms can be continued, in order to find the solution to greater precision:





from which $H = 43.1331$ cm, correct to four decimal places.
$T(43.1331) = 60.0001$ s

(b) Newton's method and explain your choice of initial value.

As in part (a), we can deduce that the required value of H is somewhere in (30, 60).
The non-linear function suggests that the value is closer to 30 than it is to 60.
Therefore select $H = 40$ as an initial guess.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{where} \quad f(x) = \frac{1}{21}\sqrt{\frac{2}{3g}}\left(2\sqrt{3}x^{2.5} + 20x^{1.5} + 30\sqrt{3}x^{0.5}\right)$$

$$\Rightarrow \quad f'(x) = \frac{1}{21}\sqrt{\frac{2}{3g}}\left(5\sqrt{3}x^{1.5} + 30x^{0.5} + 15\sqrt{3}x^{-0.5}\right)$$

n	x_n	$f(x_n)$	$f'(x_n)$	f/f'
0	40.0000000	–9.7959450	2.9603337	–3.3090679
1	43.3090679	+0.5815746	3.3140581	+0.1754871
2	**43.13**35807	+0.0016757	3.2949666	+0.0005086
3	**43.13**30722			

Therefore, correct to one decimal place,

$$\boxed{H = \mathbf{43.1} \text{ cm}}$$

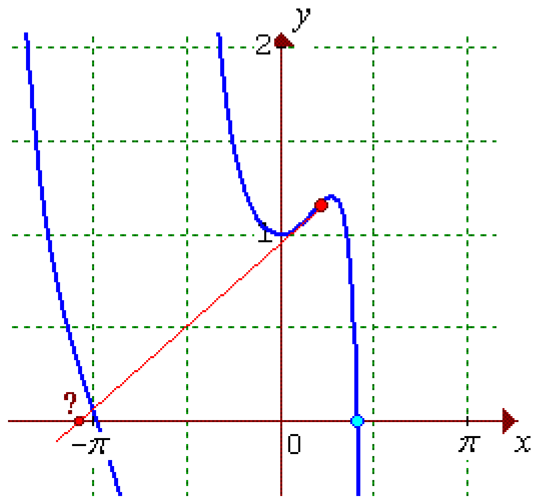
[An Excel spreadsheet file for this solution is available here.
One may experiment with different starting values.]

9. Why should Newton's Method **not** be used to find a root of $e^x = \tan x$, (except when the initial guess is very near the true value)?

Demonstrate the problem by using Newton's Method to try to find the first positive root, with initial guesses of 0.99 and 1.00.

The function $\tan x$ has an infinite number of infinite discontinuities.
It is quite possible for the tangent line from a value of x_n to cross one or more such discontinuities before reaching its x axis intercept. Newton's Method can thus become very unstable, jumping from near one root to near some other root.

Graph of $y = e^x - \tan x$



Also note the proximity to the first positive root of two turning points (where $f'(x) = 0$).

Using Newton's Method anyway, with the starting values 0.99 and 1.00:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{where} \quad f(x) = e^x - \tan x \quad \Rightarrow \quad f'(x) = e^x - \sec^2 x$$

9 (continued)

n	x_n	$f(x_n)$	$f\,'(x_n)$	$f\,/\,f\,'$
0	**0.990000**	1.167558	-0.630356	-1.852219
1	**2.842219**	17.462431	16.058515	1.087425
2	**1.754794**	11.155646	-24.091088	-0.463061
3	**2.217855**	10.511098	6.435951	1.633185
4	**0.584670**	1.132535	0.356335	3.178284
5	**-2.593614**	-0.535578	-1.297750	0.412697
6	**-3.006312**	-0.086639	-0.969053	0.089406
7	**-3.095717**	-0.000665	-0.956865	0.000695
8	**-3.096412**	0.000000	-0.956833	0.000000
9	**-3.096412**	0.000000	-0.956833	0.000000

Note that the method has converged on the wrong root!

n	x_n	$f(x_n)$	$f\,'(x_n)$	$f\,/\,f\,'$
0	**1.000000**	1.160874	-0.707237	-1.641422
1	**2.641422**	14.579664	12.734450	1.144899
2	**1.496522**	-8.972757	-177.137546	0.050654
3	**1.445868**	-3.717374	-60.162407	0.061789
4	**1.384079**	-1.302160	-25.027966	0.052028
5	**1.332051**	-0.319871	-14.092417	0.022698
6	**1.309353**	-0.033595	-11.264168	0.002982
7	**1.306370**	-0.000476	-10.947140	0.000043
8	**1.306327**	0.000000	-10.942604	0.000000
9	**1.306327**	0.000000	-10.942603	0.000000

and $x = 1.30632...$ is the first positive root.

Note how two starting values so close together have converged to very different roots. This is an illustration of how unstable Newton's Method is in this case.

[An Excel spreadsheet file for this solution is available here.
One may experiment with different starting values.]

10. Use the standard fourth order Runge-Kutta method, with a step size of $h = 0.2$, to find the value at $x = 0.6$ of the solution of the initial value problem

$$\frac{dy}{dx} = x^3\left(y^2 - y\right), \qquad y(0) = 0.2$$

Show your calculations for the constants k_1, k_2, k_3, k_4 and for y_1 in the first iteration. You may submit output from a spreadsheet program (such as Excel) for the other steps.

For this ODE, the RK4 algorithm becomes

$$k_1 = f(x_n, y_n) = x_n^{\ 3}\left(y_n^{\ 2} - y_n\right) = x_n^{\ 3} y_n (y_n - 1)$$

$$k_2 = f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h k_1\right) = \left(x_n + \tfrac{1}{2}h\right)^3 \left(y_n + \tfrac{1}{2}h k_1\right)\left(\left(y_n + \tfrac{1}{2}h k_1\right) - 1\right)$$

$$k_3 = f\left(x_n + \tfrac{1}{2}h,\ y_n + \tfrac{1}{2}h k_2\right) = \left(x_n + \tfrac{1}{2}h\right)^3 \left(y_n + \tfrac{1}{2}h k_2\right)\left(\left(y_n + \tfrac{1}{2}h k_2\right) - 1\right)$$

$$k_4 = f\left(x_n + h,\ y_n + h k_3\right) = \left(x_n + h\right)^3 \left(y_n + h k_3\right)\left(\left(y_n + h k_3\right) - 1\right)$$

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

with starting values $x_0 = 0$ and $y_0 = 0.2$.

In the first iteration,

$$k_1 = 0.0^3(0.2)(0.2 - 1) = 0$$

$$k_2 = (0.0 + 0.1)^3 (0.2 + 0.1 \times 0)\left((0.2 + 0.1 \times 0) - 1\right) = -0.000160$$

$$k_3 = (0.0 + 0.1)^3 (0.2 + 0.1 \times -0.000160)\left((0.2 + 0.1 \times -0.000160) - 1\right) = -0.000160$$

$$k_4 = (0.0 + 0.2)^3 (0.2 + 0.2 \times -0.000160)\left((0.2 + 0.2 \times -0.000160) - 1\right) = -0.001280$$

$$y(0.2) = y_1 = 1 + \frac{0.2}{6}\left(0 + 2 \times -0.000160 + 2 \times -0.000160 - 0.001280\right) = 0.199936$$

Second iteration:

$$k_1 = (0.2)^3 (0.199936)\left((0.199936) - 1\right) = -0.001280$$

$$k_2 = (0.2 + 0.1)^3 (0.199936 + 0.1 \times -0.001280)\left((0.199936 + 0.1 \times -0.001280) - 1\right) = -0.004317$$

$$k_3 = (0.2 + 0.1)^3 (0.199936 + 0.1 \times -0.004317)\left((0.199936 + 0.1 \times -0.004317) - 1\right) = -0.004312$$

$$k_4 = (0.2 + 0.2)^3 (0.199936 + 0.2 \times -0.004312)\left((0.199936 + 0.2 \times -0.004312) - 1\right) = -0.010204$$

$$y(0.4) = y_2 = 0.199936 +$$

$$\frac{0.2}{6}\left(-0.001280 + 2 \times -0.004317 + 2 \times -0.004312 - 0.010204\right) = 0.198978$$

10 (continued)

Final iteration:

$$k_1 = (0.4)^3 (0.198978)((0.198978)-1) = -0.010201$$

$$k_2 = (0.4+0.1)^3 (0.198978+0.1\times -0.010201)((0.198978+0.1\times -0.010201)-1) = -0.019846$$

$$k_3 = (0.4+0.1)^3 (0.198978+0.1\times -0.019846)((0.198978+0.1\times -0.019846)-1) = -0.019773$$

$$k_4 = (0.4+0.2)^3 (0.198978+0.1\times -0.019773)((0.198978+0.1\times -0.019773)-1) = -0.033910$$

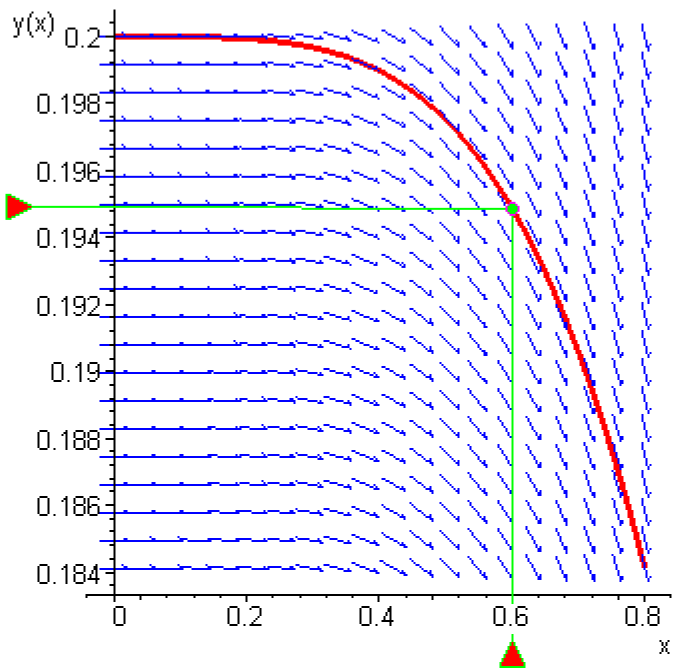
$$y(0.6) = y_3 = 0.198978 +$$

$$\frac{0.2}{6}(-0.010201+2\times -0.019846+2\times -0.019773-0.033910) = 0.194866$$

Therefore, correct to four decimal places,

$$\boxed{y(0.6) = \mathbf{0.1949}}$$

A direction field plot from Maple confirms the behaviour of the solution to this initial value problem:



[An Excel spreadsheet file for this solution is available here.]
[A Maple worksheet, from which the plot above was derived, is available here.]

10 (continued)

Note that the ODE is Bernoulli:

$$\frac{dy}{dx} + \underbrace{\boxed{x^3}}_{P} y = \underbrace{\boxed{x^3}}_{R} y^{\overset{n}{\boxed{2}}}$$

$$\Rightarrow \quad h = \int (1-n) P \, dx = -\int x^3 \, dx = -\frac{x^4}{4} \qquad \Rightarrow \quad e^h = e^{-x^4/4}$$

$$\Rightarrow \quad \int e^h R \, dx = \int x^3 e^{-x^4/4} dx = -e^{-x^4/4}$$

$$w = \frac{y^{1-n}}{1-n} = e^{-h}\left(\int e^h R \, dx + C\right) \quad \Rightarrow \quad -\frac{1}{y} = e^{+x^4/4}\left(-e^{-x^4/4} + C\right)$$

The general solution is

$$\boxed{y = \frac{1}{1 + Ae^{+x^4/4}}}$$

The initial condition $y(0) = 0.2$ leads to $A = 4$.
The exact value of $y(0.6)$ is 0.194 866 348...
The RK4 value is correct to seven decimal places!

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