ENGI 9420 Engineering Analysis
Assignment 4 Solutions

2012 Fall
[Eigenvalues; stability analysis; Chapter 4]

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1. For the matrix [20]

$$A = \begin{bmatrix} -9 & 3 \\ -8 & 1 \end{bmatrix}$$

(a) Find the eigenvalues.

The eigenvalues λ_1, λ_2 are the solutions to the equation $\det(A - \lambda I) = 0$

$$\Rightarrow \begin{vmatrix} -9-\lambda & 3 \\ -8 & 1-\lambda \end{vmatrix} = 0 \quad \Rightarrow \quad (-9-\lambda)(1-\lambda) - 3(-8) = 0$$

$$\Rightarrow \quad \lambda^2 + 8\lambda - 9 + 24 = 0 \quad \Rightarrow \quad \lambda^2 + 8\lambda + 15 = 0 \quad \Rightarrow \quad (\lambda+5)(\lambda+3) = 0$$

OR, using the formulae on page 4.30 of the Lecture Notes,

$$D = (a-d)^2 + 4bc = 100 - 96 = 4$$

$$\Rightarrow \quad \lambda = \frac{a+d \pm \sqrt{D}}{2} = \frac{-8 \pm 2}{2} = -5 \text{ or } -3$$

Therefore the eigenvalues are

$$\boxed{\lambda = -5, -3}$$

(b) Find the unit eigenvectors associated with each eigenvalue.

For each eigenvalue λ, the associated eigenvector $\vec{\mathbf{x}}$ is any non-trivial solution to $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$

Let $\vec{\mathbf{x}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

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1 (b) (continued)

$\underline{\lambda = -5}$:

$$A\vec{\mathbf{x}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \begin{bmatrix} -9+5 & 3 \\ -8 & 1+5 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left\{ \begin{matrix} -4\alpha + 3\beta = 0 \\ \text{and} \\ -8\alpha + 6\beta = 0 \end{matrix} \right\} \quad \Rightarrow \quad \beta = \tfrac{4}{3}\alpha$$

OR, using the formulae on page 4.30 of the Lecture Notes,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \dfrac{a-d-\sqrt{D}}{2} \\ c \end{bmatrix} = \begin{bmatrix} \dfrac{-9-1-2}{2} \\ -8 \end{bmatrix} = -2\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Therefore any non-zero multiple of $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = -5$.

$$\left\| \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Therefore a unit eigenvector for the eigenvalue $\lambda = -5$ is

$$\boxed{\vec{\mathbf{x}}_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}}$$

$\underline{\lambda = -3}$:

$$A\vec{\mathbf{x}} = \vec{\mathbf{0}} \quad \Rightarrow \quad \begin{bmatrix} -9+3 & 3 \\ -8 & 1+3 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left\{ \begin{matrix} -6\alpha + 3\beta = 0 \\ \text{and} \\ -8\alpha + 4\beta = 0 \end{matrix} \right\} \quad \Rightarrow \quad \beta = 2\alpha$$

OR, using the formulae on page 4.30 of the Lecture Notes,

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \dfrac{a-d+\sqrt{D}}{2} \\ c \end{bmatrix} = \begin{bmatrix} \dfrac{-9-1+2}{2} \\ -8 \end{bmatrix} = -4\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore any non-zero multiple of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for the eigenvalue $\lambda = -3$.

1 (b) (continued)

$$\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Therefore a unit eigenvector for the eigenvalue $\lambda = -3$ is

$$\boxed{\hat{\mathbf{x}}_2 = \frac{\sqrt{5}}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}$$

(c) Construct the 2×2 matrix X whose columns are the unit eigenvectors found in part (b) and evaluate the product $X^{-1}AX$.

$$X = \begin{bmatrix} 3/5 & 1/\sqrt{5} \\ 4/5 & 2/\sqrt{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 & \sqrt{5} \\ 4 & 2\sqrt{5} \end{bmatrix}$$

$$\Rightarrow \quad |X| = \frac{1}{25}\left(6\sqrt{5} - 4\sqrt{5}\right) = \frac{2\sqrt{5}}{25}$$

$$\Rightarrow \quad X^{-1} = \frac{\operatorname{adj} X}{\det X} = \frac{25\sqrt{5}}{2\times 5} \cdot \frac{1}{5} \begin{bmatrix} 2\sqrt{5} & -\sqrt{5} \\ -4 & 3 \end{bmatrix} = \frac{\sqrt{5}}{2} \begin{bmatrix} 2\sqrt{5} & -\sqrt{5} \\ -4 & 3 \end{bmatrix}$$

(or, equivalently, $X^{-1} = \frac{1}{2} \begin{bmatrix} 10 & -5 \\ -4\sqrt{5} & 3\sqrt{5} \end{bmatrix}$).

$$\Rightarrow \quad X^{-1}AX = \frac{1}{2} \begin{bmatrix} 10 & -5 \\ -4\sqrt{5} & 3\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} -9 & 3 \\ -8 & 1 \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 3 & \sqrt{5} \\ 4 & 2\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} -50 & 25 \\ 12\sqrt{5} & -9\sqrt{5} \end{bmatrix} \cdot \begin{bmatrix} 3 & \sqrt{5} \\ 4 & 2\sqrt{5} \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -50 & 0 \\ 0 & -30 \end{bmatrix}$$

$$\Rightarrow \quad X^{-1}AX = \begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix} = \Lambda$$

which is a diagonal matrix whose diagonal entries are the eigenvalues. Therefore

$$\boxed{X^{-1}AX = \Lambda = \begin{bmatrix} -5 & 0 \\ 0 & -3 \end{bmatrix}}$$

[If one elects to write the eigenvectors in the opposite order, then the columns of the matrices X, X^{-1} and Λ will all be in the opposite order.]

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1 (d) Use part (a) to determine the nature and stability of the critical point of the linear system

$$\frac{dx}{dt} + 9x = 3y \ , \quad \frac{dy}{dt} + 8x = y$$

The linear system is also $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathrm{A} \begin{bmatrix} x \\ y \end{bmatrix}$.

From part (a), the eigenvalues of the matrix A are negative and distinct.

Therefore the critical point is a

stable node.

(e) Sketch the orbits near the critical point; and

Sample eigenvectors are $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ for $\lambda = -5$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ for $\lambda = -3$

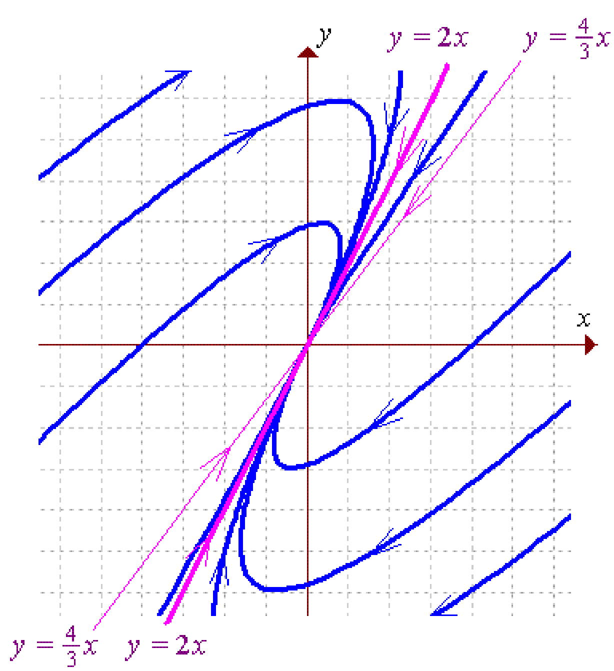
The larger eigenvalue $(\lambda = -3)$ governs behaviour near the origin, while the other eigenvalue governs behaviour asymptotically (far away from the origin).

The eigenvector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is therefore parallel to the tangent at the origin to nearly all trajectories, while the eigenvector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is parallel to nearly all trajectories at infinity.

Therefore $y = 2x$ is the tangent at the origin and $y = 4x/3$ is the asymptote.

All trajectories terminate at the origin, because the critical point is stable.

The sketch of the phase portrait then follows.



1 (f) Find the general solution of the linear system.

From the general form for a general solution,

$$\left(x(t),\, y(t)\right) = \left(c_1\alpha_1 e^{\lambda_1 t} + c_2\alpha_2 e^{\lambda_2 t},\; c_1\beta_1 e^{\lambda_1 t} + c_2\beta_2 e^{\lambda_2 t}\right)$$

we have immediately

$$\boxed{\left(x(t),\, y(t)\right) = \left(3c_1 e^{-5t} + c_2 e^{-3t},\; 4c_1 e^{-5t} + 2c_2 e^{-3t}\right)}$$

Check (not essential):

$$\left(\dot{x}(t),\, \dot{y}(t)\right) = \left(-15c_1 e^{-5t} - 3c_2 e^{-3t},\; -20c_1 e^{-5t} - 6c_2 e^{-3t}\right)$$

But

$$-9x + 3y = (-27+12)c_1 e^{-5t} + (-9+6)c_2 e^{-3t} = \dot{x}(t)$$

$$-8x + y = (-24+4)c_1 e^{-5t} + (-8+2)c_2 e^{-3t} = \dot{y}(t)$$

Therefore our general solution does satisfy the linear system of ODEs.

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2. In a system with two or more states, (for example, the energy levels of the electron shells in an atom), probabilities can be assigned for a transition from one state to another. The simplest example is a two-state model, with states 'A' and 'B'. In any time step, there is a probability a that an object in state A will stay in state A (and therefore the complementary probability $(1 - a)$ that the object will move to state B). There is a probability b that an object in state B will stay in state B (and therefore the complementary probability $(1 - b)$ that the object will move to state A). Both a and b are numbers strictly between 0 and 1. These probabilities can be expressed in a transition matrix M (also known as a Markov matrix): [20]

	from state A	B
to state A	a	$1-b$
to state B	$1-a$	b

$$\mathrm{M} = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix}$$

Note how each column of M has entries that add up to 1 exactly, no matter what choices are made for a and b.

Let the state vector $\vec{\mathbf{x}}_n$ represent the proportion of objects that are in each state at time step n:

$$\vec{\mathbf{x}}_n = \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix} \begin{matrix} \text{proportion in state A at time } n \\ \text{proportion in state B at time } n \end{matrix}$$

Again, the entries in the column add up to 1: $\alpha_n + \beta_n = 1$ ($\vec{\mathbf{x}}_n$ is a Markov vector).

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2. (continued)

The proportion of objects in each state in the next time step is related to the present proportions by

$$\vec{\mathbf{x}}_{n+1} = \mathrm{M}\,\vec{\mathbf{x}}_n \quad \Rightarrow \quad \begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix} \cdot \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}$$

As the system evolves through a long sequence of time steps, the system settles down to a steady state

$$\vec{\mathbf{x}} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \text{ such that } \mathrm{M}\vec{\mathbf{x}} = \vec{\mathbf{x}} \quad \Rightarrow \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

(a) Show that one of the eigenvalues of the transition matrix M is $\lambda = 1$, no matter what the values of the transition probabilities a and b may be.

Direct Method:

$$\left| \mathrm{M} - \lambda \mathrm{I} \right| = \begin{vmatrix} a-\lambda & 1-b \\ 1-a & b-\lambda \end{vmatrix} = (a-\lambda)(b-\lambda) - (1-a)(1-b)$$

$$= \lambda^2 - (a+b)\lambda \cancel{+ab} - 1 + (a+b) \cancel{-ab}$$

$$= \lambda^2 - (a+b)\lambda + (a+b-1)$$

$$\left| \mathrm{M} - \lambda \mathrm{I} \right| = 0 \quad \Rightarrow \quad (\lambda - 1)\big(\lambda - (a+b-1)\big) = 0 \quad \Rightarrow$$

$$\boxed{\lambda = 1 \quad \text{or} \quad \lambda = a+b-1}$$

(for any choice of a and b).

Method using formulae on Page 4.30 of the Lecture Notes:

$$D = \text{"}\left((a-d)^2 + 4bc\right)\text{"} = (a-b)^2 + 4(1-b)(1-a)$$

$$= a^2 - 2ab + b^2 + 4 - 4a - 4b + 4ab$$

$$= (a+b)^2 - 4(a+b) + 4 = (a+b-2)^2$$

$$\lambda = \text{"}\frac{(a+d) \pm \sqrt{D}}{2}\text{"} = \frac{a+b \pm (a+b-2)}{2} = 1 \text{ or } (a+b-1)$$

Also note that the steady-state solution is an eigenvector for the eigenvalue $\lambda = 1$.

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2 (b) Show that there is exactly one eigenvector of M for the eigenvalue $\lambda = 1$, whose entries add up to 1 (a Markov eigenvector) and express it in terms of a and b.

$$(\mathrm{M}-1\mathrm{I})\vec{\mathbf{x}} = \begin{bmatrix} a-1 & 1-b \\ 1-a & b-1 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (1-a)\alpha = (1-b)\beta$$

$$\Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = k \begin{bmatrix} 1-b \\ 1-a \end{bmatrix}$$

where k is *any* non-zero number.

OR, using the formulae on page 4.30:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = k \text{"}\begin{bmatrix} \frac{(a-d)-\sqrt{D}}{2} \\ c \end{bmatrix}\text{"} = k \begin{bmatrix} \frac{a-b-a-b+2}{\sqrt{\quad} \quad 2} \\ 1-a \end{bmatrix} = k \begin{bmatrix} 1-b \\ 1-a \end{bmatrix}$$

But a Markov eigenvector requires $\alpha + \beta = 1$

$$\Rightarrow k\big((1-b)+(1-a)\big) = 1 \quad \Rightarrow k = \frac{1}{(1-a)+(1-b)} \text{ only.}$$

Therefore the unique Markov eigenvector for the eigenvalue $\lambda = 1$ is

$$\boxed{\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{(1-a)+(1-b)} \begin{bmatrix} 1-b \\ 1-a \end{bmatrix}}$$

(which is the steady-state solution).

In order to verify that this is the steady-state solution, [not required], show that $\mathrm{M}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$:

$$\mathrm{M}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix} \frac{1}{(1-a)+(1-b)} \begin{bmatrix} 1-b \\ 1-a \end{bmatrix}$$

$$= \frac{1}{(1-a)+(1-b)} \begin{bmatrix} a(1-b)+(1-b)(1-a) \\ (1-a)(1-b)+b(1-a) \end{bmatrix}$$

$$= \frac{1}{(1-a)+(1-b)} \begin{bmatrix} (1-b)(a+1-a) \\ (1-a)(1-b+b) \end{bmatrix}$$

$$= \frac{1}{(1-a)+(1-b)} \begin{bmatrix} (1-b) \\ (1-a) \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

2 (c) Show that there is **no** Markov eigenvector of M for the other eigenvalue.

$$\left(\mathrm{M}-(a+b-1)\mathrm{I}\right)\bar{\mathbf{x}} = \begin{bmatrix} a-a-b+1 & 1-b \\ 1-a & b-a-b+1 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$= \begin{bmatrix} 1-b & 1-b \\ 1-a & 1-a \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \alpha+\beta = 0$$

(so that the eigenvectors are $k\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, where k is *any* non-zero number).

OR, using the formulae on page 4.30:

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = k \text{"}\begin{bmatrix} \dfrac{(a-d)+\sqrt{D}}{2} \\ c \end{bmatrix}\text{"} = k\begin{bmatrix} \dfrac{a-b+a+b-2}{2} \\ 1-a \end{bmatrix} = k\begin{bmatrix} a-1 \\ 1-a \end{bmatrix}$$

But a Markov eigenvector requires $\alpha+\beta = 1$, which is inconsistent with $\alpha+\beta = 0$.
Therefore the eigenvalue $\lambda = a+b-1$ has no Markov eigenvector.

(d) In the particular case where an object in state A has a 40% chance of staying in state A, but an object in state B has an 80% chance of staying in state B, find the steady-state proportions of objects in each state.

$$\mathrm{M} = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix} = \begin{bmatrix} 2/5 & 1/5 \\ 3/5 & 4/5 \end{bmatrix}$$

From part (b) above, the Markov eigenvector for $\lambda = 1$ is

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{(1-a)+(1-b)}\begin{bmatrix} 1-b \\ 1-a \end{bmatrix} = \frac{1}{3/5+1/5}\begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix} = \frac{5}{4}\begin{bmatrix} 1/5 \\ 3/5 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Therefore

$$\boxed{\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}}$$

In the steady state, 25% of all objects will be in state A and the other 75% will be in state B.

3. For the non-linear system [15]

$$\frac{dx}{dt} = 3x + y + 7, \quad \frac{dy}{dt} = 2x^2 + 3y + 3$$

i. determine the critical point(s);
ii. find the linear system associated with each critical point;
iii. determine the nature and stability of the critical point(s);
iv. sketch the orbits near the critical point(s); and
v. sketch the orbits on a diagram that includes all critical point(s).

At any critical points,

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \quad \Rightarrow \quad y = -3x - 7 \quad \text{and} \quad 2x^2 + 3y + 3 = 0$$

$$\Rightarrow \quad 2x^2 + 3(-3x-7) + 3 = 0 \quad \Rightarrow \quad 2x^2 - 9x - 18 = 0$$

$$\Rightarrow \quad (2x+3)(x-6) = 0 \quad \Rightarrow \quad x = -\tfrac{3}{2} \text{ or } +6$$

$$\Rightarrow \quad (x, y) = \left(-\tfrac{3}{2}, -\tfrac{5}{2}\right) \text{ or } (6, -25)$$

Near each critical point, the equivalent linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}\Bigg|_{(a,b)} \begin{pmatrix} x-a \\ y-b \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4x & 3 \end{pmatrix}\Bigg|_{(a,b)} \begin{pmatrix} x-a \\ y-b \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4a & 3 \end{pmatrix} \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

Near the critical point (–3/2, –5/2), the linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -6 & 3 \end{pmatrix} \begin{pmatrix} x+\frac{3}{2} \\ y+\frac{5}{2} \end{pmatrix}$$

Using the formulae on page 4.30,

$$D = (a-d)^2 + 4bc = 0 + 4(1)(-6) = -24$$

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{6 \pm \sqrt{-24}}{2} = 3 \pm j\sqrt{6}$$

The eigenvalues are a complex conjugate pair with a positive real part

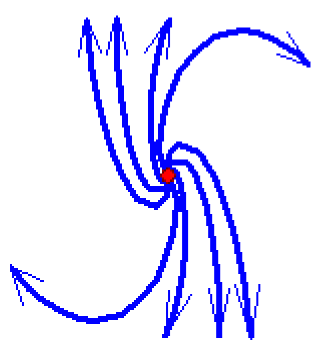
$\Rightarrow$ the critical point is an **unstable focus**.

Just above and to the right of the focus,

$\left(\text{where} \quad x + \tfrac{3}{2} > 0 \text{ and } y + \tfrac{5}{2} > 0\right)$,

$\dot{x} > 0$, so that x is increasing.

The orbits therefore spiral clockwise out from the focus.



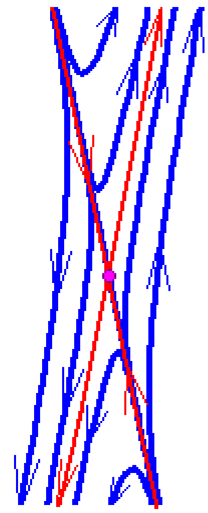
3. (continued)

Near the critical point (6, –25), the linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 24 & 3 \end{pmatrix} \begin{pmatrix} x-6 \\ y+25 \end{pmatrix}$$

$$D = (a-d)^2 + 4bc = 0 + 4(1)(24) = 96$$

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{6 \pm \sqrt{96}}{2} = 3 \pm 2\sqrt{6}$$



The eigenvalues are a real pair with opposite signs.

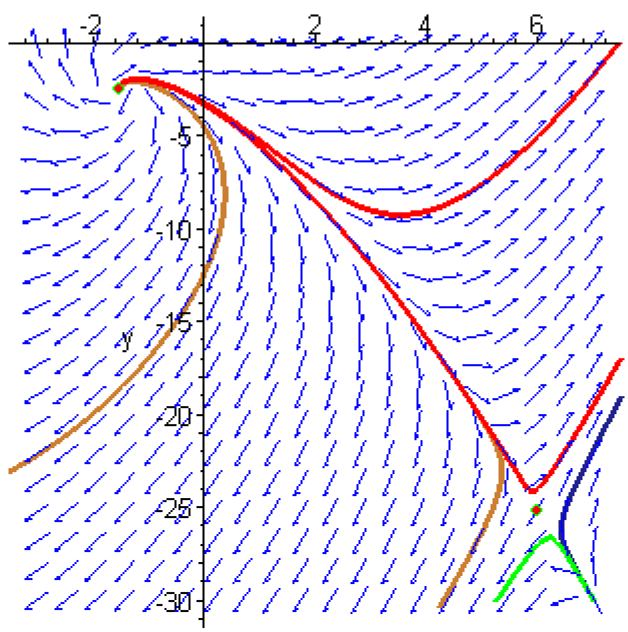
$\Rightarrow$ the critical point is an [unstable] **saddle point**.

The eigenvector that is parallel to a line with negative slope $\left(-2\sqrt{6}\right)$ is associated with the negative eigenvalue $\left(\lambda = 3 - 2\sqrt{6}\right)$.

Therefore the asymptote with negative slope is asymptotic to the incoming trajectories.

This Maple plot illustrates the behaviour of orbits in the non-linear system:

```
> with(DEtools):
> phaseportrait([diff(x(t),t) = 3*x(t) + y(t) + 7,
    diff(y(t),t) = 3*y(t) + 2*x(t)^2 + 3], [x(t), y(t)],
    t=-4..7, [[x(0)=6, y(0)=-24], [x(0)=4.5, y(0)=-29],
    [x(0)=6, y(0)=-27], [x(0)=6.5, y(0)=-25],
    [x(0)=-2, y(0)=-20], [x(0)=6, y(0)=-5]],
    x=-3..7, y=-30..0, stepsize=.01, colour=blue,
    linecolour=[red,gold,green,navy,gold,red],
    title=`Phase portrait for non-linear system`);
```



The Maple file is available at [this link].

4. For the non-linear system [15]

$$\frac{d^2x}{dt^2} + x - \frac{x^3}{3} = 0$$

i. determine the critical point(s);
ii. find the linear system associated with each critical point;
iii. determine the nature and stability of the critical point(s);
iv. sketch the orbits near the critical point(s); and
v. sketch the orbits on a diagram that includes all critical point(s).

This second order ODE is an undamped Duffing equation.
The non-linear system for this second order ODE is

$$\frac{dx}{dt} = y\,, \quad \frac{dy}{dt} = -x + \frac{x^3}{3}$$

At any critical points,

$$\frac{dx}{dt} = \frac{dy}{dt} = 0 \quad \Rightarrow \quad y = 0 \text{ and } x\left(-1 + \frac{x^2}{3}\right) = 0 \quad \Rightarrow \quad (x, y) = (0,0) \quad \text{or} \quad \left(\pm\sqrt{3}, 0\right)$$

Near the critical point (0, 0), the equivalent linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$D = (a-d)^2 + 4bc = 0 + 4(1)(-1) = -4$$

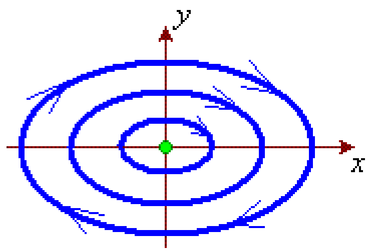
$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{0 \pm \sqrt{-4}}{2} = \pm j$$

The eigenvalues are a pure imaginary pair

$\Rightarrow$ the critical point of the linear system is a centre

$\Rightarrow$ the critical point of the non-linear system is one of a centre, a stable focus or an unstable focus.

There is no damping term (first derivative) in the original second order ODE,
making it likely that this critical point is a **centre**.



From the linear system,

$y = 0 \quad \Rightarrow \quad \dot{x} > 0 \quad \Rightarrow \quad x$ is increasing

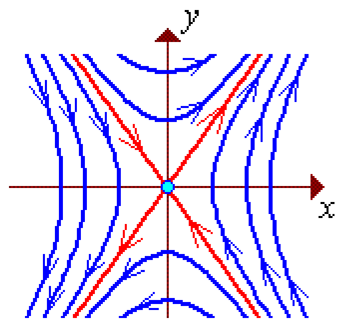
$\Rightarrow$ orbits go clockwise around the centre.

4. (continued)

Near the critical points $\left(\pm\sqrt{3}, 0\right)$, the equivalent linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}\right|_{\left(\pm\sqrt{3},0\right)} \begin{pmatrix} x \mp \sqrt{3} \\ y \end{pmatrix}$$

$$= \left.\begin{pmatrix} 0 & 1 \\ -1+x^2 & 0 \end{pmatrix}\right|_{\left(\pm\sqrt{3},0\right)} \begin{pmatrix} x \mp \sqrt{3} \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \mp \sqrt{3} \\ y \end{pmatrix}$$



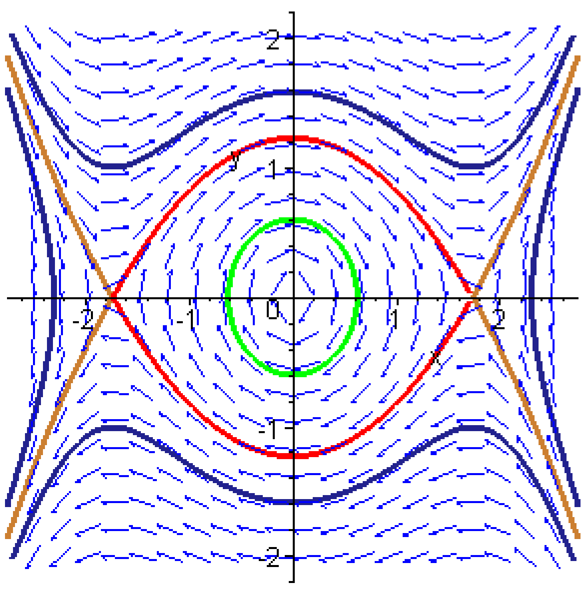
$$D = (a-d)^2 + 4bc = 0 + 4(1)(2) = +8$$

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{0 \pm \sqrt{8}}{2} = \pm\sqrt{2}$$

The eigenvalues are a real distinct pair of opposite signs
$\Rightarrow$ the critical points are **saddle points**.
From the linear system, it is clear that x is increasing when $y > 0$.
The trajectories must therefore approach the saddle point with negative slope.

As confirmation, a Maple plot for the phase portrait of this non-linear system is

```
> with(DEtools):
> phaseportrait([diff(x(t),t) = y(t), diff(y(t),t) = -x(t) + x(t)^3/3],
    [x(t), y(t)], t=-8..8, [[x(0)=-1.72, y(0)=0], [x(0)=0, y(0)=0.6],
    [x(0)=-1.74, y(0)=0], [x(0)=1.74, y(0)=0],
    [x(0)=-1.74, y(0)=1], [x(0)=1.74, y(0)=-1],
    [x(0)=-2.3, y(0)=0], [x(0)=2.3, y(0)=0]],
    x=-2.5..2.5, y=-2..2, stepsize=.01, colour=blue,
    linecolour=[red, green, gold, gold, navy, navy, navy, navy]);
```



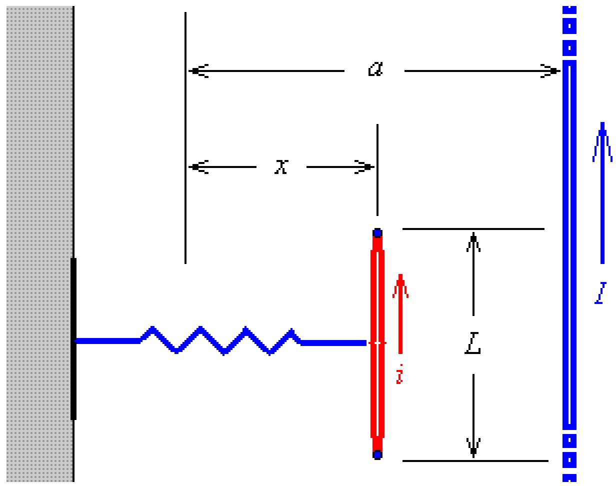
5. A mutual force of attraction is exerted between parallel current carrying wires.

The infinite wire carries current I. The finite wire of length L carries current i in the same direction and is restrained by a spring. According to the Biot-Savart law, the mutual force of attraction is

$$\frac{2IiL}{(\text{separation})} = \frac{2IiL}{a-x}$$

where $x = 0$ is the position at which the spring force is zero. The mass of the finite wire is m and the restoring constant of the spring is k. The equation of motion of the restrained wire is

$$\ddot{x} + \frac{k}{m}\left(x - \frac{b}{a-x}\right) = 0 \quad \text{where} \quad b = \frac{2IiL}{k}$$



For each of the cases $b < a^2/4$, $b = a^2/4$ and $b > a^2/4$,

(a) Locate and classify the singularities (using $dx/dt = y$).

The non-linear system for this second order ODE is

$$\dot{x} = y, \qquad \dot{y} = -\frac{k}{m}\left(x - \frac{b}{a-x}\right) = -\frac{k}{m}x + \frac{kb}{m(a-x)}$$

At any critical points,

$$\dot{x} = \dot{y} = 0 \quad \Rightarrow \quad y = 0 \quad \text{and} \quad x = \frac{b}{a-x}$$

$$\Rightarrow \quad x(a-x) = b \quad \Rightarrow \quad x^2 - ax + b = 0$$

$$\Rightarrow \quad x = \frac{a \pm \sqrt{a^2 - 4b}}{2}$$

If $b > a^2/4$, then there are no critical points at all.

If $b = a^2/4$, then there is one critical point, at $(a/2, 0)$.

5 (a) (continued)

The associated linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left.\begin{pmatrix} \frac{\partial P}{\partial x} & \frac{\partial P}{\partial y} \\ \frac{\partial Q}{\partial x} & \frac{\partial Q}{\partial y} \end{pmatrix}\right|_{\left(\frac{a}{2},0\right)} \begin{pmatrix} x-\frac{a}{2} \\ y \end{pmatrix} = \left.\begin{pmatrix} 0 & 1 \\ \frac{k}{m}\left(-1+\frac{b}{(a-x)^2}\right) & 0 \end{pmatrix}\right|_{\left(\frac{a}{2},0\right)} \begin{pmatrix} x-\frac{a}{2} \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x-\frac{a}{2} \\ y \end{pmatrix}$$

This is a degenerate linear system, whose solution is simply $y = dx/dt$ = constant, (a set of parallel horizontal lines in the phase plane). The electric and spring forces balance everywhere in the neighbourhood of the critical point.

If $b < a^2/4$, then there are two critical points, at $(x, y) = \left(\frac{a \pm \sqrt{a^2 - 4b}}{2}, 0\right)$.

The associated linear system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \left.\begin{pmatrix} 0 & 1 \\ \frac{k}{m}\left(\frac{b-(a-x)^2}{(a-x)^2}\right) & 0 \end{pmatrix}\right|_{\left(\frac{a}{2}\pm\frac{\sqrt{a^2-4b}}{2},0\right)} \begin{pmatrix} x-\frac{a}{2} \\ y \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ \frac{k}{m}\left(\frac{4b-\left(a\mp\sqrt{a^2-4b}\right)^2}{\left(a\mp\sqrt{a^2-4b}\right)^2}\right) & 0 \end{pmatrix} \begin{pmatrix} x-\frac{a}{2} \\ y \end{pmatrix}$$

Let $c = \frac{k}{m}\left(\frac{2\sqrt{a^2-4b}\left((\pm a) - \sqrt{a^2-4b}\right)}{\left(a\mp\sqrt{a^2-4b}\right)^2}\right) \quad \begin{cases} >0 & (\text{for } +a) \\ <0 & (\text{for } -a) \end{cases}$

then $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix} \begin{pmatrix} x-\frac{a}{2} \\ y \end{pmatrix}$

$$D = (a-d)^2 + 4bc = 4c$$

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{0 \pm 2\sqrt{c}}{2} = \pm\sqrt{c}$$

5 (a) (continued)

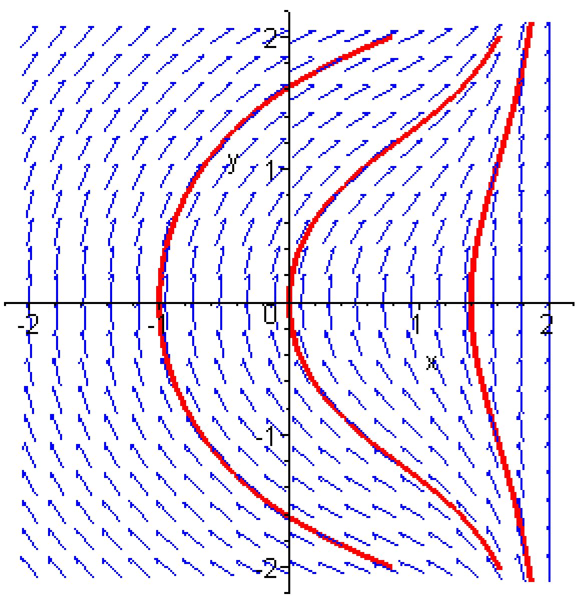
The eigenvalues are a pure imaginary pair at the critical point with the smaller value of x and a real pair of opposite sign at the other critical point. Also, there is no damping term (first derivative) in the original second order ODE.

$\Rightarrow$ the critical point at $\left(\dfrac{a - \sqrt{a^2 - 4b}}{2}, 0 \right)$ is a **centre**

and the critical point at $\left(\dfrac{a + \sqrt{a^2 - 4b}}{2}, 0 \right)$ is a **saddle point**.

(b) Sketch the phase portrait.

$b > a^2 / 4$:

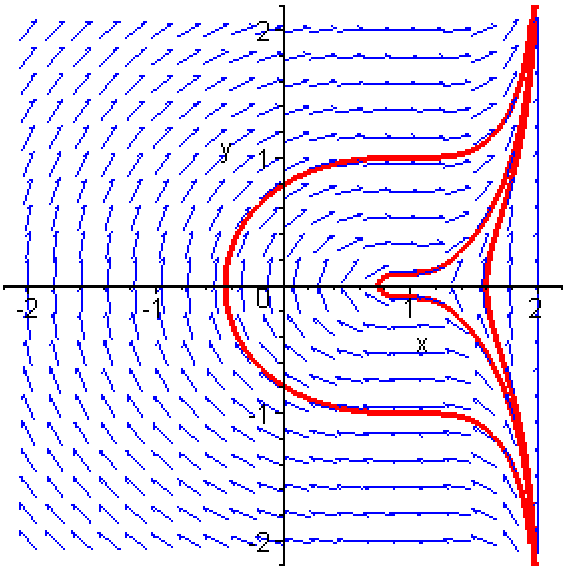


Note that high values of b correspond to cases where the electric force overwhelms the spring, forcing the short wire into contact with the infinite wire regardless of starting position and velocity.

In these diagrams,
the neutral position of the spring is at $x = 0$,
while the infinite wire is at $x = 2$.

5 (b) (continued)

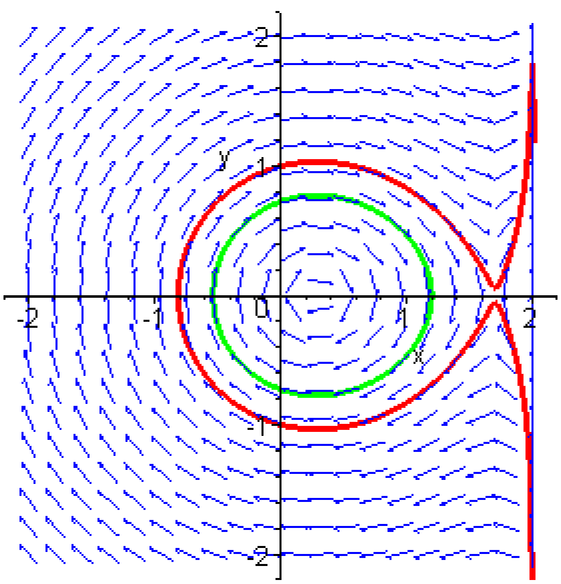
$b = a^2 / 4$:



A degenerate singularity is halfway between the neutral position of the spring and the infinitely long wire. To the left it resembles a centre, but to the right it resembles a saddle point.

The point at (a/2, 0) is a point of unstable equilibrium, where the electric and spring forces balance exactly. The slightest disturbance in either direction ultimately sends the finite wire into contact with the infinite wire at $x = a$.

$b < a^2 / 4$:



When b is small enough ($< a^2/4$), the spring can resist the electric force out as far as the separatrix, (the trajectory that passes through the saddle point).

The electric force pulls the point of stable equilibrium out to the right of $x = 0$, to the point $\left(\dfrac{a - \sqrt{a^2 - 4b}}{2}, 0 \right)$.

The other singularity (the saddle point) is a point of unstable equilibrium, at $\left(\dfrac{a + \sqrt{a^2 - 4b}}{2}, 0 \right)$

5 (c) Where it exists, find the equation of the separatrix.
[The separatrix is the orbit that separates closed orbits from open orbits.
It usually passes through at least one singularity.]

The separatrix exists only in the case $b < a^2/4$.
It is the orbit that passes through the saddle point.
For all orbits,

$$\dot{x} = y, \quad \dot{y} = -\frac{k}{m}\left(x - \frac{b}{a-x}\right) \quad \Rightarrow \quad \frac{dy}{dx} = \frac{k}{my}\left(-x + \frac{b}{a-x}\right)$$

$$\Rightarrow \quad \int y\,dy = \frac{k}{m}\int\left(-x + \frac{b}{a-x}\right)dx$$

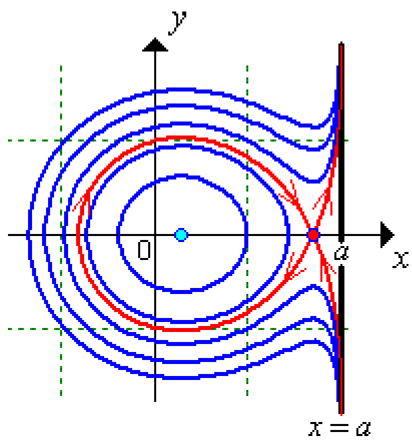
$$\Rightarrow \quad \frac{y^2}{2} = \frac{k}{m}\left(-\frac{x^2}{2} - b\ln(a-x)\right) + \frac{C}{2} \quad \Rightarrow \quad y^2 = C - \frac{k}{m}\left(x^2 + 2b\ln(a-x)\right)$$

But the separatrix passes through the saddle point at $\left(\dfrac{a + \sqrt{a^2-4b}}{2}, 0\right)$:

$$\Rightarrow \quad C = \frac{k}{m}\left(\left(\frac{a + \sqrt{a^2-4b}}{2}\right)^2 + 2b\ln\left(\frac{a - \sqrt{a^2-4b}}{2}\right)\right)$$

Therefore the equation of the separatrix is

$$y^2 = \frac{k}{m}\left(\left(\frac{a + \sqrt{a^2-4b}}{2}\right)^2 - x^2 + 2b\ln\left(\frac{a - \sqrt{a^2-4b}}{2(a-x)}\right)\right)$$



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