ENGI 9420 Engineering Analysis
Assignment 5 Solutions

2012 Fall
[Stability analysis, gradient operators; Chapters 4 and 5]

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1. Use Liénard's theorem to determine the stability of the solutions of the equation [10]

$$\frac{d^2x}{dt^2} + \left(x^2 - \frac{1}{3}\right)\frac{dx}{dt} + x^5 = 0$$

[Note that the associated linear system has a zero eigenvalue, which is an indeterminate form.]

Compare the given ODE to the standard Liénard ODE:

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0$$

Checking all the conditions of Liénard's theorem:

$f(x) = x^2 - \dfrac{1}{3}$ is even

$g(x) = x^5$ is odd and $g(x) > 0 \quad \forall x > 0$

$$F(x) = \int_0^x \left(t^2 - \frac{1}{3}\right)dt = \frac{x^3}{3} - \frac{x}{3} = \frac{x\left(x^2 - 1\right)}{3}$$

$F(x) = 0 \Rightarrow x\left(x^2-1\right) = 0 \Rightarrow$ exactly one positive root: $x = +1$

$0 < x < 1 \Rightarrow F(x) < 0; \quad x > 1 \Rightarrow F(x) > 0$ and increasing

All of the conditions of Liénard's theorem are satisfied.

Therefore $\dfrac{d^2x}{dt^2} + \left(x^2 - \dfrac{1}{3}\right)\dfrac{dx}{dt} + x^5 = 0$ does have a limit cycle

and it follows that there is an **unstable focus at the origin**.

Supporting evidence is provided by this Maple plot:

```
> with(DEtools):
> phaseportrait([diff(x(t),t) = y(t),
  diff(y(t),t) = -x(t)^5 - (x(t)^2 - 1/3)*y(t)], [x(t), y(t)],
  t=0..30, [[x(0)=0.6, y(0)=0]],
  x=-1.2..1.2, y=-1..1, stepsize=.01, colour=blue,
  linecolour=red);
> phaseportrait([diff(x(t),t) = y(t),
  diff(y(t),t) = -x(t)^5 - (x(t)^2 - 1/3)*y(t)], [x(t), y(t)],
  t=0..30, [[x(0)=1.2, y(0)=0]],
  x=-1.2..1.2, y=-1.2..1, stepsize=.01, colour=blue,
  linecolour=maroon);
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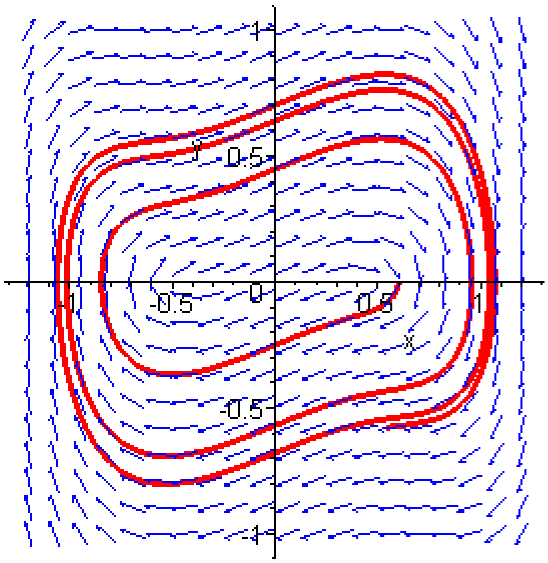
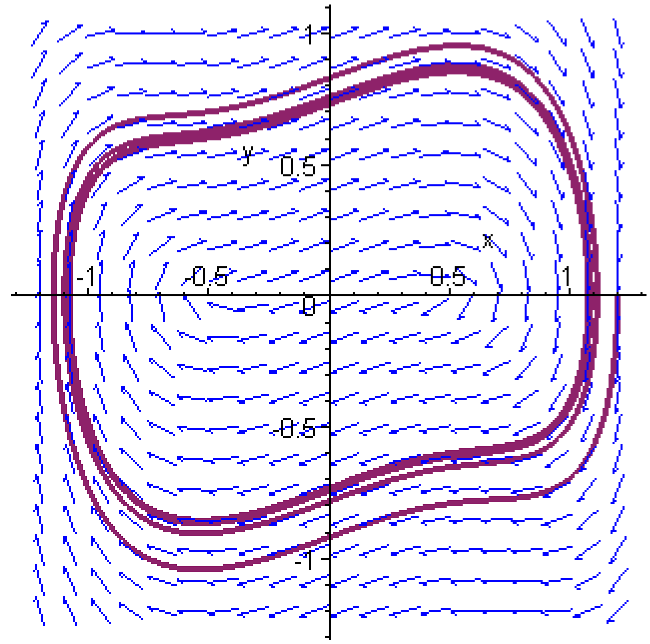
1. (continued)





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2. Find all limit cycles of the system of differential equations [10]

$$\frac{dx}{dt} = x - x^3 - xy^2 , \quad \frac{dy}{dt} = y - y^3 - yx^2$$

Hints: Compute $\frac{d}{dt}\left(x^2 + y^2\right)$; a limit cycle must be the orbit of a periodic solution that passes through no critical points of the system and encloses a critical point.]

Critical points exist where $\dot{x} = \dot{y} = 0$

$$\Rightarrow \quad x\left(1 - x^2 - y^2\right) = y\left(1 - x^2 - y^2\right) = 0$$

$$\Rightarrow \quad \left(x = 0 \quad \text{or} \quad x^2 + y^2 = 1\right) \quad \text{and} \quad \left(y = 0 \quad \text{or} \quad x^2 + y^2 = 1\right)$$

An isolated critical point exists at the origin and the entire circle, radius 1, centre the origin, is also a set of singularities.

Along any trajectory, the rate of change of (distance)^2 from the origin with respect to t is

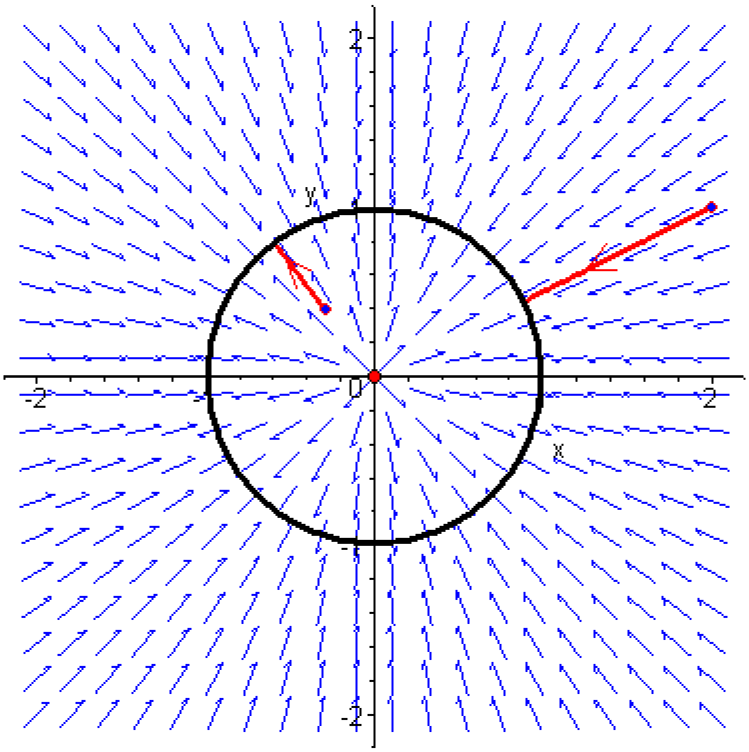
$$\frac{d}{dt}\left(x^2 + y^2\right) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$= 2x^2\left(1 - x^2 - y^2\right) + 2y^2\left(1 - x^2 - y^2\right)$$

$$= 2\left(x^2 + y^2\right)\left(1 - \left(x^2 + y^2\right)\right)$$

2. (continued)

$$\Rightarrow \quad \frac{d}{dt}\left(r^2\right) > 0 \quad \text{for} \quad r<1 \quad \text{and} \quad \frac{d}{dt}\left(r^2\right) < 0 \quad \text{for} \quad r>1$$



Therefore all orbits inside the unit circle move out towards it, all orbits outside the unit circle move in towards it and the orbit passing through any point on the unit circle remains stationary at that point, one unit away from the origin, forever. The derivatives of both x and y are zero at every point on the unit circle.

Because every point on the unit circle is itself a singularity, **there is no limit cycle for this system**, even though all trajectories terminate on the unit circle (except for the unstable equilibrium point at the origin).

This Maple plot supports this conclusion.

Additional note:

$$\frac{d}{dt}(\tan\theta) = \frac{d}{dt}\left(\frac{y}{x}\right) = \frac{d}{dx}\left(\frac{y}{x}\right)\cdot\frac{dx}{dt} = \frac{\dot{y}x - y\dot{x}}{x^2}\dot{x}$$

But $\dot{y}x - y\dot{x} = \left(xy - xy^3 - x^3y\right) - \left(xy - x^3y - xy^3\right) \equiv 0$

$$\Rightarrow \quad \frac{d}{dt}(\tan\theta) \equiv 0 \qquad \Rightarrow \quad \tan\theta = \text{constant} \quad \Rightarrow \quad \theta = \text{constant}$$

Therefore the angle θ never changes and all non-trivial trajectories are purely radial.

3. Use the Poincaré-Bendixon theorem to prove the existence of a non-trivial periodic solution of the differential equation [10]

$$\frac{d^2 z}{dt^2} + \left(z^2 - 1\right)\frac{dz}{dt} + 2\left(\frac{dz}{dt}\right)^3 + z = 0$$

Let $x = z$ and $y = \dfrac{dz}{dt}$

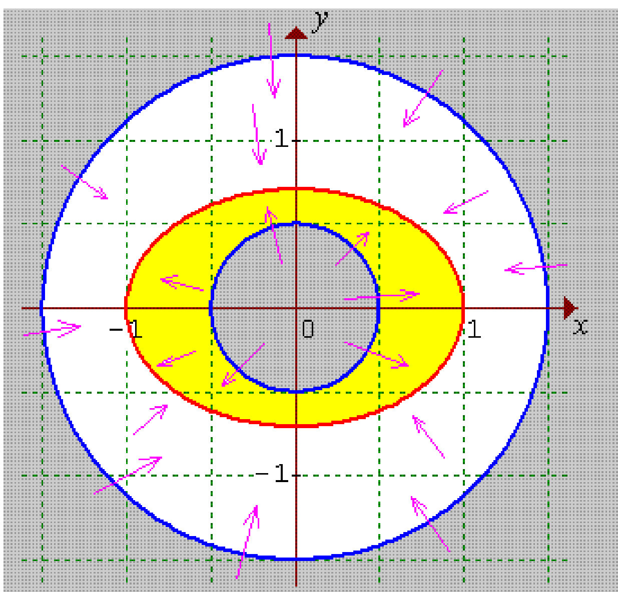
Then the ODE is equivalent to $\dot{x} = y$, $\dot{y} = -x + \left(1 - x^2\right)y - 2y^3$

Finding critical points: $\dot{x} = \dot{y} = 0 \quad \Rightarrow \quad (x, y) = (0, 0)$ only.

$$r^2 = x^2 + y^2 \quad \Rightarrow \quad r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = (xy) + \left(-xy + y^2 - x^2 y^2 - 2y^4\right)$$

$$\Rightarrow \quad r\frac{dr}{dt} = \left(1 - \left(x^2 + 2y^2\right)\right)y^2$$

Therefore the distance of any trajectory from the origin is increasing inside the ellipse $x^2 + 2y^2 = 1$, decreasing outside that ellipse and instantaneously not changing on that ellipse.



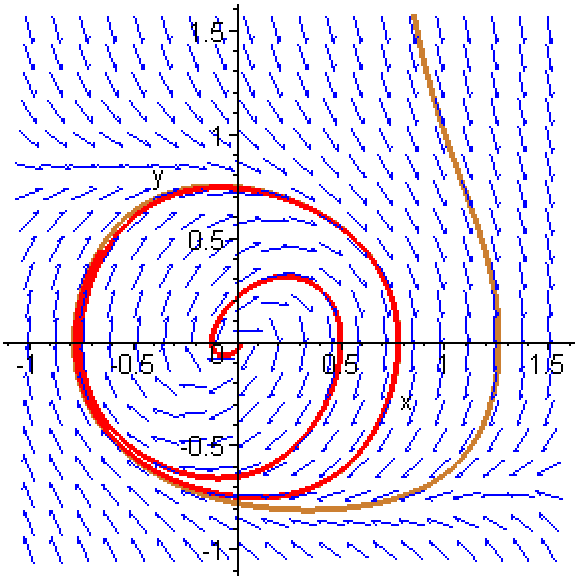
Define an annular region R between the circles centre the origin and of radii 0.5 and 1.5.

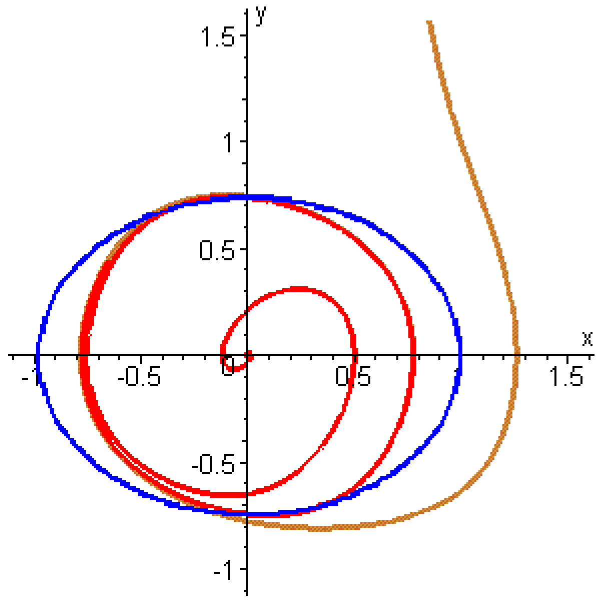
[Any radii smaller than $1/\sqrt{2}$ and larger than 1 respectively will do. We just need to define an annular region, centre the origin, that contains the ellipse completely.]

The ellipse where *dr/dt* = 0 is entirely inside R.
Everywhere on the outer circle *r* is decreasing.
Everywhere on the inner circle *r* is increasing.
All trajectories that enter R can never leave R.
There are no singularities inside R.
The inner circle encloses the singularity at the origin.
Therefore, by the Poincaré-Bendixon theorem, **a non-trivial periodic solution must exist**.

3. (continued)

The Maple phase portrait does reveal the existence of a limit cycle:





[The trajectories shown pass through (0.5, 0) and (1, 1).]
There is an unstable focus at the origin.
The limit cycle is not entirely contained within the ellipse.
The Maple file is available at this link.

4. Show that the system of differential equations [10]

$$\frac{dx}{dt} = x^3 - xy^2 - x^2 + y\,, \quad \frac{dy}{dt} = 2x^2y + y^3 + 2xy + y + 1$$

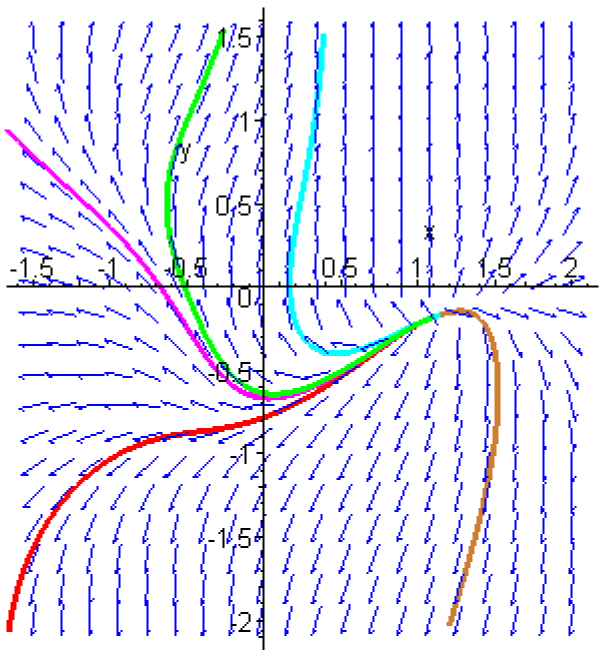
has no non-trivial periodic solution.

$$P = x^3 - xy^2 - x^2 + y\ , \quad Q = 2x^2y + y^3 + 2xy + y + 1$$

$$\Rightarrow \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \left(3x^2 - y^2 - 2x + 0\right) + \left(2x^2 + 3y^2 + 2x + 1 + 0\right)$$

$$= 5x^2 + 2y^2 + 1 > 0 \quad \forall (x, y)$$

Therefore, by the Bendixon Non-Existence Theorem,
the system has no non-trivial periodic solution.



[The trajectories shown all emanate from an unstable node near (1.15, –0.17).]
The Maple file used to generate the phase portrait above is available at this link.

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5. Show that the system of differential equations [10]

$$\frac{dx}{dt} = 5x - xy^2 - y^3 \ , \quad \frac{dy}{dt} = 4y - x^2 y + x^3$$

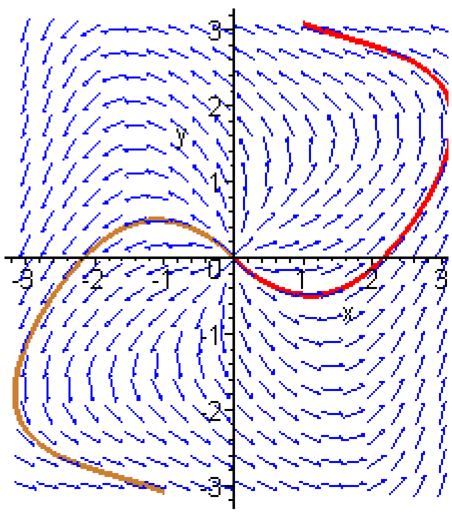
has no non-trivial periodic solution entirely inside the circle $x^2 + y^2 = 9$.

$$P = 5x - xy^2 - y^3 \ , \quad Q = 4y - x^2 y + x^3$$

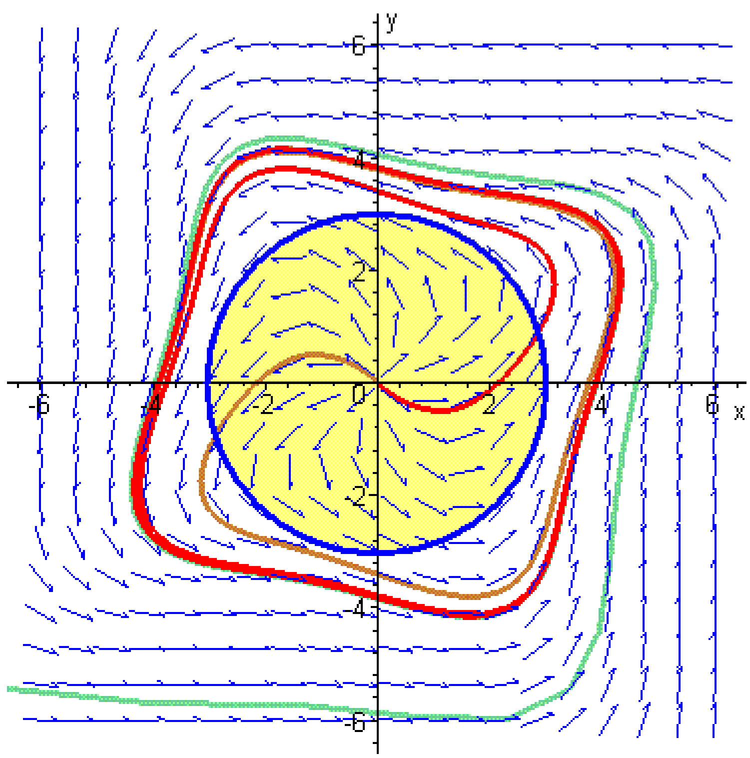
$$\Rightarrow \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 5 - y^2 + 0 + 4 - x^2 + 0 = 9 - \left(x^2 + y^2\right)$$

which is positive everywhere inside the circle $x^2 + y^2 = 9$.

Therefore, by the Bendixon Non-Existence Theorem, the system has no non-trivial periodic solution that lies entirely inside that circle.



There is an unstable focus at the origin.



A limit cycle ***does*** exist, but outside the circle.

The Maple file used to generate the phase portraits above is available at this link.

6. For the vector field $\vec{\mathbf{F}} = e^{-kr}\,\vec{\mathbf{r}}$, where $\vec{\mathbf{r}} = \left[\, x\ y\ z \,\right]^{\mathrm{T}}$ and k is a positive constant,

(a) Find the curl of $\vec{\mathbf{F}}$ in Cartesian coordinates. [3]

First note that $r^2 = x^2 + y^2 + z^2 \quad\Rightarrow\quad 2r\dfrac{\partial r}{\partial x} = 2x + 0 + 0 \quad\Rightarrow\quad \dfrac{\partial r}{\partial x} = \dfrac{x}{r}$

and, by symmetry, that $\dfrac{\partial r}{\partial y} = \dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

$$\frac{\partial}{\partial x}\left(e^{-kr}\right) = \frac{d}{dr}\left(e^{-kr}\right)\frac{\partial r}{\partial x} = -k\,e^{-kr}\,\frac{x}{r}$$

$\Rightarrow \quad \dfrac{\partial}{\partial x}\left(x\,e^{-kr}\right) = \left(1 - \dfrac{kx^2}{r}\right)e^{-kr}$ (needed in part (c) below)

and again by symmetry $\dfrac{\partial}{\partial y}\left(e^{-kr}\right) = -\dfrac{ky}{r}e^{-kr}$, $\dfrac{\partial}{\partial z}\left(e^{-kr}\right) = -\dfrac{kz}{r}e^{-kr}$,

$\dfrac{\partial}{\partial y}\left(y\,e^{-kr}\right) = \left(1 - \dfrac{ky^2}{r}\right)e^{-kr}$ and $\dfrac{\partial}{\partial z}\left(z\,e^{-kr}\right) = \left(1 - \dfrac{kz^2}{r}\right)e^{-kr}$

$$\operatorname{curl}\vec{\mathbf{F}} = \begin{vmatrix} \hat{\mathbf{i}} & \dfrac{\partial}{\partial x} & x\,e^{-kr} \\ \hat{\mathbf{j}} & \dfrac{\partial}{\partial y} & y\,e^{-kr} \\ \hat{\mathbf{k}} & \dfrac{\partial}{\partial z} & z\,e^{-kr} \end{vmatrix} = \begin{bmatrix} -\dfrac{k}{r}(zy - yz)e^{-kr} \\ -\dfrac{k}{r}(xz - zx)e^{-kr} \\ -\dfrac{k}{r}(yx - xy)e^{-kr} \end{bmatrix} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore

$$\boxed{\operatorname{curl}\vec{\mathbf{F}} \equiv \vec{\mathbf{0}}}$$

(b) Find the curl of $\vec{\mathbf{F}}$ while remaining in spherical polar coordinates throughout. [2]

$$\vec{\mathbf{F}} = e^{-kr}\,\vec{\mathbf{r}} = r\,e^{-kr}\,\hat{\mathbf{r}} \quad\Rightarrow\quad \operatorname{curl}\vec{\mathbf{F}} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & \dfrac{\partial}{\partial r} & r\,e^{-kr} \\ r\,\hat{\boldsymbol{\theta}} & \dfrac{\partial}{\partial \theta} & r\times 0 \\ r\sin\theta\,\hat{\boldsymbol{\phi}} & \dfrac{\partial}{\partial \phi} & r\sin\theta\times 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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6 (b) (continued)

Note that any purely radial vector field is automatically irrotational:

$$\vec{\mathbf{F}} = f(r)\,\hat{\mathbf{r}} \quad\Rightarrow\quad \operatorname{curl}\vec{\mathbf{F}} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & \dfrac{\partial}{\partial r} & f(r) \\ r\,\hat{\boldsymbol{\theta}} & \dfrac{\partial}{\partial\theta} & 0 \\ r\sin\theta\,\hat{\boldsymbol{\phi}} & \dfrac{\partial}{\partial\phi} & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

It is therefore no surprise that $\operatorname{curl}\vec{\mathbf{F}} = \vec{\mathbf{0}}$.

(c) Work in Cartesian coordinates to find the divergence of $\vec{\mathbf{F}}$ as a function of r. [2]

$$\operatorname{div}\vec{\mathbf{F}} = \frac{\partial}{\partial x}\left(x\,e^{-kr}\right) + \frac{\partial}{\partial y}\left(y\,e^{-kr}\right) + \frac{\partial}{\partial z}\left(z\,e^{-kr}\right)$$

$$\left(1-\frac{kx^2}{r} + 1-\frac{ky^2}{r} + 1-\frac{kz^2}{r}\right)e^{-kr} = \left(3-\frac{kr^2}{r}\right)e^{-kr}$$

Therefore

$$\boxed{\operatorname{div}\vec{\mathbf{F}} = (3-kr)\,e^{-kr}}$$

(d) Work in spherical polar coordinates to find the divergence of $\vec{\mathbf{F}}$ as a function of r. [3]

$$\operatorname{div}\vec{\mathbf{F}} = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\times r\,e^{-kr}\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta\times 0\right) + \frac{\partial}{\partial\phi}\left(r\times 0\right)\right)$$

$$= \frac{1}{r^2}\frac{d}{dr}\left(r^3e^{-kr}\right) = \frac{\left(3r^2-kr^3\right)e^{-kr}}{r^2} \quad\Rightarrow$$

$$\boxed{\operatorname{div}\vec{\mathbf{F}} = (3-kr)\,e^{-kr}}$$

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6 (e) find where div $\vec{\mathbf{F}} = 0$ and classify this surface. [2]

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div $\vec{\mathbf{F}} = 0 \quad \Rightarrow \quad 3 - kr = 0$

Therefore div $\vec{\mathbf{F}} = 0$

everywhere on the **sphere**, centre (0, 0, 0) and radius $r = \dfrac{3}{k}$.

(f) find where the magnitude F of the vector field $\vec{\mathbf{F}}$ attains its maximum value. [2]

$F = \left|\vec{\mathbf{F}}\right| = e^{-kr}\left|\vec{\mathbf{r}}\right| = r\,e^{-kr}$,

which is a function of radius r only.

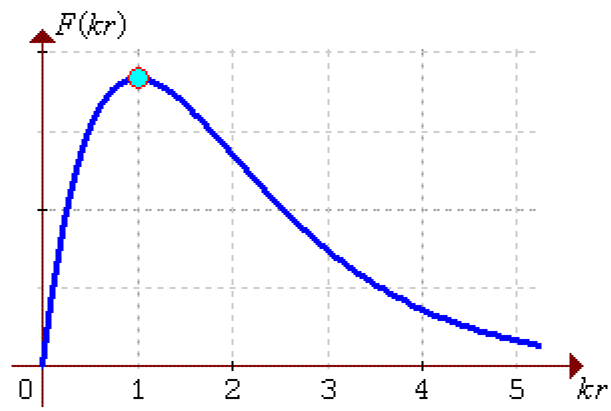
$$\frac{dF}{dr} = (1-kr)e^{-kr}$$

$$\frac{dF}{dr} = 0 \quad \Rightarrow \quad 1 - kr = 0 \quad \Rightarrow \quad r = \frac{1}{k}$$

Note that F is a differentiable positive-valued function of r for all $r > 0$.

$r = 0 \quad \Rightarrow \quad r\,e^{-kr} = 0$ and

$$\lim_{r \to \infty}\left(r\,e^{-kr}\right) = 0$$



Therefore the sole interior critical point must be an absolute maximum.

F attains its absolute maximum value

everywhere on the **sphere**, centre (0, 0, 0) and radius $r = \dfrac{1}{k}$.

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6 (g) show that $V(r) = \dfrac{-(1+kr)}{k^2}e^{-kr}$ is a potential function for the vector field $\vec{\mathbf{F}}$. [3]

In Cartesian coordinates

$$V(r) = \frac{-(1+kr)}{k^2}e^{-kr} \quad \Rightarrow \quad \vec{\nabla} V = \left[\frac{\partial}{\partial x} \ \frac{\partial}{\partial y} \ \frac{\partial}{\partial z} \right]^{\mathrm{T}} \left(\frac{-(1+kr)}{k^2}e^{-kr} \right)$$

$$\frac{\partial}{\partial x}\left(\frac{-(1+kr)}{k^2}e^{-kr} \right) = \frac{d}{dr}\left(\frac{-(1+kr)}{k^2}e^{-kr} \right)\frac{\partial r}{\partial x}$$

$$= \left(\frac{-(0+k)+k(1+kr)}{k^2}e^{-kr} \right)\frac{x}{r} = r e^{-kr}\frac{x}{r} = x e^{-kr}$$

Invoking symmetry,

$$\vec{\nabla} V = \left[x e^{-kr} \ \ y e^{-kr} \ \ z e^{-kr} \right]^{\mathrm{T}} = \vec{\mathbf{r}} e^{-kr} = \vec{\mathbf{F}}$$

OR

In spherical polar coordinates,

$$\vec{\nabla} V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\theta}}\frac{\partial V}{\partial \theta} + \frac{1}{r\sin\theta}\hat{\boldsymbol{\phi}}\frac{\partial V}{\partial \phi} = \hat{\mathbf{r}}\frac{dV}{dr} + \vec{\mathbf{0}} + \vec{\mathbf{0}}$$

$$= \hat{\mathbf{r}}\frac{d}{dr}\left(\frac{-(1+kr)}{k^2}e^{-kr} \right) = -\hat{\mathbf{r}}\frac{1}{k^2}\big(k - k(1+kr)\big)e^{-kr} = +\hat{\mathbf{r}}\frac{k^2 r}{k^2}e^{-kr} = e^{-kr}\vec{\mathbf{r}} = \vec{\mathbf{F}}$$

Therefore $V(r)$ is a potential function for the vector field $\vec{\mathbf{F}}$.

(h) find the work done to move a particle from the origin to a place where $\operatorname{div} \vec{\mathbf{F}} = 0$. [3]

[Note that work done is $\displaystyle\int_C \vec{\mathbf{F}} \bullet d\vec{\mathbf{r}} = \int_{t_0}^{t_1} \vec{\mathbf{F}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\, dt$.]

Ignoring the hint [!], where a potential function exists, the work done equals the potential difference between the ends of the path.

$$W = V\left(\frac{3}{k}\right) - V(0) = \frac{-(1+3)}{k^2}e^{-3} - \frac{-(1+0)}{k^2}e^{0}$$

Therefore

$$\boxed{W = \frac{1-4e^{-3}}{k^2}}$$

OR

6 (h) (continued)

$$W = \int_{t_0}^{t_1} e^{-kr}\,\vec{\mathbf{r}} \bullet \frac{d\vec{\mathbf{r}}}{dt}\,dt$$

The existence of a potential field implies that the vector field $\vec{\mathbf{F}}$ is conservative, so that the work done in moving from one point to another is independent of the path taken. Take a simple path, along the positive x axis, from the origin to the point $(3/k, 0, 0)$.

$$\vec{\mathbf{r}} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \Rightarrow \frac{d\vec{\mathbf{r}}}{dt} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \vec{\mathbf{r}} \bullet \frac{d\vec{\mathbf{r}}}{dt} = t \times 1 + 0 + 0 = t \quad \text{and} \quad r = t$$

The work done is

$$W = \int_0^{3/k} t\,e^{-kt}\,dt = \left[\frac{-(1+kt)e^{-kt}}{k^2}\right]_0^{3/k} = \frac{-(1+3)e^{-3} + (1+0)e^0}{k^2} = \frac{1-4e^{-3}}{k^2}$$

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7. A cylindrical parabolic coordinate system (u, v, w) is defined by

$$x = uv\,, \quad y = \frac{u^2 - v^2}{2}\,, \quad z = w$$

(a) Find the Jacobian $\dfrac{\partial(x, y, z)}{\partial(u, v, w)} = \text{abs}\left(\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix}\right)$ [3]

that allows the conversion of a volume differential dV from Cartesian coordinates to these cylindrical polar coordinates.

$$\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{pmatrix} = \begin{vmatrix} v & u & 0 \\ u & -v & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0 - 0 + 1\left(-v^2 - u^2\right)$$

Therefore the Jacobian is

$$\boxed{\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 + v^2}$$

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7 (a) (continued)

The element of volume is therefore $dV = dx\,dy\,dz = \left(u^2 + v^2\right) du\,dv\,dw$

(b) Find the scale factors h_u, h_v, h_w for this cylindrical parabolic coordinate system. [3]

$$h_u = \left|\frac{\partial \vec{\mathbf{r}}}{\partial u}\right| = \left|\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \end{bmatrix}^{\mathrm{T}}\right| = \left|\begin{bmatrix} v & u & 0 \end{bmatrix}^{\mathrm{T}}\right| = \sqrt{u^2 + v^2}$$

$$h_v = \left|\frac{\partial \vec{\mathbf{r}}}{\partial v}\right| = \left|\begin{bmatrix} \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{bmatrix}^{\mathrm{T}}\right| = \left|\begin{bmatrix} u & -v & 0 \end{bmatrix}^{\mathrm{T}}\right| = \sqrt{u^2 + v^2}$$

$$h_w = \left|\frac{\partial \vec{\mathbf{r}}}{\partial w}\right| = \left|\begin{bmatrix} \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{bmatrix}^{\mathrm{T}}\right| = \left|\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}\right| = 1$$

(c) Let the unit vectors of the cylindrical parabolic coordinate system be $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$. [3]
Find an expression for the gradient in this cylindrical parabolic coordinate system.

$$\boxed{\vec{\nabla} V = \frac{\hat{\mathbf{u}}}{h_u}\frac{\partial V}{\partial u} + \frac{\hat{\mathbf{v}}}{h_v}\frac{\partial V}{\partial v} + \frac{\hat{\mathbf{w}}}{h_w}\frac{\partial V}{\partial w} = \frac{1}{\sqrt{u^2+v^2}}\left(\hat{\mathbf{u}}\frac{\partial V}{\partial u} + \hat{\mathbf{v}}\frac{\partial V}{\partial v}\right) + \hat{\mathbf{w}}\frac{\partial V}{\partial w}}$$

(except on the w axis)

(d) Find an expression for the Laplacian $\nabla^2 f$ in this cylindrical parabolic coordinate system. [3]

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left(\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_w h_u}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right)$$

$$= \frac{1}{u^2+v^2}\left(\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\left(u^2+v^2\right)\frac{\partial f}{\partial w}\right)\right) \Rightarrow$$

$$\boxed{\nabla^2 f = \frac{1}{u^2+v^2}\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right) + \frac{\partial^2 f}{\partial w^2}}$$

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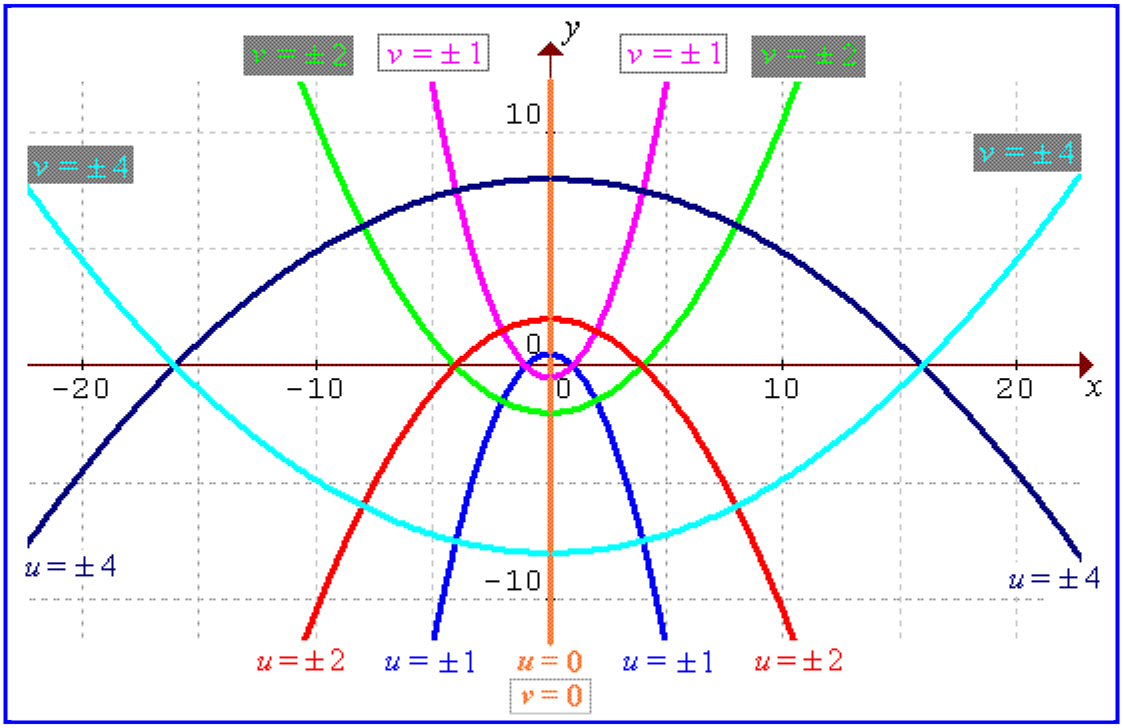
7 (e) Sketch on the same *x*-*y* plane any three members of each of the two families of coordinate curves u = constant and v = constant . [3]



8. The location of a particle at any time $t > 0$ is given in cylindrical polar coordinates by
$$\rho(t) = 2 - e^{-t}, \quad \phi(t) = t, \quad z(t) = e^{-t}.$$
Find the acceleration $\vec{\mathbf{a}}(t)$ in cylindrical polar coordinates. [5]

For this particle, $\dot{\rho}(t) = +e^{-t}, \quad \dot{\phi}(t) = 1, \quad \dot{z}(t) = -e^{-t}$

$$\Rightarrow \frac{d\hat{\boldsymbol{\rho}}}{dt} = \frac{d\phi}{dt}\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}, \quad \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\frac{d\phi}{dt}\hat{\boldsymbol{\rho}} = -\hat{\boldsymbol{\rho}} \quad \text{and} \quad \frac{d\hat{\mathbf{k}}}{dt} = \vec{\mathbf{0}}$$

$$\vec{\mathbf{r}} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{k}} = \left(2 - e^{-t}\right)\hat{\boldsymbol{\rho}} + e^{-t}\hat{\mathbf{k}}$$

$$\Rightarrow \quad \vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{d\rho}{dt}\hat{\boldsymbol{\rho}} + \rho\frac{d\hat{\boldsymbol{\rho}}}{dt} + \frac{dz}{dt}\hat{\mathbf{k}} = e^{-t}\hat{\boldsymbol{\rho}} + \left(2 - e^{-t}\right)\hat{\boldsymbol{\phi}} - e^{-t}\hat{\mathbf{k}}$$

$$\Rightarrow \quad \vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \left(\frac{d}{dt}e^{-t}\right)\hat{\boldsymbol{\rho}} + e^{-t}\frac{d\hat{\boldsymbol{\rho}}}{dt} + \left(\frac{d}{dt}\left(2 - e^{-t}\right)\right)\hat{\boldsymbol{\phi}} + \left(2 - e^{-t}\right)\frac{d\hat{\boldsymbol{\phi}}}{dt} - \left(\frac{d}{dt}e^{-t}\right)\hat{\mathbf{k}}$$

$$= \left(-e^{-t}\right)\hat{\boldsymbol{\rho}} + e^{-t}\hat{\boldsymbol{\phi}} + e^{-t}\hat{\boldsymbol{\phi}} - \left(2 - e^{-t}\right)\hat{\boldsymbol{\rho}} + e^{-t}\hat{\mathbf{k}} \quad \Rightarrow$$

$$\boxed{\vec{\mathbf{a}}(t) = -2\hat{\boldsymbol{\rho}} + 2e^{-t}\hat{\boldsymbol{\phi}} + e^{-t}\hat{\mathbf{k}}}$$

OR

One may quote a general form for the acceleration vector in cylindrical polar coordinates
$$\vec{\mathbf{a}} = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\hat{\boldsymbol{\rho}} + \left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right)\hat{\boldsymbol{\phi}} + \ddot{z}\hat{\mathbf{k}}$$

$$\Rightarrow \quad \vec{\mathbf{a}} = \left(-e^{-t} - \left(2 - e^{-t}\right)1\right)\hat{\boldsymbol{\rho}} + \left(0 + 2e^{-t}1\right)\hat{\boldsymbol{\phi}} + e^{-t}\hat{\mathbf{k}} = -2\hat{\boldsymbol{\rho}} + 2e^{-t}\hat{\boldsymbol{\phi}} + e^{-t}\hat{\mathbf{k}}$$

[As $t \to \infty$ the particle spirals out asymptotically to a circular path of radius 2, centre the origin, in the *x*-*y* plane, while the *z* coordinate decreases asymptotically from 1 to 0.]

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9. The location of a particle at any time $t > 0$ is given in spherical polar coordinates by

$$r(t) = 2, \quad \theta(t) = \frac{\pi}{6}, \quad \phi(t) = t\,.$$

(a) Find the velocity $\vec{\mathbf{v}}(t)$ and speed $v(t)$ in spherical polar coordinates. [4]

For this particle, $\dot{r}(t) = \dot{\theta}(t) = 0, \quad \dot{\phi}(t) = 1$

$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}} = 2\,\hat{\mathbf{r}}$

$$\Rightarrow \quad \vec{\mathbf{v}} = \frac{d\vec{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \frac{dr}{dt}\hat{\mathbf{r}} + 2\left(\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{d\phi}{dt}\sin\theta\,\hat{\boldsymbol{\phi}}\right) = 0\,\hat{\mathbf{r}} + 2\left(0\,\hat{\boldsymbol{\theta}} + 1\sin\frac{\pi}{6}\,\hat{\boldsymbol{\phi}}\right)$$

$\Rightarrow$

$$\boxed{\vec{\mathbf{v}} = 1\hat{\boldsymbol{\phi}}} \quad \Rightarrow \quad \boxed{v = 1}$$

(b) Find the acceleration $\vec{\mathbf{a}}(t)$ in spherical polar coordinates. [3]

$$\vec{\mathbf{a}} = \frac{d\vec{\mathbf{v}}}{dt} = \frac{d\hat{\boldsymbol{\phi}}}{dt} = -\frac{d\phi}{dt}\left(\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right) = -1\left(\sin\frac{\pi}{6}\hat{\mathbf{r}} + \cos\frac{\pi}{6}\hat{\boldsymbol{\theta}}\right) \quad \Rightarrow$$

$$\boxed{\vec{\mathbf{a}} = -\frac{1}{2}\left(\hat{\mathbf{r}} + \sqrt{3}\,\hat{\boldsymbol{\theta}}\right)}$$

(c) Describe the motion of the particle (what sort of path does it follow?) [3]

$z = r\cos\theta = 2\cos\frac{\pi}{6} = \sqrt{3} =$ constant $\quad \Rightarrow \quad$ the path lies in one plane, parallel to the x-y plane. The distance from the z-axis is $r\sin\theta = 2\sin\frac{\pi}{6} = 1 =$ constant.

Therefore the particle travels **at constant speed around the circle of radius 1, centre the *z*-axis, in the plane $z = \sqrt{3}$.**

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