1. For the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x\,\partial y} - 3\frac{\partial^2 u}{\partial y^2} = 4y$$

(a) Classify the partial differential equation as one of elliptic, parabolic or hyperbolic. [2]

(b) Find the general solution. [7]

(c) Find the complete solution, given the additional information [11]

$$u(0,y)=0\,,\quad u_x(0,y)=y^2$$

(a) $D = B^2 - 4AC = 4 + 12 = 16 > 0 \Rightarrow$ the PDE is

hyperbolic

(b) $\lambda = \dfrac{-B \pm \sqrt{D}}{2A} = \dfrac{-2 \pm 4}{2} = -3, +1$

A.E.: $u_C = f(y-3x) + g(y+x)$

P.S.: The right side of the PDE is a first order polynomial.
The left side of the PDE involves second order derivatives only.
Therefore try $u_P = ax^3 + bx^2y + cxy^2 + dy^3$

$\Rightarrow \quad (u_P)_x = 3ax^2 + 2bxy + cy^2 + 0$ and $(u_P)_y = 0 + bx^2 + 2cxy + 3dy^2$

$\Rightarrow \quad (u_P)_{xx} = 6ax + 2by\,,\ (u_P)_{xy} = 2bx + 2cy$ and $(u_P)_{yy} = 2cx + 6dy$

Substituting into the PDE,

$(u_P)_{xx} + 2(u_P)_{xy} - 3(u_P)_{yy} = 6ax + 2by + 4bx + 4cy - 6cx - 18dy = 4y$

$\Rightarrow \quad (6a + 4b - 6c)x + (2b + 4c - 18d)y = 0x + 4y$

$\Rightarrow \quad 3a + 2b - 3c = 0$ and $b + 2c - 9d = 2$

The general solution is

$$u(x,y) = f(y-3x) + g(y+x) + ax^3 + bx^2y + cxy^2 + dy^3$$
where $3a + 2b - 3c = 0$ and $b + 2c - 9d = 2$

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1 (c) $u(x,y) = f(y-3x) + g(y+x) + ax^3 + bx^2y + cxy^2 + dy^3$

$\Rightarrow \quad u_x(x,y) = -3f'(y-3x) + g'(y+x) + 3ax^2 + 2bxy + cy^2 + 0$

$u(0,y) = 0 \quad \Rightarrow \quad f(y) + g(y) + 0+0+0+dy^3 = 0$ **(A)**

$\frac{d}{dy}(\mathbf{A}) \quad \Rightarrow \quad f'(y) + g'(y) = -3dy^2$ **(B)**

$u_x(0,y) = y^2 \quad \Rightarrow \quad -3f'(y) + g'(y) + 0+0+cy^2 = y^2$ **(C)**

$(\mathbf{B})-(\mathbf{C}) \quad \Rightarrow \quad 4f'(y)+0 = (c-3d-1)y^2 \quad \Rightarrow \quad f'(y) = \frac{(c-3d-1)y^2}{4}$

Now choose $c = 3d+1 \quad \Rightarrow \quad f'(y) = 0 \quad \Rightarrow \quad f(y) = 0$

The two constraints on the four arbitrary constants now become

$b + 2(3d+1) - 9d = 2 \quad \Rightarrow \quad b = 3d$

and $3a + 2(3d) - 3(3d+1) = 0 \quad \Rightarrow \quad 3a = 3d+3 \quad \Rightarrow \quad a = d+1$

$(\mathbf{A}) \quad \Rightarrow \quad g(y) = -f(y) - dy^3 = -dy^3$

Choose $d = 0 \quad \Rightarrow \quad g(y) = 0, \quad a = 1, \quad b = 0, \quad c = 1$

The solution becomes

$u(x,y) = 0 + 0 + 1x^3 + 0 + 1xy^2 + 0 \quad \Rightarrow$

$$\boxed{u(x,y) = x^3 + xy^2}$$

Check (not required):

$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x\,\partial y} - 3\frac{\partial^2 u}{\partial y^2} = (6x+0) + 2(0+2y) - 3(0+2x) = 4y$ ✓

$u(0,y) = 0+0 = 0$ ✓

$u_x(0,y) = \left(3x^2+y^2\right)\Big|_{x=0} = y^2$ ✓

Note that if no values are chosen for the four coefficients a, b, c, d, then the complete solution becomes

$$u(x,y) = \frac{(c-3d-1)}{12}(y-3x)^3 + \frac{(1-9d-c)}{12}(y+x)^3 + ax^3 + bx^2y + cxy^2 + dy^3$$

together with $3a + 2b - 3c = 0$ and $b + 2c - 9d = 2$

It takes some effort to show that this reduces to $u(x,y) = x^3 + xy^2$!

2 (a) Show that the only intersection of the curves $y = e^{-x^2}$ and $y = x$ must occur for some value of x in the interval $0 < x < 1$. [7]

(b) Use Newton's method with a reasonable initial value x_0 to estimate, correct to five decimal places, the value of x at which $f(x) = 0$, where [8]

$$f(x) = x - e^{-x^2}$$

(a) $y = e^{-x^2} > 0 \quad \forall x \quad \Rightarrow$ the curves can intersect in the first quadrant only.

$\frac{d}{dx} e^{-x^2} = -2x e^{-x^2} < 0 \quad \Rightarrow \quad y = e^{-x^2}$ is decreasing for all $x > 0$,

while $y = x$ is increasing for all $x > 0$.

Therefore any intersection of these two curves is unique.

Define $f(x) = x - e^{-x^2}$, then $f(0) = 0 - e^0 = -1 < 0$, while $f(1) = 1 - e^{-1} > 0$

The unique solution of $f(x) = 0$ must therefore be in the interval $0 < x < 1$.

(b) A reasonable first guess is $x_0 = 0.5$

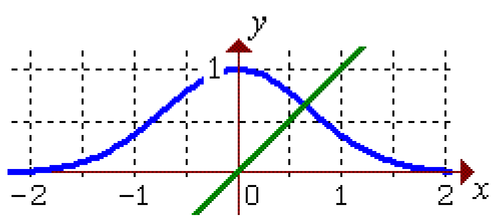
$f(x) = x - e^{-x^2} \quad \Rightarrow \quad f'(x) = 1 + 2x e^{-x^2}$

Newton's method is based on the algorithm $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

x_n	$f(x_n)$	$f'(x_n)$	$f(x_n)/f'(x_n)$
0.500 000 000	-0.278 800 783	1.778 800 783	-0.156 735 249
0.656 735 249	0.007 072 037	1.853 313 461	0.003 815 888
0.652 919 360	0.000 001 334	1.852 605 641	0.000 000 720
0.652 918 640	0.000 000 000		

Correct to five decimal places,

$$\boxed{x = 0.65292}$$



3. The non-linear second order ordinary differential equation

$$\frac{d^2x}{dt^2} + (1-x)\frac{dx}{dt} + 4x - x^2 = 0$$

can be represented by the system of first order ordinary differential equations

$$\dot{x} = y$$
$$\dot{y} = (x-1)y - 4x + x^2$$

(a) Find the locations of both critical points. [4]

(b) For each critical point, identify its nature (node, centre, focus or saddle point) and stability. [7]

(c) Find the equations of the asymptotes for the linear approximation at any node or saddle point. [7]
[Note: the general solution is ***not*** required.]

(d) Sketch the phase portrait in the [linear] neighbourhood of each critical point. [6]

(e) Sketch the phase portrait for the non-linear system, including both critical points. [6]

BONUS QUESTION

(f) Find the equation of the separatrix (the curve that separates trajectories that terminate in a stable critical point from trajectories that recede to infinity). [+5]

(a) $\dot{x} = 0 \quad \Rightarrow \quad y = 0$

$\dot{y} = 0 \quad \Rightarrow \quad 0 - 4x + x^2 = 0 \quad \Rightarrow \quad x(x-4) = 0$

Therefore the only critical points are

$$\boxed{(x, y) = (0, 0) \text{ and } (4, 0)}$$

(b) Near a critical point (a, b) the linear approximation to the non-linear system is

$$\dot{\vec{\mathbf{x}}} = A\vec{\mathbf{x}}, \quad \text{where} \quad A = \left.\begin{bmatrix} P_x & P_y \\ Q_x & Q_y \end{bmatrix}\right|_{(a,b)}$$

$$\begin{aligned} P &= y \\ Q &= (x-1)y - 4x + x^2 \end{aligned} \quad \Rightarrow \quad A = \begin{bmatrix} 0 & 1 \\ b - 4 + 2a & a - 1 \end{bmatrix}$$

<u>(0, 0)</u>:

$$A = \begin{bmatrix} 0 & 1 \\ -4 & -1 \end{bmatrix} \quad \Rightarrow \quad D = (a-d)^2 + 4bc = 1 - 16 = -15 < 0$$

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-1 \pm \sqrt{-15}}{2}$$ - a complex conjugate pair with negative real part

$\Rightarrow$ there is a **stable focus** at (0, 0).

3 (b) (continued)

(4, 0):

$$A = \begin{bmatrix} 0 & 1 \\ +4 & 3 \end{bmatrix} \quad \Rightarrow \quad D = (a-d)^2 + 4bc = 9 + 16 = 25 > 0$$

$$\lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{3 \pm \sqrt{25}}{2} = \frac{3 \pm 5}{2} = -1, +4$$ - real with opposite signs

$\Rightarrow$ there is a **saddle point (unstable)** at (4, 0).

(c) At the saddle point the eigenvectors are

for $\lambda = -1$:

$$A\vec{\mathbf{x}} = \lambda \vec{\mathbf{x}} \quad \Rightarrow \quad \begin{bmatrix} 0+1 & 1 \\ 4 & 3+1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} =$ any non-zero multiple of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

The inward asymptote is $y - 0 = -1(x-4) \quad \Rightarrow \quad \boxed{y = 4 - x}$

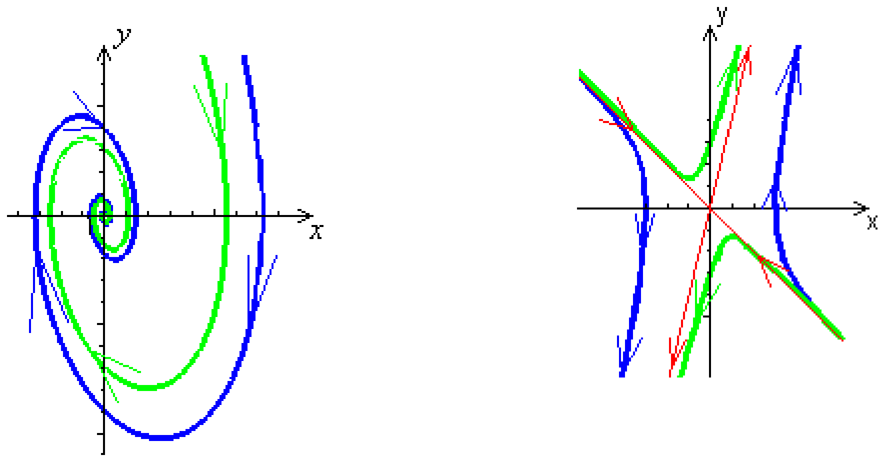
for $\lambda = +4$:

$$A\vec{\mathbf{x}} = \lambda \vec{\mathbf{x}} \quad \Rightarrow \quad \begin{bmatrix} 0-4 & 1 \\ 4 & 3-4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

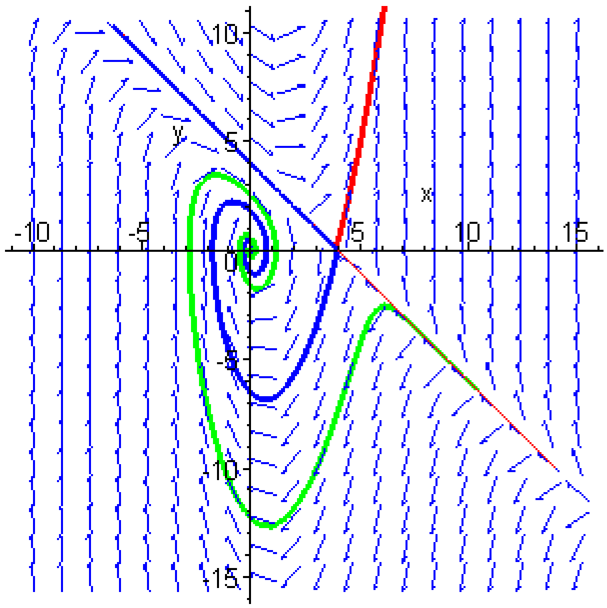
$\Rightarrow \begin{bmatrix} \alpha \\ \beta \end{bmatrix} =$ any non-zero multiple of $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$

The outward asymptote is $y - 0 = 4(x-4) \quad \Rightarrow \quad \boxed{y = 4x - 16}$

(d) Near (0, 0) Near (4, 0)



3 (e) Clearly the saddle point dominates in most of the first quadrant and part of the fourth quadrant and the stable focus dominates in the third quadrant and at least half of the second quadrant. What happens elsewhere is not so obvious. A phase portrait from Maple is displayed here.



This plot suggests that the incoming asymptote $y = 4 - x$ to the saddle point may also be the separatrix between orbits that terminate at the focus and orbits that recede to infinity.

(f) We are seeking a solution to the non-linear system that passes through the saddle point.

However, the exact solution to the ODE $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} = \frac{(x-1)y - 4x + x^2}{y}$

is not obvious!

Check to see if the incoming asymptote $y = 4 - x$ in the linear approximation near the saddle point is also a solution to the non-linear system of ODEs:

$$y = 4 - x \quad \Rightarrow \quad \frac{dy}{dx} = -1$$

But, by the chain rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

$$\frac{dy}{dt} = (x-1)y - 4x + x^2 = (x-1)(4-x) - x(4-x) = (4-x)(x-1-x) = -(4-x)$$

$$\frac{dx}{dt} = y = 4 - x \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-(4-x)}{4-x} = -1 = \frac{d}{dx}(4-x)$$

Therefore $y = 4 - x$ ***is*** a solution to the non-linear system of ODEs.

All trajectories above this line recede to infinity.

3 (f) (continued)

The other asymptote $y = 4x - 16$ to the saddle point in the linear approximation near the saddle point is ***not*** a solution to the non-linear system of ODEs:

$$y = 4x - 16 \quad \Rightarrow \quad \frac{dy}{dx} = 4$$

$$\frac{dy}{dt} = (x-1)y - 4x + x^2 = (x-1)(4x-16) - x(4-x)$$

$$= (x-4)(4x-4+x) = (3x-4)(x-4)$$

$$\frac{dx}{dt} = y = 4x - 16 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{(3x-4)(x-4)}{4(x-4)} = \frac{3x-4}{4} \neq 4$$

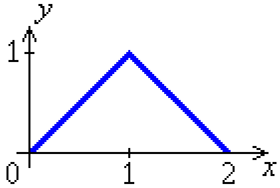
Orbits below $y = 4 - x$ can cross the line $y = 4x - 16$.

All trajectories in this region are swept into the stable focus.

Therefore the separatrix is the line

$$\boxed{y = 4 - x}$$

4. A perfectly elastic frictionless string is fixed at $x = 0$ and $x = 2$. It is stretched in a triangular configuration [15]



$$y(x,0) = f(x) = \begin{cases} x & (0 \le x < 1) \\ 2-x & (1 \le x \le 2) \end{cases}$$

as illustrated and is released from rest.
The speed of waves on the string is $c = 6$.

Find a Fourier series expression for the subsequent displacement $y(x,t)$ of the string. You may quote

$$y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(u)\sin\left(\frac{n\pi u}{L}\right)du\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi c t}{L}\right).$$

$L = 2$

$$c_n = \int_0^2 f(u)\sin\left(\frac{n\pi u}{2}\right)du$$

$$= \int_0^1 u\sin\left(\frac{n\pi u}{2}\right)du + \int_1^2 (2-u)\sin\left(\frac{n\pi u}{2}\right)du$$

Integrations by parts:

	D	I
u	$2-u$	$\sin\frac{n\pi u}{2}$
	$+$	
1	-1	$\frac{-2}{n\pi}\cos\frac{n\pi u}{2}$
	$-$	
0	0	$-\left(\frac{2}{n\pi}\right)^2\sin\frac{n\pi u}{2}$

$$c_n = \left[u\left(-\frac{2}{n\pi}\right)\cos\left(\frac{n\pi u}{2}\right) + \left(-\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi u}{2}\right)\right]_0^1$$

$$+ \left[(2-u)\left(-\frac{2}{n\pi}\right)\cos\left(\frac{n\pi u}{2}\right) - \left(-\frac{2}{n\pi}\right)^2\sin\left(\frac{n\pi u}{2}\right)\right]_1^2$$

$$= \left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right) - (0+0)$$

$$+ (0-0) - \left(-\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$\Rightarrow \quad c_n = \frac{8}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \quad \Rightarrow \quad \boxed{y(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{2}\right)\cos(3n\pi t)}$$

which can be rewritten as

$$\boxed{y(x,t) = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin\left(\frac{(2k-1)\pi x}{2}\right)\cos(3(2k-1)\pi t)}$$

5. Find the path $y = f(x)$ between the points $(0,1)$ and $(1, 2e^3)$ that provides an extremum for the value of the integral [20]

$$I = \int_0^1 \left((y')^2 + 9y^2 + 12y\,e^{3x} \right) dx$$

The Euler equation for extremals is $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$.

F is a function of all three of $x, y, y' \Rightarrow$ none of the special cases applies.

$$\frac{\partial F}{\partial y'} = \frac{\partial}{\partial y'}\left((y')^2 + 9y^2 + 12y\,e^{3x} \right) = 2y' + 0 + 0 \Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 2y''$$

$$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left((y')^2 + 9y^2 + 12y\,e^{3x} \right) = 0 + 18y + 12e^{3x}$$

$$\Rightarrow \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 2y'' - 18y - 12e^{3x} = 0$$

The extremal path is therefore a solution to the ODE $\frac{d^2y}{dx^2} - 9y = 6e^{3x}$

A.E.: $\lambda^2 - 9 = 0 \Rightarrow \lambda = \pm 3$

C.F.: $y_C = Ae^{3x} + Be^{-3x}$

P.S.: $r = 6e^{3x} = 6y_1$, part of the C.F. Therefore we cannot try $y_P = ce^{3x}$.

Try $y_P = cxe^{3x} \Rightarrow y_P' = c(1+3x)e^{3x} \Rightarrow y_P'' = c(6+9x)e^{3x}$

$\Rightarrow y_P'' - 9y_P = c(6+9x-9x)e^{3x} = 6e^{3x} \Rightarrow 6c = 6 \Rightarrow c = 1$

$\Rightarrow y_P = xe^{3x}$

G.S.: $y = (A+x)e^{3x} + Be^{-3x}$

B.C.: Curve passes through $(0,1) \Rightarrow 1 = A + B$ **(A)**

Curve passes through $(1, 2e^3) \Rightarrow 2e^3 = (A+1)e^3 + Be^{-3} \Rightarrow 1 = A + Be^{-6}$ **(B)**

(A) – (B) $\Rightarrow 0 = B\left(1 - e^{-6}\right) \Rightarrow B = 0 \Rightarrow A = 1$

C.S.: $y = (1+x)e^{3x}$

Therefore the extremal path is

$$\boxed{y = (1+x)e^{3x}}$$

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