1) Find the intersection of the planes whose equations in Cartesian coordinates are [10]

$$\begin{aligned} x+y+z &= 6 \\ 2y-z &= 1 \\ 3x+y+4z &= 17 \end{aligned}$$

Show your working.
What type of geometric object is the intersection? (point? line? etc.)

BONUS QUESTION

Explain, in geometrical terms, why the linear system has no solution if the equation of the third plane is changed to $3x+y+4z = 18$. [+4]

Employing the method of Gaussian elimination to reduced row-echelon form:

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 3 & 1 & 4 & 17 \end{array}\right] \xrightarrow[R_3-3R_1]{} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & 1 & -1 \end{array}\right]$$

At this point we can see that the third equation is not independent of the other two equations.

$$\xrightarrow[R_3+R_2]{} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[R_2\div 2]{} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & \boxed{1} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[R_1-R_2]{} \left[\begin{array}{ccc|c} \boxed{1} & 0 & \frac{3}{2} & \frac{11}{2} \\ 0 & \boxed{1} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$ which is in reduced row-echelon form.

The solution is $x+\frac{3}{2}z=\frac{11}{2}$, $y-\frac{1}{2}z=\frac{1}{2}$, with z free to be any real number.
Equivalently,

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 11-3t \\ 1+t \\ 2t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 11 \\ 1 \\ 0 \end{bmatrix} + \frac{t}{2}\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \quad (t\in\mathbb{R})}$$

Geometrically, the three planes intersect in a **line**.

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1. (continued)

If the system changes to

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 3 & 1 & 4 & 18 \end{array}\right]$$

then the reduction proceeds

$$\left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 3 & 1 & 4 & 18 \end{array}\right] \xrightarrow[R_3 - 3R_1]{} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & -2 & 1 & 0 \end{array}\right] \xrightarrow[R_2 + R_3]{} \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 6 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right]$$

The leading '1' in the rightmost column is the hallmark of an inconsistent system ("0 = 1").

Geometrically, the third plane has been translated away from the line of intersection of the other two planes and is now parallel to that line. Each pair of planes still meets in a line, but the set of three planes no longer has any mutual intersection at all.

2) By any valid method, find the function $x(t)$ whose Laplace transform is [10]

$$X(s) = \frac{5\,e^{-2s}}{s^2+7s+12}.$$

Hence (or otherwise) solve the initial value problem for a mass-spring system

$$\frac{d^2x}{dt^2}+7\frac{dx}{dt} + 12x = 5\delta(t-2), \qquad x(0)=x'(0)=0,$$

where $\delta(t-2)$ is the Dirac delta function, (modelling a sudden hammer blow to the system at time $t = 2$ seconds). Sketch a graph of the solution.
Is this system under-damped, critically damped or over-damped?

Let $G(s) = \dfrac{5}{s^2+7s+12} = \dfrac{5}{(s+3)(s+4)} = \dfrac{a}{s+3} + \dfrac{b}{s+4}$

By the cover-up rule,

$$a = \frac{5}{\cancel{(-3+3)}\,(-3+4)} = \frac{5}{1} = 5 \quad \text{and} \quad b = \frac{5}{(-4+3)\,\cancel{(-4+4)}} = \frac{5}{-1} = -5$$

$$\Rightarrow \quad G(s) = \frac{5}{s^2+7s+12} = \frac{5}{s+3} - \frac{5}{s+4} = \mathcal{L}\left\{5e^{-3t}-5e^{-4t}\right\}$$

$$\Rightarrow \quad x(t) = \mathcal{L}^{-1}\left\{G(s)\,e^{-2s}\right\} = 5\left(e^{-3t}-e^{-4t}\right)\Big|_{t\to t-2} H(t-2) \quad \text{(second shift theorem)}$$

Therefore

$$\boxed{x(t) = 5\left(e^{-3(t-2)}-e^{-4(t-2)}\right)H(t-2)}$$

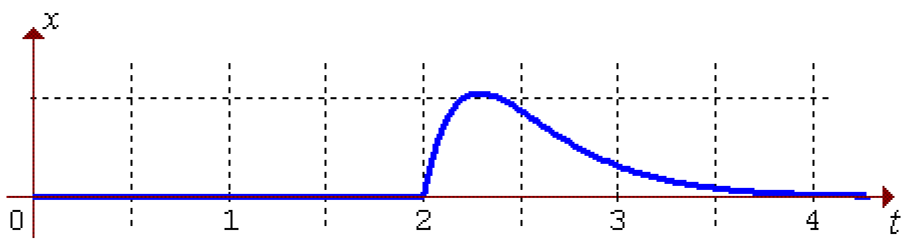
The Laplace transform of the initial value problem is

$$\left(s^2X-0s-0\right)+7\left(sX-0\right)+12X = 5e^{-2s} \quad \Rightarrow \quad X(s) = \frac{5\,e^{-2s}}{s^2+7s+12}$$

The solution to the initial value problem is therefore the same function $x(t)$ as above:

$$\boxed{x(t) = 5\left(e^{-3(t-2)}-e^{-4(t-2)}\right)H(t-2)}$$

The system is at rest at equilibrium until the hammer blow arrives.
It is set into motion suddenly at time $t = 2$ and settles back to equilibrium without oscillations.
The system is clearly **over-damped**.



3) Determine the nature (node, saddle point, centre or focus) and stability of the critical point for the linear system of differential equations [10]

$$\frac{dx}{dt} = 2x - y$$
$$\frac{dy}{dt} = 7x - 6y$$

Sketch the orbits near the critical point and label any asymptotes.

The coefficient matrix is $\mathrm{A} = \begin{bmatrix} 2 & -1 \\ 7 & -6 \end{bmatrix}$

The discriminant $D = (a-d)^2 + 4bc = (2+6)^2 + 4(-1)(7) = 64 - 28 = 36 > 0$

$$\Rightarrow \quad \lambda = \frac{(a+d) \pm \sqrt{D}}{2} = \frac{-4 \pm 6}{2} = -5, +1$$

The eigenvalues are a real pair of values of opposite sign

$\Rightarrow$ the critical point is a **saddle point** (which is an **unstable** critical point).

The eigenvector for $\lambda = -5$ is

$$\alpha = \frac{(a-d) - \sqrt{D}}{2} = \frac{8-6}{2} = 1, \quad \beta = c = 7 \quad \Rightarrow \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$$

The inward asymptote is the line $y = 7x$

The eigenvector for $\lambda = +1$ is

$$\alpha = \frac{(a-d) + \sqrt{D}}{2} = \frac{8+6}{2} = 7, \quad \beta = c = 7 \quad \Rightarrow \quad \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The outward asymptote is the line $y = x$

The general solution is $\left(x(t), y(t)\right) = \left(Ae^{-5t} + Be^{t},\ 7Ae^{-5t} + Be^{t}\right)$

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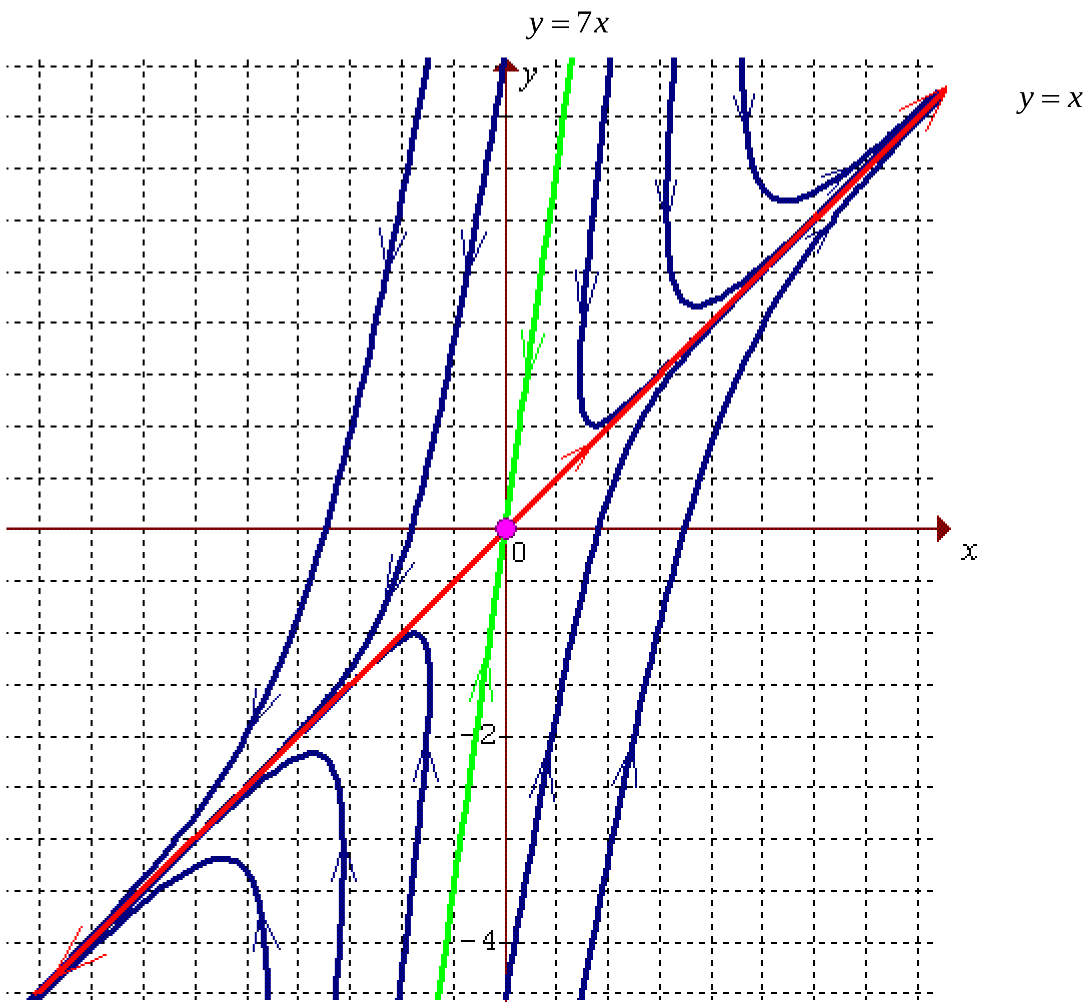
3. (continued)

Plot of some trajectories near the saddle point:



The equation of the inward (green, steeper) asymptote is $y = 7x$
The equation of the outward (red, shallower) asymptote is $y = x$

4) Use a Frobenius series method to find the general solution of the ordinary differential equation [10]

$$x^2 \frac{d^2 y}{dx^2} + 6x\frac{dy}{dx} + 4y = 18x^2$$

Compare the ODE to the standard linear form $P(x)\frac{d^2 y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = F(x)$:

$\frac{x \cdot Q(x)}{P(x)} = \frac{x \cdot 6x}{x^2} = 6$, $\frac{x^2 \cdot R(x)}{P(x)} = \frac{x^2 \cdot 4}{x^2} = 4$ and $\frac{F(x)}{P(x)} = \frac{18x^2}{x^2} = 18$ are all analytic for all x, including at $x = 0$. Therefore $x = 0$ is a regular singular point.

Try a series solution of the form $y = \sum_{n=0}^{\infty} c_n x^{n+r}$

$$\Rightarrow \quad y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} \quad \Rightarrow \quad y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

Substitute into the ODE:

$$\Rightarrow \quad x^2 y'' + 6xy' + 4y = \sum_{n=0}^{\infty} \big((n+r)(n+r-1) + 6(n+r) + 4\big) c_n x^{n+r} = 18x^2$$

The initial term in the series is

$$\big((0+r)(0+r-1) + 6(0+r) + 4\big) c_0 x^{0+r} = \left(r^2 + 5r + 4\right) c_0 x^r = (r+4)(r+1) c_0 x^r$$

If $r = 2$ then

$$(6)(3) c_0 x^2 = 18x^2 \quad \Rightarrow \quad c_0 = 1$$

and, for all other n, $\big((n+2)(n+1) + 6(n+2) + 4\big) c_n x^{n+r} = 0x^{n+r} \quad \Rightarrow \quad \left(n^2 + 9n + 18\right) c_n = 0$

But $n^2 + 9n + 18 > 0$ for all positive $n \;\Rightarrow\; c_n = 0 \quad \forall n > 0$

If $r = 2$ then the series collapses to the single term $y = x^2$ (which is the particular solution).

If $r \neq 2$ then

$(r+4)(r+1) c_0 = 0$. The requirement $c_0 \neq 0 \;\Rightarrow\; (r+4)(r+1) = 0$ (the indicial equation)

$\Rightarrow \; r = -4$ or $r = -1$

If $r = -4$ then

$$\sum_{n=0}^{\infty} \big((n-4)(n-4-1) + 6(n-4) + 4\big) c_n x^{n-4} = 18x^2$$

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4. (continued)

$$\Rightarrow \quad \sum_{n=0}^{\infty} \left(n^2 - 3n\right) c_n x^{n-4} = \sum_{n=0}^{\infty} n\left(n-3\right) c_n x^{n-4} = 18x^2$$

Matching coefficients:

$n=0$: $0c_0 = 0 \quad \Rightarrow \quad c_0$ is arbitrary

$n=1$: $-2c_1 = 0 \quad \Rightarrow \quad c_1 = 0$

$n=2$: $-2c_2 = 0 \quad \Rightarrow \quad c_2 = 0$

$n=3$: $0c_3 = 0 \quad \Rightarrow \quad c_3$ is arbitrary

$n=6$: $18c_6 = 18 \quad \Rightarrow \quad c_6 = 1$

For all other $n > 3$: $\left(\text{non-zero}\right) c_n = 0 \quad \Rightarrow \quad c_n = 0$

The series becomes $y = c_0 x^{-4} + c_3 x^{-1} + x^2$ (which includes the $r = 2$ case).

If $r = -1$ then

$$\sum_{n=0}^{\infty} \left(\left(n-1\right)\left(n-1-1\right) + 6\left(n-1\right) + 4\right) c_n x^{n-1} = 18x^2$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} \left(n^2 + 3n\right) c_n x^{n-1} = \sum_{n=0}^{\infty} n\left(n+3\right) c_n x^{n-1} = 18x^2$$

Matching coefficients:

$n=0$: $0c_0 = 0 \quad \Rightarrow \quad c_0$ is arbitrary

$n=1$: $4c_1 = 0 \quad \Rightarrow \quad c_1 = 0$

$n=2$: $10c_2 = 0 \quad \Rightarrow \quad c_2 = 0$

$n=3$: $18c_3 = 18 \quad \Rightarrow \quad c_3 = 1$

$n>3$: $\left(\text{non-zero}\right) c_n = 0 \quad \Rightarrow \quad c_n = 0$

The series becomes $y = c_0 x^{-1} + x^2$, all of which is part of the solution when $r = -4$.

Therefore the general solution of the ODE is the finite series

$$\boxed{y = \frac{A}{x^4} + \frac{B}{x} + x^2}$$

[This is a Cauchy-Euler ODE, for which an analytical method of solution is available.]

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