Families of Solutions to Bernoulli ODEs

In the family of solutions to the differential equation

$$\frac{dy}{dx} + p\,y = r\,y^n$$

it is shown that variation of the initial condition $y(0) = a$ causes a horizontal shift in the solution curve $y = f(x)$, rather than the vertical shift that one might anticipate. The complete solution is obtained in the case where the coefficients p and r are both constant. The behaviour of the solutions is explored in the case $p = r$ as a varies for several values of n.

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1. Behaviour of exponential functions

Among the features of the exponential function $y = e^{kx}$ is that a stretch of its graph by a factor of a constant a $(a > 0)$ in the y direction is identical to a translation by $c = -\frac{1}{k}\ln a$ in the x direction. Put another way, the horizontal distance between any two points on the graphs of $y = e^{kx}$ and $y = a\,e^{kx}$ $(a > 0)$ that share the same y coordinate is a constant $c = -\frac{1}{k}\ln a$.

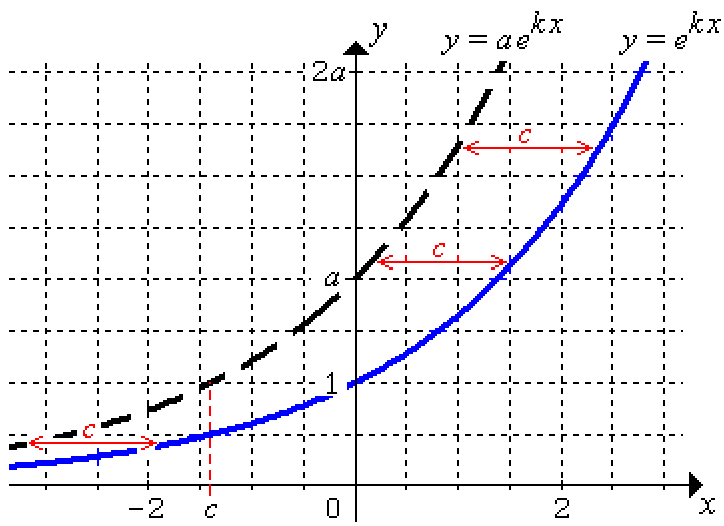

Figure 1 – Graphs of the exponential family $y = a\,e^{kx}$

The proof is simple. Setting the values of the scaled and translated functions equal at each x,

$$a\,e^{kx} \equiv e^{k(x-c)} \equiv e^{kx}e^{-kc} \quad \Rightarrow \quad a = e^{-kc} \quad \Rightarrow \quad -kc = \ln a$$

Therefore the horizontal shift c is related to the vertical stretch factor a by $c = -\frac{1}{k}\ln a$.

If $k > 0$ and $a > 1$ (as illustrated), then $c < 0$ and the translation is to the left.

If $a < 0$, then there is a reflection in the x-axis, together with a horizontal shift of $c = -\frac{1}{k}\ln|a|$.

This property of the exponential function explains the behaviour of the graphs of families of solutions to first order Bernoulli initial value problems with constant coefficients.

2. Complete solution of the first order Bernoulli initial value problem with constant coefficients:

The ordinary differential equation (ODE)

$$\frac{dy}{dx} + p(x)\,y(x) = r(x)\big(y(x)\big)^n$$

is the general first order Bernoulli ODE, where n is some real constant and $p(x)$ and $r(x)$ are integrable real functions of the real variable *x*. In this paper we restrict our attention to the case of constant coefficients. Let a, n, p and r all be independent real constants.

Consider the initial value problem (IVP)

$$\frac{dy}{dx} + p\,y = r\,y^n\,, \quad y(0) = a$$

If $n \geq 0$ then $y \equiv 0$ is a solution to the ordinary differential equation. $y \equiv 0$ may be part of another family of solutions, depending on the values of p, r and n. In most cases, $y \equiv 0$ is a singular solution to the ordinary differential equation. If in addition $a = 0$, then the complete solution of the IVP includes $y(x) \equiv 0$.

If $n = 1$ then the IVP is linear and homogeneous: $\dfrac{dy}{dx} + (p - r)\,y = 0\,, \quad y(0) = a$, for which the complete solution is quickly found to be $y = a\,e^{(r-p)x}$. $y \equiv 0$ is clearly the member of this family of solutions for which $a = 0$. There is therefore no singular solution in this case, which is no surprise: linear ordinary differential equations never have singular solutions. If in addition $p = r$, then the solution reduces to $y \equiv a$. Otherwise, using the result from the previous section, a change in the value of the initial condition from $y(0) = a_1$ to $y(0) = a_2$ (where a_1 and a_2 have the same sign) results in a horizontal translation of the solution curve by $c = \dfrac{1}{p-r} \ln \dfrac{a_2}{a_1}$.

If $n \neq 1$ then the change of variables $w = \dfrac{y^{1-n}}{1-n}$ transforms the ODE into a linear form.

$$w = \frac{y^{1-n}}{1-n} \;\Rightarrow\; \frac{dw}{dx} = \frac{d}{dy}\left(\frac{y^{1-n}}{1-n}\right)\cdot\frac{dy}{dx} = y^{-n}\frac{dy}{dx} \;\Rightarrow\; \frac{dy}{dx} = y^n \frac{dw}{dx}$$

$$\frac{dy}{dx} + p\,y = r\,y^n \;\Rightarrow\; y^n\frac{dw}{dx} + p\,y = r\,y^n \;\Rightarrow\; \frac{dw}{dx} + p\,y^{1-n} = r \;\Rightarrow$$

$$\frac{dw}{dx} + p(1-n)\,w = r$$

Solving this linear ODE,

$$h = \int P\,dx = \int p(1-n)\,dx = p(1-n)x \qquad \Rightarrow \quad e^h = e^{(1-n)px} \quad \text{(integrating factor)}$$

$$\Rightarrow \quad \int e^h R\,dx = \int e^{(1-n)px} r\,dx = \frac{r}{(1-n)p} e^{(1-n)px}$$

(unless $p=0$, in which case $\int e^h R\,dx = \int r\,dx = rx$)

The general solution of the ODE for $p \neq 0$ and $n \neq 1$ is

$$\frac{y^{1-n}}{1-n} = w = e^{-h}\left(\int e^h R\,dx + C\right) = e^{-(1-n)px}\left(\frac{r}{(1-n)p} e^{(1-n)px} + C\right)$$

$$\Rightarrow \quad y^{1-n} = \frac{r}{p} + (1-n)Ce^{-(1-n)px} \qquad \Rightarrow \quad y(x) = \left(\frac{r}{p} + (1-n)Ce^{-(1-n)px}\right)^{1/(1-n)}$$

Imposing the initial condition,

$$y(0) = a \quad \Rightarrow \quad a^{1-n} = \frac{r}{p} + (1-n)C \quad \Rightarrow \quad (1-n)C = a^{1-n} - \frac{r}{p}$$

Therefore the complete solution to the general Bernoulli initial value problem with constant coefficients in the case $n \neq 1$ and $p \neq 0$ is

$$\boxed{y(x) = \left(\frac{r}{p} + \left(a^{1-n} - \frac{r}{p}\right)e^{-(1-n)px}\right)^{1/(1-n)}}$$

together with the singular solution $y \equiv 0$ in the case $(n > 0 \text{ and } a = 0)$.

3. Special cases:

If $n > 0$ then $y \equiv 0$ is a singular solution of $\dfrac{dy}{dx} = r\,y^n$.

The general solution of $\dfrac{dy}{dx} = r\,y^n$ is $y(x) = \big((1-n)(rx+C)\big)^{1/(1-n)}$

Imposing the initial condition,

$y(0) = \big((1-n)C\big)^{1/(1-n)} = a \quad \Rightarrow \quad (1-n)C = a^{1-n}$

The complete solution in the case $p = 0$ is therefore

$$y(x) = \left((1-n)rx + a^{1-n}\right)^{1/(1-n)}$$

This can be re-written as $y(x) = \big((1-n)r(x-c)\big)^{1/(1-n)}$, where $c = \dfrac{a^{1-n}}{(n-1)r} \quad (r \neq 0)$.

The case $p = r = 0$ has the simple solution $y \equiv a$. In all other cases for which $p = 0$, each member of the family can be found from another solution curve by a horizontal shift. There is no solution curve when a^{1-n} is not real.

One can deduce this exception as the limit as $p \to 0$ of the more general $p \neq 0$ case:

$$\lim_{p\to 0}\left(\frac{r}{p} + \left(a^{1-n} - \frac{r}{p}\right)e^{-(1-n)px}\right)^{1/(1-n)}$$

$$= \lim_{p\to 0}\left(a^{1-n}e^{-(1-n)px} + \frac{r}{p}\left(1 - e^{-(1-n)px}\right)\right)^{1/(1-n)}$$

$$= \lim_{p\to 0}\left(a^{1-n}e^{-(1-n)px} + \frac{r}{p}\left(1 - \left(1 + \frac{-(1-n)\,px}{1!} + \frac{\left(-(1-n)\,px\right)^2}{2!} + \dots\right)\right)\right)^{1/(1-n)}$$

(using the Maclaurin series for the exponential function)

$$= \lim_{p\to 0}\left(a^{1-n}e^{-(1-n)px} + r\left(+(1-n)x - \frac{p\left(-(1-n)x\right)^2}{2} - \dots\right)\right)^{1/(1-n)}$$

$$= \left(a^{1-n} + r\big((1-n)x\big)\right)^{1/(1-n)}$$

The complete solution $y(x)$ is constant $\left(y \equiv a\right)$ only if $r = a^{1-n} p$ and/or ($a = 0$ and $n > 0$). In the ordinary solution the term a^{1-n} is well-defined at $a = 0$ only if $1 - n > 0 \quad \Rightarrow \quad n < 1$. When the initial condition is $y(0) = 0$, the ordinary solution exists only if $n < 1$ and the singular solution exists only if $n > 0$. Therefore the complete solution includes both the ordinary and singular solutions when $a = 0$ and $0 < n < 1$. In the following sections we will look at two such cases: $n = \frac{1}{2}$ and $n = \frac{2}{3}$, then at one case $(n = 2)$ where only the singular solution exists at $a = 0$ and finally at one case $(n = -1)$ where the singular solution does not exist.

Furthermore, if $(1 - n)$ is a fraction that, in its lowest terms, has an even denominator, then there is no real solution when $a < 0$ and there may be a distinct pair of real solutions when $a > 0$.

The inhomogeneous linear IVP is a special case ($n = 0$),

$$\frac{dy}{dx} + p\,y = r\,, \quad y(0) = a$$

for which the complete solution is

$$y(x) = \frac{r}{p} + \left(a - \frac{r}{p}\right)e^{-px} \quad .$$

From the differential equation $\frac{dy}{dx} + p\,y = r\,y^n$ one can deduce the existence of a horizontal asymptote shared by many ordinary solutions:

As $\frac{dy}{dx} \to 0\,, \quad p\,y \to r\,y^n \quad \Rightarrow \quad y \equiv 0$ or $y^{n-1} \to \frac{p}{r}$

Therefore, whenever $\left(\frac{p}{r}\right)^{1/(n-1)}$ is a real number, all non-singular solutions share the horizontal asymptote $y = \left(\frac{p}{r}\right)^{1/(n-1)}$ (and, for some values of n, $y = -\left(\frac{p}{r}\right)^{1/(n-1)}$).

4. Example for $n = \frac{1}{2}$

The initial value problem

$$\frac{dy}{dx} + 2y = 2\sqrt{y}\,, \qquad y(0) = a$$

is an example of a first order Bernoulli ODE with $p = r = 2$ and $n = \frac{1}{2}$.
Let us explore the evolution of the graphs of the solution as the value of the initial condition a varies. From the general case above, the complete solution is

$$y(x) = \begin{cases} \left(1 + \left(\pm\sqrt{a} - 1\right)e^{-x}\right)^2 & (a > 0) \\ \left(1 - e^{-x}\right)^2 \text{ or } 0 & (a = 0) \\ \text{no real solution} & (a < 0) \end{cases}$$

Note that the initial value problem has a singular solution $(y \equiv 0)$ in addition to the ordinary solution when (and only when) $a = 0$. At $a = 1$ one branch of the ordinary solution becomes another constant solution, $y \equiv 1$. An interesting result emerges upon investigating the behaviour of the solution to each side of $a = 1$:

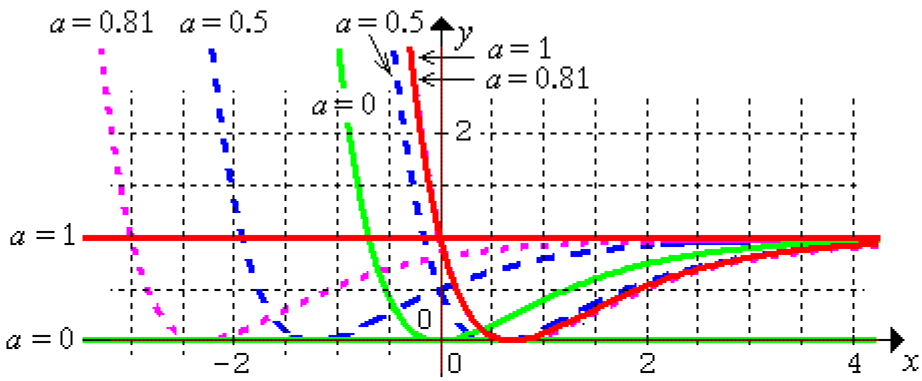

Figure 2 – Family of solutions for $p = r = 2,\ n = \frac{1}{2},\ 0 \le a < 1$

All of the solution graphs for $0 < a < 1$ are similar to each other, one branch shifting further to the left as $a \to 1^-$, the other shifting further to the right. They all share the same limiting behaviour $\lim\limits_{x\to\infty} y(x) = 1$, with the exception of the singular solution $y \equiv 0$.

Whenever $p = r$, all of the ordinary solutions share the same horizontal asymptote,
$y = \left(\frac{p}{r}\right)^{1/(n-1)} = 1$. Figure 3 displays some of the solution graphs for $a \ge 1$.

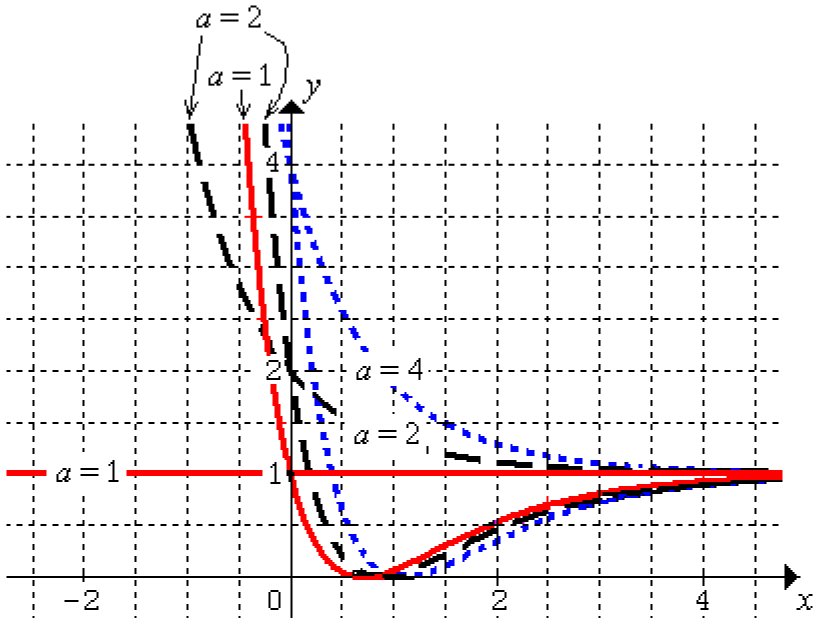

Figure 3 – Family of solutions for $p = r = 2,\ n = \frac{1}{2},\ a \geq 1$

All of the solution graphs for one branch of $a > 1$ are similar to each other, shifting further to the left as $a \to 1^{+}$ and to the right as $a \to \infty$. The other branch is identical to the graphs for $0 < a < 1$, shifting further to the right as $a \to \infty$.

Similarity is established easily.

For one branch of $0 < a < 1$,

$$y = \left(1 + \left(\sqrt{a} - 1\right)e^{-x}\right)^2 = \left(1 - \left(1 - \sqrt{a}\right)e^{-x}\right)^2 = \left(1 - e^{+c}e^{-x}\right)^2 = \left(1 - e^{-(x-c)}\right)^2,$$

where $e^c = 1 - \sqrt{a} \quad \Rightarrow \quad c = \ln\left(1 - \sqrt{a}\right)$.

The same y-axis intercept (0, a) is attained by a horizontal translation of $c = \ln\left(1 - \sqrt{a}\right)$.

For $0 < a < 1$, $\ln\left(1 - \sqrt{a}\right) < 0$ so that the shift is to the left. $c = 0$ when $a = 0$.

For the other branch of $0 < a < 1$,

$$y = \left(1 + \left(-\sqrt{a} - 1\right)e^{-x}\right)^2 = \left(1 - \left(1 + \sqrt{a}\right)e^{-x}\right)^2 = \left(1 - e^{+c}e^{-x}\right)^2 = \left(1 - e^{-(x-c)}\right)^2,$$

where $e^c = 1 + \sqrt{a} \quad \Rightarrow \quad c = \ln\left(1 + \sqrt{a}\right)$.

The same y-axis intercept (0, a) is attained by a horizontal translation of $c = \ln\left(1 + \sqrt{a}\right)$.

$\ln\left(1 + \sqrt{a}\right) > 0$ so that the shift is to the right. $c = 0$ when $a = 0$.

Therefore all of the solution curves for $0 < a < 1$ are pairs of identical copies of $y = \left(1 - e^{-x}\right)^2$, shifted to the right by an amount $c = \ln\left(1 \pm \sqrt{a}\right)$ (which is actually a shift to the left for the branch $c = \ln\left(1 - \sqrt{a}\right)$). For one branch, $c \to -\infty$ as $a \to 1^-$.

For one branch of $a > 1$,

$$y = \left(1 + \left(\sqrt{a} - 1\right)e^{-x}\right)^2 = \left(1 + e^{+c}e^{-x}\right)^2 = \left(1 + e^{-(x-c)}\right)^2,$$

where $e^c = \sqrt{a} - 1 \quad \Rightarrow \quad c = \ln\left(\sqrt{a} - 1\right)$.

$c = 0$ when $\sqrt{a} - 1 = 1 \quad \Rightarrow \quad \sqrt{a} = 2 \quad \Rightarrow \quad a = 4$.

Therefore all of the solution curves for this branch of $a > 1$ are identical copies of $y = \left(1 + e^{-x}\right)^2$, (the solution when $y(0) = a = 4$), shifted to the right by an amount $c = \ln\left(\sqrt{a} - 1\right)$. The shift is actually to the left when $1 < a < 4$. $c \to -\infty$ as $a \to 1^+$ and $c \to +\infty$ as $a \to \infty$.

For the other branch of $a > 1$,

$$y = \left(1 + \left(-\sqrt{a} - 1\right)e^{-x}\right)^2 = \left(1 - \left(1 + \sqrt{a}\right)e^{-x}\right)^2 = \left(1 - e^{+c}e^{-x}\right)^2 = \left(1 - e^{-(x-c)}\right)^2,$$

where $e^c = 1 + \sqrt{a} \quad \Rightarrow \quad c = \ln\left(1 + \sqrt{a}\right)$.

Therefore all of the solution curves for this branch of $a > 1$ are identical copies of $y = \left(1 - e^{-x}\right)^2$, shifted to the right by an amount $c = \ln\left(1 + \sqrt{a}\right)$. For this branch, $c \to +\infty$ as $a \to \infty$.

Therefore the accommodation of varying values of the initial condition $y(0)$ is achieved by *horizontal* shifts in the solution graph rather than vertical shifts, except that a second distinct shape emerges for $a \geq 1$ and that there is no solution graph at all for $a < 0$.

5. Example for $n = \frac{2}{3}$

If we now take $p = r = 3$ and $n = \frac{2}{3}$, then the initial value problem becomes

$$\frac{dy}{dx} + 3y = 3y^{2/3}, \qquad y(0) = a$$

with complete solution

$$y(x) = \begin{cases} \left(1 + \left(\sqrt[3]{a} - 1\right)e^{-x}\right)^3 & (a \neq 0) \\ \left(1 - e^{-x}\right)^3 \text{ or } 0 & (a = 0) \end{cases}$$

Again the behaviour differs on each side of $a = 1$, (where the solution simplifies to $y \equiv 1$) and there is a singular solution for $a = 0$ in addition to the ordinary solution.

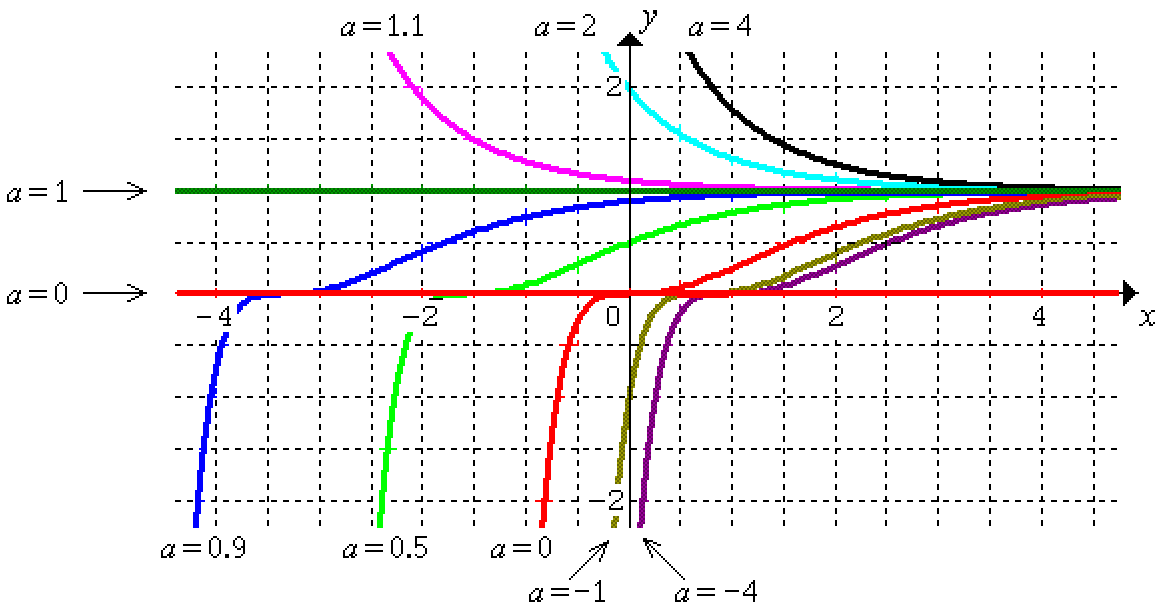

Figure 4 – Family of solutions for $p = r = 3,\ n = \frac{2}{3}$

All of the solution curves for $a < 1$ are horizontal translations of $y = \left(1 - e^{-x}\right)^3$, the ordinary solution curve for $a = 0$, by an amount c, where $1 - \sqrt[3]{a} = e^c \;\Rightarrow\; c = \ln\left(1 - \sqrt[3]{a}\right)$. The translation is to the left when $0 < a < 1$ and to the right when $a < 0$.

All of the solution curves for $a > 1$ are horizontal translations of $y = \left(1 + e^{-x}\right)^3$, the ordinary solution curve for $a = 8$, by an amount c, where $\sqrt[3]{a} - 1 = e^c \;\Rightarrow\; c = \ln\left(\sqrt[3]{a} - 1\right)$. The translation is to the left when $1 < a < 8$ and to the right when $a > 8$.

The solution curve for $a = 1$ is a limiting case of both families and arises from the ordinary solution. Only the singular solution $y \equiv 0$ fails to approach $y = 1$ as $x \to \infty$.

These examples from sections 4 and 5 are both in the range $0 < n < 1$, for which both singular and ordinary solutions exist for $a = y(0) = 0$.

6. Example for $n = 2$

If we now take $p = r = 1$ and $n = 2$, then the initial value problem becomes

$$\frac{dy}{dx} + y = y^2 , \qquad y(0) = a$$

with complete solution

$$y(x) = \begin{cases} \dfrac{1}{1 + \left(\dfrac{1}{a} - 1\right) e^x} & (a \neq 0) \\ 0 & (a = 0) \end{cases}$$

This time only the singular solution exists for $a = y(0) = 0$, although it is easy to show that

$\lim_{a \to 0} \dfrac{1}{1 + \left(\dfrac{1}{a} - 1\right) e^x} = 0 \quad \forall x$, identical to the singular solution.

If and only if $\dfrac{1}{a} - 1 < 0 \quad \Rightarrow \quad \dfrac{1-a}{a} < 0 \quad \Rightarrow \quad a < 0$ or $a > 1$, then the solution curve has an infinite discontinuity at $e^x = \dfrac{-a}{1-a} \quad \Rightarrow \quad x = \ln\left(\dfrac{a}{a-1}\right)$.

We again obtain two families of solutions, one for $0 < a < 1$ and the other outside that range.
Again we can show that every member of each family is a simple horizontal translation of either $y(x) = \dfrac{1}{1+e^x}$ or $y(x) = \dfrac{1}{1-e^x}$ respectively.

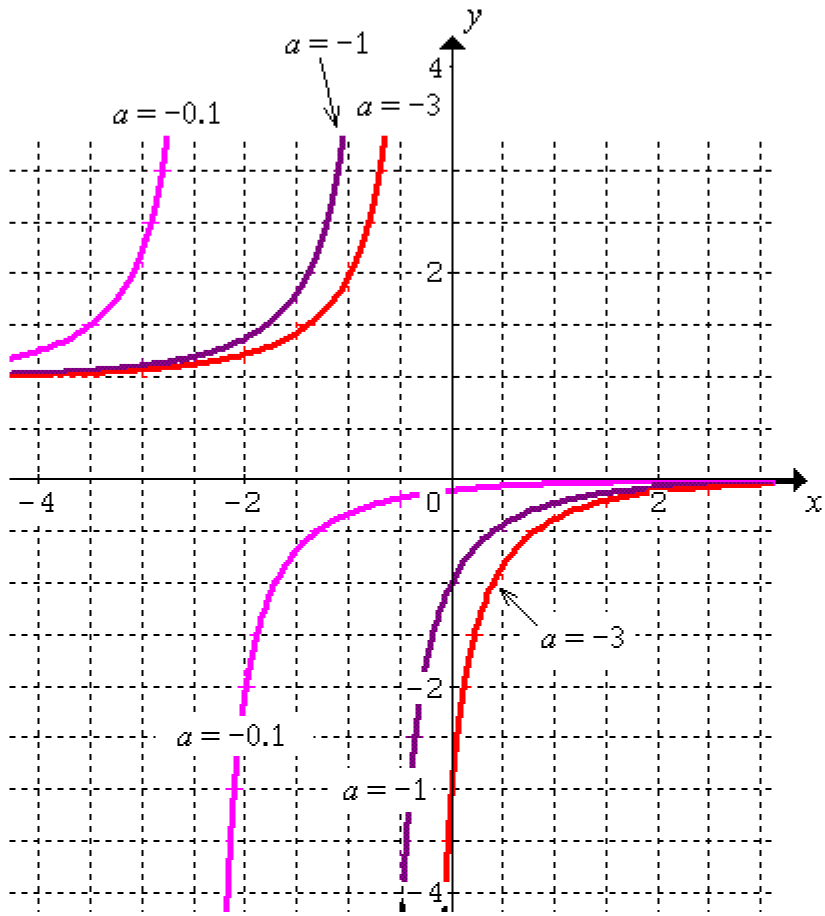

Figure 5 – Family of solutions for $p = r = 1,\ n = 2,\ a < 0$

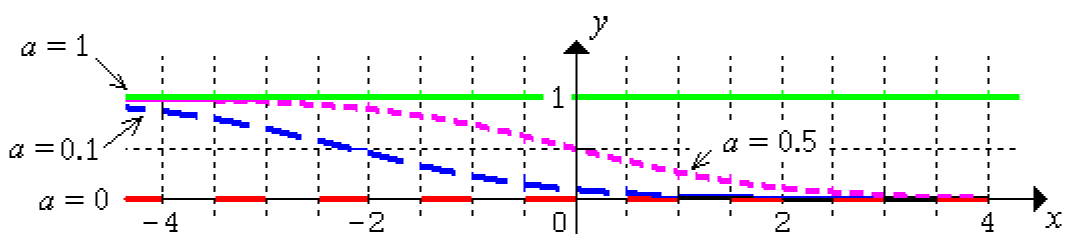

Figure 6 – Family of solutions for $p = r = 1,\ n = 2,\ 0 \le a \le 1$

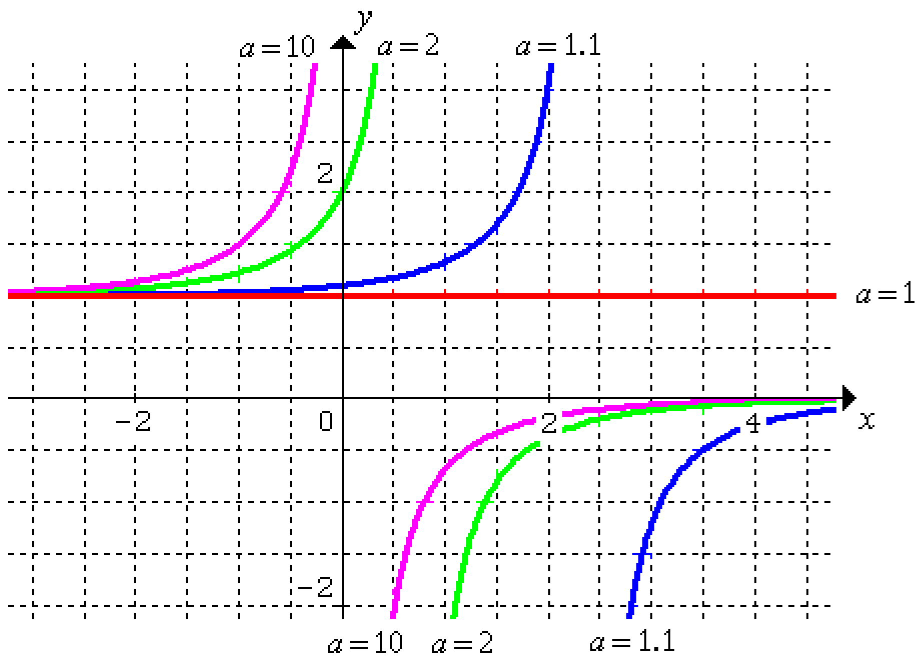

Figure 7 – Family of solutions for $p = r = 1,\ n = 2,\ a \ge 1$

7. Example for $n = -1$ with an asymptote

If we now take $p = r = \frac{1}{2}$ and $n = -1$, then the initial value problem becomes

$$\frac{dy}{dx} + \frac{1}{2} y = \frac{1}{2y}, \qquad y(0) = a$$

with complete solution

$$y(x) = \begin{cases} +\sqrt{1 + \left(a^2 - 1\right) e^{-x}} & (a \ge 0) \\ -\sqrt{1 + \left(a^2 - 1\right) e^{-x}} & (a \le 0) \end{cases}$$

This time there is no singular solution. However, there are two distinct solutions when (and only when) $a = 0$.

The solution is real for all x if and only if $|a| \ge 1$.
There are two constant solutions: $y \equiv 1$ when $a = 1$ and $y \equiv -1$ when $a = -1$.

All of the solution graphs share the same limiting behaviour $\lim_{x \to \infty} |y(x)| = 1$.

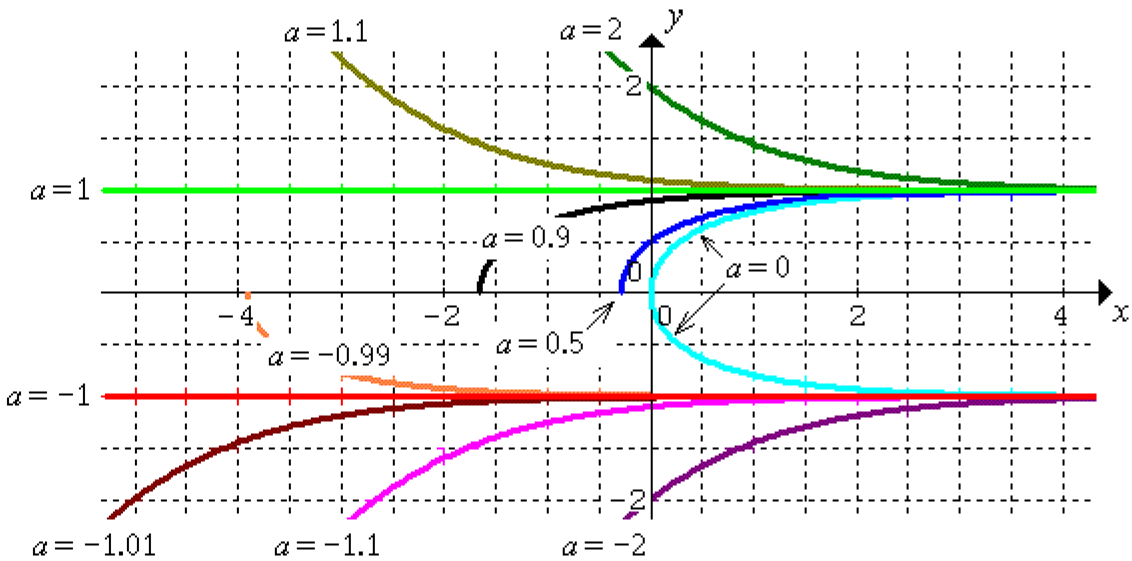

Figure 8 – Family of solutions for $p = r = \frac{1}{2},\ n = -1$

Again we have two families of solution curve:

horizontal translations of $y(x) = \operatorname{sgn}(a)\sqrt{1 - e^{-x}}$ (for $-1 < a < +1$) and

horizontal translations of $y(x) = \operatorname{sgn}(a)\sqrt{1 + e^{-x}}$ (otherwise).

In all of the examples here I have set $p = r$.

The effect of other positive choices for $\dfrac{p}{r}$ is to re-scale the limiting value of $y(x)$ from 1 to $\left(\dfrac{r}{p}\right)^{1/(1-n)}$ (or $\pm\left(\dfrac{r}{p}\right)^{1/(1-n)}$, as in the case $n = -1$ above). If $\left(\dfrac{r}{p}\right)^{1/(1-n)}$ is not a real number, then there is no horizontal asymptote.

8. Example for $n = -1$ without an asymptote

If we now take $p = -\frac{1}{2},\ r = \frac{1}{2}$ and $n = -1$, then the initial value problem becomes

$$\frac{dy}{dx} - \frac{1}{2}y = \frac{1}{2y}, \qquad y(0) = a$$

with complete solution

$$y(x) = \begin{cases} +\sqrt{(a^2+1)e^x - 1} & (a \ge 0) \\ -\sqrt{(a^2+1)e^x - 1} & (a \le 0) \end{cases}$$

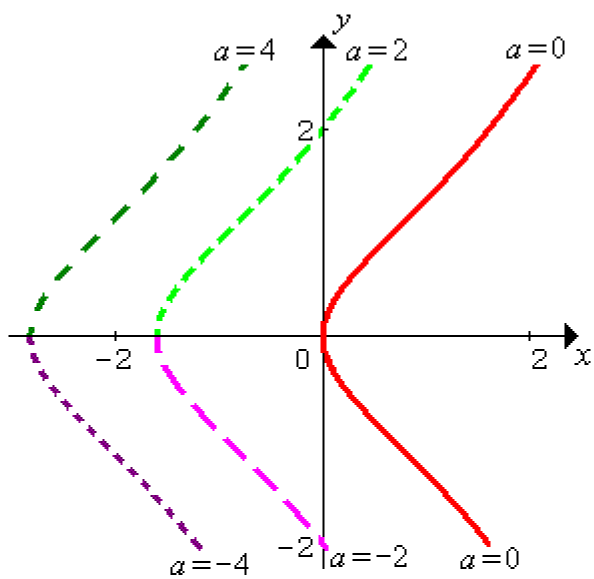

Figure 9 – Family of solutions for $p = -\frac{1}{2},\ r = +\frac{1}{2},\ n = -1$

This time there are no asymptotes, because $\left(\dfrac{r}{p}\right)^{1/(1-n)} = (-1)^{1/2}$ is not a real number.

There are two families of solution curve:
horizontal translations of $y(x) = +\sqrt{e^x - 1}$ (for $a \ge 0$) and
horizontal translations of $y(x) = -\sqrt{e^x - 1}$ (for $a \le 0$).
There are two distinct solutions when and only when $a = 0$. There is no singular solution.

9. The General Case

It is easy to show that this phenomenon, (vertical translations in the initial condition cause horizontal translations in the solution curves), must occur for all non-singular solutions of any first order Bernoulli initial value problem with constant coefficients (except for $p = r = 0$ or $n = 1$).

In the complete solution $y(x) = \left(\frac{r}{p} + \left(a^{1-n} - \frac{r}{p} \right) e^{-(1-n)px} \right)^{1/(1-n)}$, equate the translation $e^{-(1-n)p(x-c)}$ to the term $\left(a^{1-n} - \frac{r}{p} \right) e^{-(1-n)px}$ when $a^{1-n} > \frac{r}{p}$

(or to $-\left(\frac{r}{p} - a^{1-n} \right) e^{-(1-n)px}$ when $a^{1-n} < \frac{r}{p}$).

In the first case,

$$e^{-(1-n)px} e^{+(1-n)pc} \equiv \left(a^{1-n} - \frac{r}{p} \right) e^{-(1-n)px} \quad \Rightarrow \quad e^{+(1-n)pc} \equiv \left(a^{1-n} - \frac{r}{p} \right)$$

$$\Rightarrow \quad +(1-n)pc = \ln\left(a^{1-n} - \frac{r}{p} \right) \quad \Rightarrow \quad c = \frac{1}{(1-n)p} \ln\left(a^{1-n} - \frac{r}{p} \right)$$

A similar calculation in the second case leads to $c = \frac{1}{(1-n)p} \ln\left(\frac{r}{p} - a^{1-n} \right)$.

Therefore *all* non-singular solutions are simple translations of

$y(x) = \left(\frac{r}{p} + e^{-(1-n)px} \right)^{1/(1-n)}$ to the right by $c = \frac{1}{(1-n)p} \ln\left(a^{1-n} - \frac{r}{p} \right)$ when $a^{1-n} > \frac{r}{p}$

or of

$y(x) = \left(\frac{r}{p} - e^{-(1-n)px} \right)^{1/(1-n)}$ to the right by $c = \frac{1}{(1-n)p} \ln\left(\frac{r}{p} - a^{1-n} \right)$ when $a^{1-n} < \frac{r}{p}$.

In each case, the translation is actually to the left if $c < 0$.

The behaviour of the exponential function (as mentioned in section 1) is the fundamental reason for this feature of the solutions to the Bernoulli initial value problem.

The boundary case between the two families, the horizontal asymptote $y(x) \equiv \left(\frac{r}{p} \right)^{1/(1-n)}$, is also the limiting case of an infinite translation for both families.

All of the non-singular solutions tend to the same limiting value of $\left(\frac{r}{p}\right)^{1/(1-n)}$ as $x \to +\infty$ in the case $(1-n)\,p > 0$ (or as $x \to -\infty$ in the case $(1-n)\,p < 0$), except when $\left(\frac{r}{p}\right)^{1/(1-n)}$ is not real.

This behaviour persists even for $p = 0$:
For $p = 0$, $r \neq 0$ and $n \neq 1$ the general solution may be re-written as
$y(x) = \left((1-n)rx + a^{1-n}\right)^{1/(1-n)} = \left((1-n)r(x-c)\right)^{1/(1-n)}$, where $c = \dfrac{-a^{1-n}}{(1-n)r}$

If $r = p = 0$, then the ODE becomes the trivial $\dfrac{dy}{dx} = 0$, whose solution is just $y \equiv a$ for all *n*.
If $(n = 1 \text{ and } r = p)$ then again the ODE becomes $\dfrac{dy}{dx} = 0$, whose solution is $y \equiv a$ for all *n*.
Together with the singular solutions, these are the only exceptions to the following conclusion.

A vertical translation in the initial value $a = y(0)$ results in a horizontal translation of another solution curve to the Bernoulli ODE with constant coefficients, $\dfrac{dy}{dx} + p\,y = r\,y^n$.

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