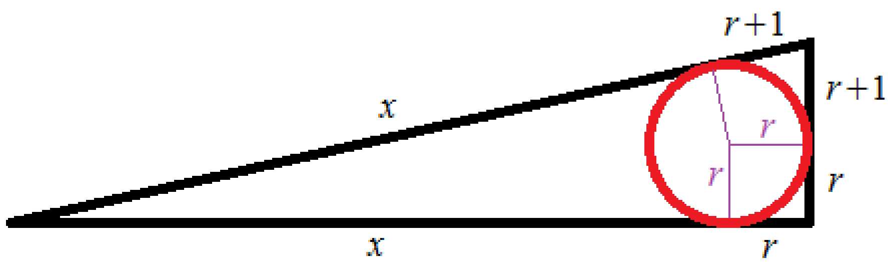
Circles in Triangles

A circle of integer radius r just touches the three sides of a right-angled triangle internally.
The hypotenuse of the triangle is exactly one unit longer than the next longest side.
Find the lengths of the three sides of the triangle in terms of r.

The tangents from a point to a circle are of equal length (as shown in the diagram).



The three sides of the triangle have lengths $(x+r+1)$, $(x+r)$ and $(2r+1)$.

Therefore $(x+r+1)^2 = (x+r)^2 + (2r+1)^2$

$$\Rightarrow \quad \cancel{(x+r)^2} + 2(x+r) + \cancel{1} = \cancel{(x+r)^2} + \left(4r^2 + 4r + \cancel{1}\right) \quad \Rightarrow \quad 2(x+r) = 4r^2 + 4r$$

$$\Rightarrow \quad x + r = 2r^2 + 2r \quad \Rightarrow \quad x = 2r^2 + r$$

Therefore the three sides of the triangle have lengths

$$\boxed{2r^2 + 2r + 1,\ 2r^2 + 2r\ ,\ 2r + 1}$$

r	***a***	***b***	**hyp**
1	3	4	5
2	5	12	13
3	7	24	25
4	9	40	41
5	11	60	61
6	13	84	85
7	15	112	113
8	17	144	145
9	19	180	181
10	21	220	221

The $r = 4$ case is illustrated.

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