Coffee Cooling

An example of first order ordinary differential equations

Does white coffee cool faster if one adds the milk immediately or if one delays adding the milk until later? Let

$T_c =$ the initial temperature of the coffee (no milk)

$T_m =$ the temperature of the milk when added

$T_R =$ the ambient (room) temperature

$T(t) =$ the temperature of the coffee (with milk after it has been added) at time t

$h =$ the time to halve the temperature difference between coffee and air (the half life)

$k =$ the cooling coefficient $= (\ln 2) \div h$

$t_1 =$ the time at which the milk is added [assuming instant complete mixing]

$V_c =$ the volume of the coffee

$V_m =$ the volume of the milk

Newton's law of cooling is

$$\frac{dT}{dt} = k\left(T - T_R\right)$$

the solution of which, (before the milk is added), is

$$T(t) = T_R + \left(T_c - T_R\right) e^{-kt} = T_R + \left(T_c - T_R\right) 2^{-t/h}$$

If the milk is added immediately (at $t = 0$), then

$$T(t) = T_R + \left(\frac{V_c T_c + V_m T_m}{V_c + V_m} - T_R\right) e^{-kt}$$

If the milk is added at some subsequent time $t = t_1$, then

$$T\left(t_1^-\right) = T_R + \left(T_C - T_R\right) e^{-kt_1}$$

$$T\left(t_1^+\right) = \frac{V_c T\left(t_1^-\right) + V_m T_m}{V_c + V_m}$$

After the milk is added $\left(t > t_1\right)$,

$$T(t) = T_R + \left(T\left(t_1^+\right) - T_R\right) e^{-k\left(t - t_1\right)}$$

$$\Rightarrow \quad T(t) = T_R + \left(\frac{V_c\left(T_R + \left(T_c - T_R\right) e^{-kt_1}\right) + V_m T_m}{V_c + V_m} - T_R\right) e^{-k\left(t - t_1\right)}$$

$$= T_R + \frac{V_c\left(T_c - T_R\right) e^{-kt}}{V_c + V_m} + \left(\frac{V_c T_R + V_m T_m}{V_c + V_m} - T_R\right) e^{-k\left(t - t_1\right)}$$

Define the difference between these two temperatures at time $t > t_1$ to be
$\Delta T =$ (temp. if milk added later) $-$ (temp. if milk added immediately), that is

$$\Delta T = \frac{-\left(V_c T_R + V_m T_m\right) e^{-kt}}{V_c + V_m} + T_R\left(e^{-kt} - e^{-k(t-t_1)}\right) + \left(\frac{V_c T_R + V_m T_m}{V_c + V_m}\right) e^{-k(t-t_1)}$$

$$= \frac{-\left(V_c T_R + V_m\left(T_R + T_m - T_R\right)\right) e^{-kt}}{V_c + V_m} + T_R\left(e^{-kt} - e^{-k(t-t_1)}\right) + \left(\frac{V_c T_R + V_m\left(T_R + T_m - T_R\right)}{V_c + V_m}\right) e^{-k(t-t_1)}$$

$$= \frac{-\left(\left(V_c + V_m\right) T_R + V_m\left(T_m - T_R\right)\right) e^{-kt}}{V_c + V_m} + T_R\left(e^{-kt} - e^{-k(t-t_1)}\right) + \left(\frac{\left(V_c + V_m\right) T_R + V_m\left(T_m - T_R\right)}{V_c + V_m}\right) e^{-k(t-t_1)}$$

$$= -T_R e^{-kt} - \frac{V_m\left(T_m - T_R\right)}{V_c + V_m} e^{-kt} + T_R e^{-kt} - T_R e^{-k(t-t_1)} + T_R e^{-k(t-t_1)} + \frac{V_m\left(T_m - T_R\right)}{V_c + V_m} e^{-k(t-t_1)}$$

$$\Rightarrow \boxed{\Delta T = -\frac{V_m\left(T_R - T_m\right)}{V_c + V_m}\left(e^{-k(t-t_1)} - e^{-kt}\right)}$$

If the milk is kept at room temperature, $\left(T_m = T_R\right)$, then this simplifies to $\Delta T = 0$.

Therefore the time t_1 at which one adds milk at room temperature to cooling coffee has ***no effect at all*** on the subsequent temperature of the milk/coffee mixture! The temperature is

$$\boxed{T(t) = \begin{cases} T_R + \left(T_c - T_R\right) e^{-kt} & \left(t < t_1\right) \\ T_R + \left(\dfrac{V_c T_c + V_m T_R}{V_c + V_m} - T_R\right) e^{-kt} & \left(t > t_1\right) \end{cases}}$$

This independence disappears if the milk is kept at a temperature other than room temperature. If the milk is colder than room temperature, then delaying its addition cools the coffee down faster. If the milk is warmer than room temperature, then immediate addition maximizes the cooling rate.

An Excel file illustrates this situation.

This work is inspired by the article “Keeping the Coffee Warm, The Mathematics of Cooling”, by Stella Dudzic, Mathematics in Schools, vol. **42(3)**, pp. 9-10, 2013 May.

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