88.76 Testing for the independence of three events

In an introductory course in probability, it does not take long to encounter the statement that two events A and B are [stochastically] independent if and only if

$$\mathrm{P}[A \text{ and } B] = \mathrm{P}[A] \times \mathrm{P}[B].$$

I have noticed that some students then go on to assume that three events A, B and C are 'independent' if and only if

$$\mathrm{P}[A \text{ and } B \text{ and } C] = \mathrm{P}[A] \times \mathrm{P}[B] \times \mathrm{P}[C]. \quad (1)$$

The n events $\{A_1, A_2, \ldots, A_n\}$ are mutually independent if, and only if, for every k ($k = 2, 3, \ldots, n$) and every subset $\{i_1, i_2, \ldots, i_k\}$ of k distinct values drawn from the set of the first n natural numbers,

$$\mathrm{P}\left[A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}\right] = \mathrm{P}\left[A_{i_1}\right] \times \mathrm{P}\left[A_{i_2}\right] \times \ldots \times \mathrm{P}\left[A_{i_k}\right]. \quad (2)$$

Equation (1) is therefore a necessary, but not sufficient, condition for the mutual independence of three events.

An efficient way to cause students to think again about equation (1) is to present the following counterexample:

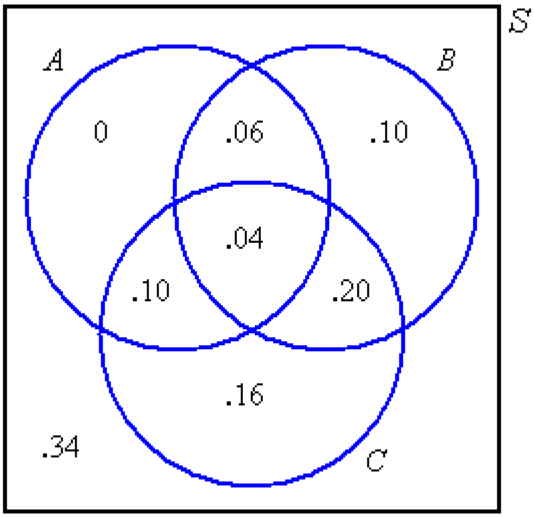

FIGURE 1

One can soon see that, although

$$\mathrm{P}[A \text{ and } B \text{ and } C] = \mathrm{P}[A] \times \mathrm{P}[B] \times \mathrm{P}[C] = .04,$$

no two of the three events are pair-wise independent:

$$\mathrm{P}[A \text{ and } B] = .10, \quad \text{but} \quad \mathrm{P}[A] \times \mathrm{P}[B] = .2 \times .4 = .08$$

$$\mathrm{P}[B \text{ and } C] = .24, \quad \text{but} \quad \mathrm{P}[B] \times \mathrm{P}[C] = .4 \times .5 = .20$$

$$\mathrm{P}[C \text{ and } A] = .14, \quad \text{but} \quad \mathrm{P}[C] \times \mathrm{P}[A] = .5 \times .2 = .10$$

Acknowledgement
I wish to thank an anonymous referee for a suggestion that has improved this note.

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