Limits and second order ordinary differential equations

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Introduction

In the initial value problem

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = ke^{-ax}, \qquad y(0) = y_0,\ y'(0) = y'_0$$

(where p, q, k, a, y_0, y'_0 are all constants), a new student is often surprised at the change in form of the solution

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x} + ce^{-ax}$$

when at least two of $\lambda_1, \lambda_2, -a$ are equal.

Valuable practice in finding limits (especially l'Hôpital's rule) and practice in matrix algebra arise in a demonstration that these changes of form must occur.

However, the complete solutions must be used, not the general solutions. While it is true that the general solution of

$$\frac{d^2y}{dx^2} + (m + n)\frac{dy}{dx} + mny = ke^{-ax} \qquad (m, n, a \text{ all distinct})$$

is $y = Ae^{-mx} + Be^{-nx} + \dfrac{k}{(a - m)(a - n)}e^{-ax}$ and the general solution of

$$\frac{d^2y}{dx^2} + (m + n)\frac{dy}{dx} + mny = ke^{-nx} \qquad (m, n \text{ distinct})$$

is

$$y(x) = Ce^{-mx} + De^{-nx} + \frac{k}{m - n}xe^{-nx},$$

the limit as $a \to n$ of the particular solution for $a \neq n$ is *not* the particular solution for $a = n$ (unless $k = 0$):

$$\lim_{a \to n} \frac{k}{(a - m)(a - n)}e^{-ax} \neq \frac{k}{m - n}xe^{-nx}.$$

In fact, this limit does not exist at all!

Complete solutions

The initial value problem

$$\frac{d^2y}{dx^2} + (m + n)\frac{dy}{dx} + mny = ke^{-ax} \qquad (1)$$

(m, n, a all distinct), $y(0) = y_0$, $y'(0) = y'_0$, can be shown to have the complete solution

$$y(x) = \frac{1}{m-n}\left(\left(y'_0 + my_0 + \frac{k}{a-n}\right)e^{-nx} - \left(y'_0 + ny_0 + \frac{k}{a-m}\right)e^{-mx}\right)$$
$$+ \frac{k}{(a-m)(a-n)}e^{-ax}. \quad (2)$$

By taking appropriate limits of this complete solution, one can verify that the complete solution to

$$\frac{d^2y}{dx^2} + (m+n)\frac{dy}{dx} + mny = ke^{-nx}, \qquad \left(m \neq n, y(0) = y_0, y'(0) = y'_0\right) \quad (3)$$

is

$$y(x) = \frac{1}{m-n}\left(\left(y'_0 + my_0 - \frac{k}{m-n} + kx\right)e^{-nx} - \left(y'_0 + ny_0 - \frac{k}{m-n}\right)e^{-mx}\right) \quad (4)$$

and that the complete solution to

$$\frac{d^2y}{dx^2} + 2n\frac{dy}{dx} + n^2y = ke^{-ax}, \qquad (a \neq n, y(0) = y_0, y'(0) = y'_0) \quad (5)$$

is

$$y(x) = \left(\left(y'_0 + ny_0 - \frac{k}{n-a}\right)x + \left(y_0 - \frac{k}{(n-a)^2}\right)\right)e^{-nx} + \frac{k}{(n-a)^2}e^{-ax} \quad (6)$$

and that the complete solution to

$$\frac{d^2y}{dx^2} + 2n\frac{dy}{dx} + n^2y = ke^{-nx}, \qquad \left(y(0) = y_0, y'(0) = y'_0\right) \quad (7)$$

is

$$y(x) = \left(\frac{1}{2}kx^2 + \left(y'_0 + ny_0\right)x + y_0\right)e^{-nx}. \quad (8)$$

These solutions have been incorporated into an Excel spreadsheet, see [1]. This spreadsheet file can assist in the setting of practice and test questions and the rapid verification of their solutions.

Below are derivations of (2) (the general case), which provides practice with 2×2 matrices, and of (8), which provides practice with limits.

General case (m, n, a all distinct)

$$\frac{d^2y}{dx^2} + (m+n)\frac{dy}{dx} + mny = ke^{-ax}, \qquad \left(y(0) = y_0, y'(0) = y'_0\right). \quad (9)$$

The general solution is easily found to be

$$y(x) = Ae^{-mx} + Be^{-nx} + \frac{k}{(a-m)(a-n)}e^{-ax}.$$

Applying the initial conditions $y(0) = y_0, y'(0) = y'_0$,

$$y(0) = y_0 \Rightarrow y_0 = A + B + \frac{k}{(a-m)(a-n)}$$

$$y'(x) = -mAe^{-mx} - nBe^{-nx} - \frac{ak}{(a-m)(a-n)}e^{-ax}$$

$$y'(0) = y'_0 \Rightarrow y'_0 = -mA - nB - \frac{ak}{(a-m)(a-n)}.$$

In matrix form,

$$\begin{bmatrix} 1 & 1 \\ -m & -n \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} y_0 - \dfrac{k}{(a-m)(a-n)} \\ y'_0 + \dfrac{ak}{(a-m)(a-n)} \end{bmatrix}$$

the solution of which is

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{m-n}\begin{bmatrix} -n & -1 \\ m & 1 \end{bmatrix}\begin{bmatrix} y_0 - \dfrac{k}{(a-m)(a-n)} \\ y'_0 + \dfrac{ak}{(a-m)(a-n)} \end{bmatrix}$$

$$\Rightarrow A = \frac{1}{m-n}\left(-ny_0 + \frac{nk}{(a-m)(a-n)} - y'_0 - \frac{ak}{(a-m)(a-n)}\right)$$

$$= -\frac{1}{m-n}\left(y'_0 + ny_0 + \frac{(a-n)k}{(a-m)(a-n)}\right)$$

$$\Rightarrow A = -\frac{1}{m-n}\left(y'_0 + ny_0 + \frac{k}{a-m}\right)$$

and

$$B = \frac{1}{m-n}\left(my_0 - \frac{mk}{(a-m)(a-n)} + y'_0 + \frac{ak}{(a-m)(a-n)}\right)$$

$$= \frac{1}{m-n}\left(y'_0 + my_0 + \frac{(a-m)k}{(a-m)(a-n)}\right)$$

$$\Rightarrow B = \frac{1}{m-n}\left(y'_0 + my_0 + \frac{k}{a-n}\right).$$

The complete solution for (1) (m, n, a all distinct) is

$$y(x) = \frac{1}{m-n}\left(\left(y_0' + my_0 + \frac{k}{a-n}\right)e^{-nx} - \left(y_0' + ny_0 + \frac{k}{a-m}\right)e^{-mx}\right) + \frac{k}{(a-m)(a-n)}e^{-ax}. \tag{10}$$

If any of m, n, a are equal, then the form of the particular solution changes.

Case $n = m = a$ (*directly*):

$$\frac{d^2y}{dx^2} + 2n\frac{dy}{dx} + n^2y = ke^{-nx}. \tag{11}$$

The general solution is $y(x) = \left(\tfrac{1}{2}kx^2 + Ax + B\right)e^{-nx}$.

Applying the initial conditions,

$$y(0) = (0 + 0 + B) = y_0 \Rightarrow B = y_0$$

$$y'(x) = \left(kx + A - \tfrac{1}{2}nkx^2 - nAx - nB\right)e^{-nx}$$

$$\Rightarrow y'(0) = (0 + A - 0 - 0 - ny_0) = y_0' \Rightarrow A = y_0' + ny_0.$$

The complete solution is

$$y(x) = \left(\frac{1}{2}kx^2 + \left(y_0' + ny_0\right)x + y_0\right)e^{-nx}. \tag{12}$$

Case $n = m = a$ (*by limits*):

Now take the limit of (6) to the case $m = n \neq a$ as $a \to n$:

$$y(x) = \left((y_0' + ny_0)x + y_0\right)e^{-nx} + \left(\left(-\frac{k}{n-a}\right)x - \left(\frac{k}{(n-a)^2}\right)\right)e^{-nx} + \frac{k}{(n-a)^2}e^{-ax}$$

$$= \left((y_0' + ny_0)x + y_0\right)e^{-nx} + \frac{k}{(n-a)^2}\left(e^{-ax} - \left((n-a)x + 1\right)e^{-nx}\right).$$

The latter term involves a 0/0 type of indeterminacy, for which we may apply l'Hôpital's rule twice.

$$\lim_{a \to n}\left(\frac{k}{(n-a)^2}\left(e^{-ax} - \left((n-a)x + 1\right)e^{-nx}\right)\right)$$

$$\overset{H}{=} \lim_{a \to n}\left(\frac{k}{-2(n-a)}\left(-xe^{-ax} - (-x)e^{-nx}\right)\right)$$

$$\overset{H}{=} \lim_{a \to n}\left(\frac{k}{+2}\left(+x^2e^{-ax} - 0\right)\right) = \frac{1}{2}kx^2e^{-nx}.$$

Therefore the complete solution in the case $n = m = a$ is

$$y(x) = \left(\frac{1}{2}kx^2 + (y_0' + ny_0)x + y_0\right)e^{-nx}. \tag{13}$$

An alternative is to take the limit of (4) to the case $a = n \neq m$ as $m \to n$:

$$y(x) = \frac{1}{m-n}\left(\left(y_0' + my_0 - \frac{k}{m-n} + kx\right)e^{-nx} - \left(y_0' + ny_0 - \frac{k}{m-n}\right)e^{-mx}\right)$$

$$= \frac{1}{(m-n)^2}\left(\left(\left(y_0' + my_0 + kx\right)(m-n) - k\right)e^{-nx} - \left(\left(y_0' + ny_0\right)(m-n) - k\right)e^{-mx}\right).$$

Again, this is a 0/0 type of indeterminacy as $m \to n$

$$\lim_{m \to n} y(x) \overset{H}{=}$$

$$\lim_{m\to n} \frac{1}{2(m-n)}\left(\left(2my_0 + y_0' + kx - ny_0\right)e^{-nx} - \left(y_0' + ny_0 - x\left(\left(y_0' + ny_0\right)(m-n) - k\right)\right)e^{-mx}\right)$$

$$\overset{H}{=} \lim_{m\to n} \frac{1}{2}\left(2y_0e^{-nx} - \left(-x\left(y_0' + ny_0\right) - x\left(y_0' + ny_0 - x\left(y_0' + ny_0\right)(m-n) - k\right)\right)e^{-mx}\right)$$

$$= \frac{1}{2}\left(2y_0 + 2x\left(y_0' + ny_0\right) + kx^2\right)e^{-nx}$$

$$\Rightarrow \lim_{m\to n} y(x) = \left(y_0 + x\left(y_0' + ny_0\right) + \frac{1}{2}kx^2\right)e^{-nx}$$

which is (8) again.

The other cases are left to the interested reader.

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Reference

1. Available at http://www.engr.mun.ca/~ggeorge/ODE2_solver.xlsx

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