Comment on the Article

"Picturing Newton's Formula for π"

by Hasan Ünal

Mathematics in School, 2012 Nov., vol. 41, no. 5, page 13

In a curious turn of events, there is a simple error in the published letter of 2013 January which invalidates the derivation of the generalised form of Newton's formula, yet that generalised form remains correct! I am grateful to Brian Pottle, a student in the Bachelor of Engineering programme at Memorial University, for bringing this error to my attention.

The published letter reads:

Mathematics in School, 2013 Jan., vol. 42, no. 1, page 39

I enjoyed reading the extension of Newton's formula to

$$2\tan^{-1}\left(\frac{a-x}{a}\right) + \tan^{-1}\left(\frac{x}{a-1}\right) + \tan^{-1}\left(\frac{1}{a}\right) = \frac{\pi}{2} \quad \text{with} \quad x = a\sqrt{2}$$

However, I noticed a further generalisation.

$x = a\sqrt{2} \quad \Rightarrow \quad a = \dfrac{x^2}{2}$ ☜ [the error is here – a failure to square a]

which is an integer if and only if x is an even integer, $x = 2n$

$$\Rightarrow \qquad a = \frac{(2n)^2}{2} = 2n^2 \quad \Rightarrow \quad a - x = 2n^2 - 2n = 2n(n-1),$$

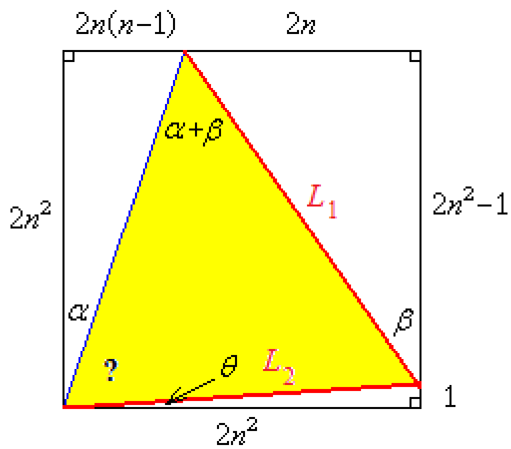
$$\frac{a-x}{a} = \frac{2n(n-1)}{2n^2} = \frac{n-1}{n}, \quad \frac{x}{a-1} = \frac{2n}{2n^2-1} \quad \text{and} \quad \frac{1}{a} = \frac{1}{2n^2}$$

The generalised form of Newton's formula then becomes, for all natural numbers n,

$$2\tan^{-1}\left(\frac{n-1}{n}\right) + \tan^{-1}\left(\frac{2n}{2n^2-1}\right) + \tan^{-1}\left(\frac{1}{2n^2}\right) = \frac{\pi}{2}$$

Starting instead with a generalisation of the diagram in the note by Hasan Ünal, [next page]

Generalised Newton's Formula for π



The square is of side $2n^2$. Two of the vertices of the shaded triangle divide the top and right sides as shown. The angles α, β and θ are defined to be the angles as illustrated in the left, right and bottom unshaded right-angled triangles respectively, so that

$$\tan\alpha = \frac{2n(n-1)}{2n^2} = \frac{n-1}{n}, \quad \tan\beta = \frac{2n}{2n^2-1} \quad \text{and} \quad \tan\theta = \frac{1}{2n^2}$$

The angle at the top vertex of the shaded triangle is clearly $\alpha+\beta$.

A key step is to show that the shaded triangle is isosceles.
The square of the length of its right side is (from the right unshaded triangle)

$$L_1^2 = (2n)^2 + \left(2n^2-1\right)^2 = 4n^2 + 4n^4 - 4n^2 + 1 = 4n^4 + 1$$

The square of the length of its lower side is (from the bottom unshaded triangle)

$$L_2^2 = (2n)^2 + (1)^2 = 4n^4 + 1 = L_1^2$$

Therefore these two sides of the shaded triangle are equal.
Therefore the angle at the lower left vertex of the triangle is $\alpha+\beta$.

Adding up the angles at the lower left corner of the square, $\alpha + (\alpha+\beta) + \theta = \frac{\pi}{2} \quad \Rightarrow$

$$2\tan^{-1}\left(\frac{n-1}{n}\right) + \tan^{-1}\left(\frac{2n}{2n^2-1}\right) + \tan^{-1}\left(\frac{1}{2n^2}\right) = \frac{\pi}{2}$$

Setting $n=2$ recovers Newton's formula.
The generalised form holds even for $n = 1$!

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