6

ELECTRICAL SYSTEMS

Except at quite high frequencies, electrical circuits can usually be considered as an interconnection of lumped elements. In such cases, which include a large and very important portion of the applications of electrical phenomena, we can model a circuit by using ordinary differential equations and we can apply the solution techniques discussed in this book.

In this chapter, we consider fixed linear circuits using the same approach we used for mechanical systems. We introduce the element and interconnection laws and then combine them to form procedures for finding the model of a circuit. After developing a general technique for finding the input-output model, we present several specialized results for the important case of resistive circuits. We then discuss systematic procedures for obtaining the model as a set of state-variable equations. After a discussion of operational amplifiers, the chapter concludes with two computer simulations.

6.1 VARIABLES

The variables most commonly used to describe the behavior of electrical circuits are

\[ e, \text{ voltage in volts (V)} \]
\[ i, \text{ current in amperes (A)} \]

The related variables

\[ q, \text{ charge in coulombs (C)} \]
\[ \phi, \text{ flux in webers (Wb)} \]
\[ \Lambda, \text{ flux linkage in weber-turns} \]

may be used on occasion. Current is the time derivative of charge, so \( i \) and \( q \) are related by the expressions

\[ i = \frac{dq}{dt} \quad (1) \]

and

\[ q(t) = q(t_0) + \int_{t_0}^{t} i(\lambda) d\lambda \quad (2) \]

Flux and flux linkage are related by the number of turns \( N \) in a coil of wire, such that if all the turns are linked by all the flux, then \( \Lambda = N\phi \).

Figure 6.1 Conventions for denoting current. (a) Acceptable. (b) Preferred.

We represent the current into and out of a circuit element by arrows drawn on the circuit diagram as shown in Figure 6.1. The arrows point in the direction in which positive charge—that is, positive ions—flows when the current has a positive value. Equivalently, a positive current can also correspond to electrons (which have a negative charge) flowing in the opposite direction.

Because a net charge cannot exist within any circuit element, the current entering one end of a two-terminal element must leave the other end. Hence \( i_1 = i_2 \) in Figure 6.1(a) at all times, so only one current arrow need be shown, as in Figure 6.1(b).

The voltage at a point in a circuit is a measure of the difference between the electrical potential of that point and the potential of an arbitrarily established reference point called the ground node, or ground for short. The ground associated with a circuit is denoted by the symbol shown in the lower part of Figure 6.2(a). Any point in the circuit that has the same potential as the ground has a voltage of zero, by definition. The voltage \( e_1 \) shown in Figure 6.2(a) is positive if the point with which it is associated is at a higher potential than the ground; it is negative if the potential of the point with which it is associated is lower than that of the ground.

We can define the voltages of the two terminals of a circuit element individually with respect to ground by writing appropriate symbols next to the terminals, as shown in Figure 6.2(b). We define the voltage between the terminals of an element by placing a symbol next to the element and plus and minus signs on either side of the element or at the terminals, as shown in Figure 6.2(c). When the element voltage \( e_a \) is positive, the terminal marked with the plus sign is at a higher potential than the other terminal.

In Figure 6.2(d), \( e_1 \) and \( e_2 \) denote the terminal voltages with respect to ground, and \( e_a \) is the voltage across the element, with its positive sense indicated by the plus and minus

Figure 6.2 Conventions for denoting voltages.
signs. These three voltages are related by the equation
\[ e_a = e_1 - e_2 \]
Interchanging the plus and minus signs reverses the sign of the voltage \( e_a \) in any equation in which it appears.

When we define the positive senses of the current and voltage associated with a circuit element as shown in Figure 6.3, such that a positive current is assumed to enter the element at the terminal designated by the plus sign, then the power supplied to the element is
\[ p = ei \]
which has units of watts. If at some instant \( p \) is negative, then the circuit element is supplying power to the rest of the circuit at that instant. Because power is the time derivative of energy, the energy supplied to the element over the interval \( t_0 \) to \( t_1 \) is
\[ \int_{t_0}^{t_1} p(t) \, dt \]
which has units of joules, where 1 joule = 1 volt-ampere-second.

\[ \text{6.2 ELEMENT LAWS} \]

The elements in the electrical circuits that we shall consider are resistors, capacitors, inductors, and sources. The first three of these are referred to as \textbf{passive elements} because, although they can store or dissipate energy that is present in the circuit, they cannot introduce additional energy. They are analogous to the dashpot, mass, and spring for mechanical systems. In contrast, sources are \textbf{active elements} that can introduce energy into the circuit and that serve as the inputs. They are analogous to the force or displacement inputs for mechanical systems.

\textbf{Resistor}

A \textbf{resistor} is an element for which there is an algebraic relationship between the voltage across its terminals and the current through it—that is, an element that can be described by a curve of \( e \) versus \( i \). A linear resistor is one for which the voltage and current are directly proportional to each other—that is, one described by \textbf{Ohm's law}:
\[ e = Ri \]
\[ i = \frac{e}{R} \]
where \( R \) is the \textbf{resistance} in ohms (\( \Omega \)). A resistor and its current and voltage are denoted as shown in Figure 6.4. If we reversed either the current arrow or the voltage polarity (but not both) in the figure, we would introduce a minus sign into (4) and (5). The resistance of a body of length \( l \) and constant cross-sectional area \( A \) made of a material with resistivity \( \rho \) is \( R = \rho l / A \).

A resistor dissipates any energy supplied to it by converting it into heat (in this it is analogous to the frictional element of mechanical systems). We can write the power \( ei \) dissipated by a linear resistor as
\[ p = R i^2 = \frac{1}{R} e^2 \]

\textbf{Capacitor}

A \textbf{capacitor} is an element that obeys an algebraic relationship between the voltage and the charge, where the charge is the integral of the current. We use the symbol shown in Figure 6.5 to represent a capacitor. For a linear capacitor, the charge and voltage are related by
\[ q = C e \]
where \( C \) is the \textbf{capacitance} in farads (F). For a fixed linear capacitor, the capacitance is a constant. If (6) is differentiated and \( q \) replaced by \( i \), the element law for a fixed linear capacitor becomes
\[ i = \frac{C}{dt} \frac{de}{dt} \]
To express the voltage across the terminals of the capacitor in terms of the current, we solve (7) for \( de/dt \) and then integrate, getting
\[ e(t) = e(t_0) + \frac{1}{C} \int_{t_0}^{t} i(\lambda) \, d\lambda \]
where \( e(t_0) \) is the voltage corresponding to the initial charge, and where the integral is the charge delivered to the capacitor between the times \( t_0 \) and \( t \).

One form of a capacitor consists of two parallel metallic plates, each of area \( A \), separated by a dielectric material of thickness \( d \). Provided that fringing of the electric field is negligible, the capacitance of this element is \( C = \varepsilon A / d \), where \( \varepsilon \) is the permittivity of the dielectric material. The values of practical capacitances are typically expressed in micro-
farads ($\mu F$), where $1 \, \mu F = 10^{-6} \, F$. However, for numerical convenience we may use farads in our examples.

The energy supplied to a capacitor is stored in its electrical field and can affect the response of the circuit at future times. For a fixed linear capacitor, the stored energy is

$$w = \frac{1}{2} Ce^2$$

Because the energy stored is a function of the voltage across its terminals, the initial voltage $e(t_0)$ of a capacitor is one of the conditions we need in order to find the complete response of a circuit for $t \geq t_0$.

**Inductor**

An **inductor** is an element for which there is an algebraic relationship between the voltage across its terminals and the derivative of the flux linkage. The symbol for an inductor and the convention for defining its current and voltage are shown in Figure 6.6. For a linear inductor,

$$e = \frac{d}{dt}(Li)$$

where $L$ is the **inductance** with units of henries (H). For a fixed linear inductor, $L$ is constant and we can write the element law as

$$e = LI$$

We can find an expression for the current through the inductor by using (9) to integrate $di/dt$, giving

$$i(t) = i(t_0) + \frac{1}{L} \int_{t_0}^{t} e(\lambda) \, d\lambda$$

where $i(t_0)$ is the initial current through the inductor.

For a linear inductor made by winding $N$ turns of wire around a toroidal core of a material having a constant permeability $\mu$, cross-sectional area $A$, and mean circumference $\ell$, the inductance is $L = \mu N^2 A/\ell$. Typical values of inductance are usually less than 1 henry and are often expressed in millihenries (mH).

The energy supplied to an inductor is stored in its magnetic field, and for a fixed linear inductor this energy is given by

$$w = \frac{1}{2} Li^2$$

To find the complete response of a circuit for $t \geq t_0$, we need to know the initial current $i(t_0)$ for each inductor.

**Sources**

The inputs for electrical circuit models are provided by ideal voltage and current sources. A **voltage source** is any device that causes a specified voltage to exist between two points in a circuit, regardless of the current that may flow. A **current source** causes a specified current to flow through the branch containing the source, regardless of the voltage that may be required. The symbols used to represent general voltage and current sources are shown in Figure 6.7(a) and Figure 6.7(b). We often represent physical sources by the combination of an ideal source and a resistor, as shown in parts (c) and (d) of Figure 6.7.

A voltage source that has a constant value for all time is often represented as shown in Figure 6.7(e). The symbol $E$ denotes the value of the voltage, and the terminal connected to the longer line is the positive terminal. A battery is often represented in this fashion.

**Open and Short Circuits**

An **open circuit** is any element through which current cannot flow. For example, a switch in the open position provides an open circuit, as shown in Figure 6.8(a). Likewise, we can consider a current source that has a value of $i(t) = 0$ over a nonzero time interval an open circuit and can draw it as shown in Figure 6.8(b).

A **short circuit** is any element across which there is no voltage. A switch in the closed position, as shown in Figure 6.9(a), is an example of a short circuit. Another example is a voltage source with $e(t) = 0$, as indicated in Figure 6.9(b).
6.3 Interconnection Laws

Two interconnection laws are used in conjunction with the appropriate element laws in modeling electrical circuits. These laws are known as Kirchhoff’s voltage law and Kirchhoff’s current law.

Kirchhoff’s Voltage Law

When a closed path—that is, a loop—is traced through any part of a circuit, the algebraic sum of the voltages across the elements that make up the loop must equal zero. This property is known as Kirchhoff’s voltage law. It may be written as

\[ \sum_{j} e_j = 0 \quad \text{around any loop} \quad (11) \]

where \( e_j \) denotes the voltage across the \( j \)th element in the loop.

It follows that summing the voltages across individual elements in any two different paths from one point to another will give the same result. For instance, in the portion of a circuit sketched in Figure 6.10(a), summing the voltages around the loop, going in a counterclockwise direction, and taking into account the polarities indicated on the diagram give

\[ e_1 + e_2 - e_3 - e_4 = 0 \]

Reversing the direction in which the loop is traversed yields

\[ e_4 + e_3 - e_2 - e_1 = 0 \]

Likewise, going from point A to point B by each of the two paths shown gives

\[ e_1 + e_2 = e_4 + e_3 \]

which is, of course, equivalent to both of the foregoing loop equations. In fact, we invoked (11) for the circuit element shown in Figure 6.2(d), which is repeated in Figure 6.10(b), when we stated that \( e_a = e_1 - e_2 \), because it follows from the voltage law that \( e_2 + e_a - e_1 = 0 \).

Kirchhoff’s Current Law

When the terminals of two or more circuit elements are connected together, the common junction is referred to as a node. All the joined terminals are at the same voltage and can be considered part of the node. Because it is not possible to accumulate any net charge at a node, the algebraic sum of the currents at any node must be zero at all times. This property is known as Kirchhoff’s current law. It may be written as

\[ \sum_{j} i_j = 0 \quad \text{at any node} \quad (12) \]

where the summation is over the currents through all the elements joined to the node.

In applying (12), we must take into account the directions of the current arrows. We shall use a plus sign in (12) for a current arrow directed away from the node being considered and a minus sign for a current arrow directed toward the node. This is consistent with the fact that the current \( i \) entering a node is equivalent to the current \(-i\) leaving the node.\(^1\) For the partial circuit shown in Figure 6.11, applying (12) at the node to which the three elements are connected gives \( i_1 + i_2 + i_3 = 0 \). If we wish, we can also use Kirchhoff’s current law in (12) for any closed surface that surrounds part of the circuit.

It is a common practice to write the current-law equation directly in terms of the element values and the voltages of the nodes. Consider, for example, the circuit segment shown in Figure 6.12, where \( e_A, e_B, e_D, \) and \( e_F \) represent the voltages of the nodes with respect to ground. By Kirchhoff’s voltage law, the voltage across the resistor is \( e_A - e_B \); by the element law, the current through the resistor is \( i_1 = (e_A - e_B)/R \). Similarly, the current through the inductor is

\[ i_2 = i_2(0) + \frac{1}{L} \int_0^t (e_A - e_D) \, dt \]

\(^1\) Instead of interpreting the left side of (12) as the algebraic sum of the currents leaving the node, it would also be correct to use the algebraic sum of the currents entering the node.
The element laws (4), (8), and (9) give expressions for \(e_R\), \(e_C\), and \(e_L\):
\[
\begin{align*}
  e_R &= Ri \\
  e_C &= e_C(0) + \frac{1}{C} \int_0^t i(\lambda) d\lambda \\
  e_L &= L \frac{di}{dt}
\end{align*}
\]  
(14)

where the initial time has been taken as \(t_0 = 0\) in (8). Substituting (14) into (13) and rearranging give the circuit model as the integral-differential equation

\[
L \frac{di}{dt} + Ri + \frac{1}{C} \int_0^t i(\lambda) d\lambda = e_i(t) - e_C(0)
\]  
(15)

To eliminate the constant term and the integral, we differentiate (15) term by term, which yields
\[
L \frac{d^2i}{dt^2} + Ri + \frac{1}{C} \frac{di}{dt} = \frac{d}{dt}(e_i(t))
\]
a second-order differential equation for the current \(i\) with the derivative of the applied voltage acting as the forcing function.

**EXAMPLE 6.2**

Obtain the input-output differential equation relating the input \(i_i(t)\) to the output \(e_o\) for the circuit shown in Figure 6.14.

**SOLUTION** Each of the four circuit elements in Figure 6.14 has one terminal connected to the ground node and the other terminal connected to another common node. By a trivial application of Kirchhoff's voltage law, we see that the voltage across each element is \(e_o\). Hence we say that the elements are connected in **parallel**.

Because the circuit has a single node whose voltage is unknown, we shall apply Kirchhoff's current law at that node in order to obtain the circuit model. We could also apply the current law at the ground node, but we would obtain no new information. The currents through the three passive elements are \(i_C\), \(i_R\), and \(i_L\). As indicated by the arrows in Figure 6.14, each of these currents is considered positive when it flows from the upper node to the ground node.

Figure 6.14 Parallel RLC circuit with a current source.
Applying Kirchhoff’s current law by summing the currents leaving the upper node, we write
\[ i_C + i_R + i_L - i(t) = 0 \]  \hspace{1cm} (16)
From the element laws given by (5), (7), and (10), we have
\[ i_R = \frac{1}{R} e_o \]
\[ i_C = C e_o \]
\[ i_L = i_L(0) + \frac{1}{L} \int_0^t e_o(\lambda) d\lambda \]  \hspace{1cm} (17)
where the initial time has been taken as \( t_0 = 0 \).
Substituting (17) into (16) and rearranging the result give the model as
\[ C e_o + \frac{1}{R} e_o + \frac{1}{L} \int_0^t e_o(\lambda) d\lambda = i(t) - i_L(0) \]  \hspace{1cm} (18)
Differentiating (18) term by term eliminates the constant term and the integral, resulting in the input-output differential equation
\[ C \frac{d e_o}{dt} + \frac{1}{R} e_o + \frac{1}{L} e_o = \frac{di}{dt} \]

6.4 OBTAINING THE INPUT-OUTPUT MODEL

Two general procedures for developing input-output models of electrical circuits are the node-equation method and the loop-equation method. Example 6.1 was actually a simple illustration of the loop-equation method, and Example 6.2 used the node-equation method. In the loop-equation method, a rather trivial application of the current law enables us to express the current through every element in terms of one or more loop currents. We then write an appropriate set of simultaneous equations by using the voltage law and the element laws. In the node-equation method, we use Kirchhoff’s voltage law in a trivial way to express the voltage across every element in terms of node voltages. Then we write a set of simultaneous equations by using Kirchhoff’s current law and the element laws.

We shall emphasize the node-equation method, partly because in some circuits the loop-equation method requires us to use fictitious loop currents that do not correspond to measurable currents through individual elements. Furthermore, the node-equation method is well suited to handling the current sources that appear in models of transistor circuits. (References that cover both methods in detail are listed in Appendix F.)

When we use the node-equation method, we start by labeling the voltage of each node with respect to the ground node. If a voltage source is connected between a particular node and ground, the voltage of that node is the known source voltage. Where the sources are needed, we introduce symbols to define the voltages of the other nodes with respect to ground.

Once we have done this, we can express the voltage across each passive element in terms of the node voltages by a trivial application of Kirchhoff’s voltage law, as illustrated by the discussion of Figure 6.2(d) and Figure 6.12. We write a current-law equation for each of the nodes whose voltage is unknown, using the element laws to express the currents through the passive elements in terms of the node voltages. We need only combine the resulting set of equations into input-output form to complete the model.

**EXAMPLE 6.3**

Derive the input-output equation for the circuit shown in Figure 6.15(a), using the node-equation method. The input and output voltages are \( e_i(t) \) and \( e_o \), respectively.

**SOLUTION**

The first step is to define all unknown node voltages and redraw the circuit diagram with all node voltages shown, as in Figure 6.15(b). We use the heavy lines to emphasize that the ground node extends across the bottom of the entire circuit and that the node whose voltage is \( e_o \) extends from \( L \) to \( R_3 \). We show the source voltage \( e_i(t) \) at the upper left node and denote the voltage of the remaining node with respect to ground by \( e_A \). Because \( e_A \) and \( e_o \) are unknown node voltages, we shall write a current-law equation at each of these nodes, using the appropriate element laws.

To assist in writing the equations, we can draw separate sketches for each node, as shown in Figure 6.16 (analogous to the free-body diagrams drawn for mechanical systems). For each element, the voltage across its terminals is shown in terms of the node voltages, with the plus sign placed at the node in question. Then we use the appropriate element law to write an expression for the current leaving the node.

We can apply Kirchhoff’s current law to each of the nodes shown in Figure 6.16 by setting the algebraic sum of the currents leaving each node equal to zero. The result is the pair of equations

\[ \frac{e_A - e_i(t)}{R_1} + C_1 e_A + \frac{e_A - e_o}{R_2} = 0 \]  \hspace{1cm} (19a)
\[ \frac{e_o - e_A}{R_2} + i_L(0) + \frac{1}{L} \int_0^t e_o(\lambda) d\lambda + C_2 e_o + \frac{e_o}{R_3} = 0 \]  \hspace{1cm} (19b)

With a bit of experience, the reader should be able to write these equations directly from the circuit diagram, without drawing the sketches shown in Figure 6.16. It is worthwhile to note that the current through \( R_2 \) is labeled \( (e_A - e_o)/R_2 \) in the sketch for node \( A \) and is
Several results in the last example should be specifically noted. The order of an input-output equation is generally the same as the number of energy-storing elements—that is, it is the number of capacitors plus the number of inductors. Thus we could have anticipated that (21) would be third order. In unusual cases, the order of the input-output equation might be less than the number of energy-storing elements. This can happen when the specified output does not depend on the values of some of the passive elements or when the capacitor voltages or inductor currents are not all independent. An illustration of this will be considered in Example 6.10.

In order to avoid drawing partial circuits like those in Figure 6.16, we can use the following rule for the current leaving a node through a passive element. The voltage that appears in the basic element law is replaced by the voltage of the node being considered minus the voltage at the other end of the passive element. The reader should examine (19a) for node A and (19b) for node O to see how the terms can be written directly from Figure 6.15(b). If a current source is attached to the node, we can easily include the known source when writing Kirchhoff’s current law. If, however, a voltage source is connected directly to the node, its current will not be known until after the circuit has been completely solved. To prevent introducing an additional unknown, we try to avoid summing currents at nodes to which voltage sources are attached. Thus, in the last example, we would not write a current-law equation at the junction of $R_1$ and $e(t)$. 

Finally, consider the simplified node equations that remain after any integral signs have been eliminated by differentiation and after like terms have been collected together. In (20a) for node A, all the terms involving $e_A$ and its derivatives have the same sign. Similarly in (20b) for node O, all the terms with $e_O$ and its derivatives have the same sign. Some insight into the reason for this can be found in Section 8.2, but the reader may wish to use this general property now as a check on the work.

The next example involves a voltage source neither end of which is connected to ground. In such cases, we must take special care when labeling the node voltages and writing the current-law equations.

**EXAMPLE 6.4**

Find the input-output differential equation for the circuit shown in Figure 6.17(a) where the inputs are the two voltage sources $e_1(t)$ and $e_2(t)$ and the output is the voltage $e_o$.

**SOLUTION**

The voltage of each node with respect to the ground is shown in Figure 6.17(b). Two of the nodes are labeled $e_1(t)$ and $e_2(t)$, corresponding to the left voltage source and the output voltage, respectively. In labeling the node to the right of $L$ and $R_1$, we do not introduce a new symbol (such as $e_3$) but take advantage of the fact that $e_2(t)$ is a source voltage. By Kirchhoff’s voltage law, this remaining node voltage is $e_o + e_1(t)$, as shown on the diagram. This approach avoids the introduction of unnecessary variables whenever there is a voltage source not connected to ground.

The directions of the current reference arrows included in Figure 6.17(b) are arbitrary, but our equations must be consistent with the directions selected. Applying Kirchhoff’s current law to the upper right node, we have

$$i_C + i_{R_2} + i_2 = 0$$

(22)

Although we can express $i_C$ and $i_{R_2}$ in terms of the node voltage $e_o$ by the element laws, the current $i_2$ through the voltage source cannot be directly related to $e_2(t)$. However, by applying the current law to the node labeled $e_o + e_1(t)$, we see that

$$i_2 = i_{R_1} + i_{R_2} + i_1.$$
which is a second-order differential equation. To solve it, we must know the two initial conditions $e_o(0)$ and $i(t)$, in addition to the source voltages $e_1(t)$ and $e_2(t)$.

The solution of a circuit model, such as the one in (25), is discussed in the next chapter. We may sometimes wish to see how the nature of the response changes when a source branch is disconnected. Even with all the sources disconnected, there will still be some output due to any energy that has been previously stored in the capacitors and inductors. The circuit in the following example contains only one energy storing element, so we should expect that it will be described by a first-order input-output equation.

**EXAMPLE 6.5**

Find the differential equation for the output voltage $e_o$ in Figure 6.18(a) when the switch is closed. Numerical values are given for the resistors but not for the capacitor. Repeat the problem when the left branch is disconnected by opening the switch.

**SOLUTION**

The node voltages with respect to ground, with the switch closed, are labeled in Figure 6.18(b). Summing currents at nodes $A$ and $O$ gives the following pair of equations:

$$ C(\dot{e}_A - \dot{e}_o) + \frac{1}{3}(e_A - e_o) + \frac{1}{4}e_A + \frac{3}{4}(e_A - e(t)) = 0 \tag{26a} $$

$$ C(\dot{e}_o - \dot{e}_A) + \frac{1}{3}(e_o - e_A) + \frac{1}{2}e_o = 0 \tag{26b} $$

Substituting (24) into (23) and rearranging terms give the integral-differential equation

$$ Ce_o + \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right)e_o + \frac{1}{L} \int_0^t e_o \, d\lambda = 0 $$

By differentiating this expression term by term, we obtain the desired input-output model:

$$ Ce_o + \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right)e_o + \frac{1}{L} \dot{e}_o = \frac{1}{R_1} \dot{e}_1 + \frac{1}{R_2} \dot{e}_2 + \frac{1}{R_3} (e_1(t) - e_2(t)) \tag{25} $$

Figure 6.17  Circuit for Example 6.4. (a) As specified by the example statement. (b) With currents and node voltages defined.

Figure 6.18  (a) Circuit for Example 6.5. (b) With the switch closed. (c) With the switch open.
We could collect like terms in each of these equations and then use the \( p \)-operator method to eliminate \( e_A \) and to obtain an equation in \( e_o \) and its derivatives. However, by adding (26a) and (26b) we see that
\[
\frac{1}{2} e_o + e_A - \frac{3}{4} e_i(t) = 0
\]
Replacing \( e_A \) in (26b) by \(-\frac{1}{2} e_o + \frac{3}{4} e_i(t)\), we have
\[
C \left( e_o + \frac{1}{2} e_o - \frac{3}{4} e_i \right) + \frac{1}{3} \left[ e_o + \frac{1}{2} e_o - \frac{3}{4} e_i(t) \right] + \frac{1}{2} e_o = 0
\]
from which
\[
\frac{3}{2} C e_o + e_o = \frac{3}{4} C e_i + \frac{1}{4} e_i(t)
\]
(27)

With the switch open, the circuit reduces to the one shown in Figure 6.18(c). Once again summing currents at nodes \( A \) and \( O \), we obtain
\[
C(e_A - e_o) + \frac{1}{3} (e_A - e_o) + \frac{1}{4} e_A = 0
\]
(28)

Following the same procedure as before, we find that for part (c) of Figure 6.18,
\[
2 C e_o + e_o = 0
\]
Note that, as is the case in most examples, disconnecting the source has not only caused the input terms to disappear but has also changed the coefficients on the left side of the differential equation.

### 6.5 RESISTIVE CIRCUITS

There are many useful circuits that contain only resistors and sources, with no energy-storing elements. Such circuits are known as resistive circuits and are modeled by algebraic rather than differential equations. In this section we shall develop rules for finding the voltages and currents in such circuits and for replacing certain combinations of resistors by a single equivalent resistor.

The analysis of resistive circuits is important for other reasons as well. Even for circuits with energy-storing elements, we are often interested primarily in the steady-state response to a constant input, after any initial transients have died away. We shall see how any circuit reduces to a resistive one under these circumstances.

Even when we need to find the complete response of a general electrical system, a combination of two or more resistors will frequently be connected to the remainder of the circuit by a single pair of terminals, as shown in Figure 6.19(a). In such situations, it is possible to replace the entire combination of resistors by a single equivalent resistor \( R_{eq} \), as shown in Figure 6.19(b). Provided that \( R_{eq} \) is selected such that \( e = R_{eq} i \) is satisfied, the response of the remainder of the circuit is identical in both cases. Once we have found \( R_{eq} \), it is easier to analyze the complete circuit because there are fewer nodes and thus fewer

\[ R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \]  

(30)

Using (29) with the expressions for \( e_1 \) and \( e_2 \), we see that
\[
e_1 = \left( \frac{R_1}{R_1 + R_2} \right) e
\]
(31a)
\[
e_2 = \left( \frac{R_2}{R_1 + R_2} \right) e
\]
(31b)

![Figure 6.19 Replacement of a combination of resistors by an equivalent resistance.](image1)

![Figure 6.20](image2)
which is known as the voltage divider rule. From (31), the ratio of the individual resistor voltages is

$$\frac{e_1}{e_2} = \frac{R_1}{R_2}$$  \hspace{1cm} (32)

**Resistors in Parallel**

Two resistors are in parallel when each terminal of one resistor is connected to a separate terminal of the other resistor, as shown in Figure 6.21(a). It is apparent that two resistors in parallel must have the same voltage across their terminals.

From Ohm's law, the individual currents are \(i_1 = (1/R_1)e\) and \(i_2 = (1/R_2)e\). From Kirchhoff's current law, \(i = i_1 + i_2\). Thus

$$i = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)e \hspace{1cm} (33)$$

For the equivalent resistance shown in Figure 6.21(b), we have \(i = (1/R_{eq})e\). From (33), we see that for the parallel combination in Figure 6.21(a),

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

or

$$R_{eq} = \frac{R_1R_2}{R_1 + R_2} \hspace{1cm} (34)$$

To relate \(i_1\) to the total current, we can write \(i_1 = (1/R_1)e = (R_{eq}/R_1)i\). Doing this for \(i_2\) and then using (34) to express \(R_{eq}\) in terms of \(R_1\) and \(R_2\), we obtain

$$i_1 = \left(\frac{R_2}{R_1 + R_2}\right)i \hspace{1cm} (35a)$$

$$i_2 = \left(\frac{R_1}{R_1 + R_2}\right)i \hspace{1cm} (35b)$$

which is known as the current divider rule. From (35), the ratio of the individual resistor currents is \(i_1/i_2 = R_2/R_1\).

![Figure 6.21](image)

**Figure 6.21** (a) Two resistors in parallel. (b) Equivalent resistance.

Calculating equivalent resistances and solving for the currents and voltages in most types of resistive networks can be simplified by using these rules for series and parallel combinations, as demonstrated in the following example.

**EXAMPLE 6.6**

The resistive circuit shown in Figure 6.22(a) consists of a voltage source connected to a combination of seven resistors. The output is the voltage \(e_o\). Find the equivalent resistance \(R_{eq}\) of the seven-resistor combination and evaluate \(e_o\).

**SOLUTION** To obtain \(R_{eq}\), we use (30) and (34) repeatedly to combine series or parallel combinations of resistors into single equivalent resistors. Starting with the original circuit in Figure 6.22(a), we replace the 6-Ω and 3-Ω resistors that are in parallel with a single 2-Ω resistor. We also combine the 10-Ω and 2-Ω resistors in series into a 12-Ω resistor, which yields the intermediate circuit diagram shown in Figure 6.22(b). Note that the output
voltage \( e_o \) does not appear on this diagram. Next we replace the parallel combination of the 4-\( \Omega \) and 12-\( \Omega \) resistors with a 3-\( \Omega \) resistor, and obtain Figure 6.22(c). The series combination of 2 \( \Omega \) and 3 \( \Omega \) gives a 5-\( \Omega \) resistor, which is in parallel with another 5-\( \Omega \) branch, yielding the resistor of \( \frac{1}{2} \) \( \Omega \) that is shown in Figure 6.22(d). Thus the equivalent resistance connected across the voltage source is
\[
R_{eq} = \frac{1}{2} + \frac{5}{3} = 3 \Omega
\]

To find the output voltage \( e_o \), we make repeated use of the voltage-divider rule given by (31a) to obtain, in turn, \( e_A, e_B \), and finally \( e_o \). From Figure 6.22(d),
\[
e_A = \left( \frac{\frac{3}{2}}{\frac{3}{2} + \frac{5}{3}} \right) e_i(t) = \frac{5}{6} e_i(t)
\]
and from Figure 6.22(c),
\[
e_B = \left( \frac{\frac{3}{2} + \frac{5}{3}}{\frac{5}{3}} \right) e_A = \frac{1}{2} e_i(t)
\]
Then, from the original circuit diagram,
\[
e_o = \left( \frac{\frac{2}{3} + \frac{10}{3}}{2 + 10} \right) e_B = \frac{1}{12} e_i(t)
\]

Although the rules for combining series and parallel resistors often simplify the process of modeling a circuit, there are circuits in which the resistances do not occur in series or parallel combinations. In such situations, we can find an equivalent resistance by writing and solving the appropriate node equations, which will be strictly algebraic when only resistors and sources are involved.

The Steady-State Response

In the next two chapters, we present in detail general methods for obtaining the complete response of a linear system that contains energy-storing elements. The response normally consists of a transient part that dies out with time and a steady-state component that has a form similar to that of the input. If the input is constant over a long period of time, then the steady-state response will also be a constant.

If we just want to find the steady-state response to a constant input, then we can assume that all variables, including the input and output, are constants. Because the derivatives of constants are zero, we can, under these special circumstances, set all the derivatives in the modeling equations equal to zero. The equations then reduce to algebraic equations. Reconsider, for example, the circuit in Figure 6.18(a). Assume that after the switch closes, the voltage source has a constant value of 24 V, and that we want only the steady-state value of \( e_o \), which we denote by \( e_o \)\(_{ss} \). The input-output equation was found in (27). When we set all the derivatives equal to zero, the equation reduces to
\[
(e_o)_{ss} = \frac{1}{4} (24) = 6 V
\]

For an electrical system, it is not necessary to find the general modeling equations if we only want the steady-state response to a constant input. We know that all currents and voltages then become constants. Under these circumstances, \( di_i/dt = 0 \) and \( de_C/dt = 0 \). Then from the respective element laws, we set that the voltage across any inductor and the current through any capacitor must be zero. Hence in the steady-state, an inductor acts like a short circuit and a capacitor becomes an open circuit.

When finding the steady-state response to a constant input, we may redraw the circuit with any inductor replaced by a short circuit and any capacitor replaced by an open circuit. The original circuit then reduces to a purely resistive one. In the case of Figure 6.18(a), with the switch closed and with \( e_i(t) \) having a constant value of 24 V, we can draw the steady-state equivalent circuit shown in Figure 6.23(a). By the rules for series and parallel resistors, the part of the circuit to the right of the \( \frac{1}{2} \) \( \Omega \) resistor can be replaced by an equivalent resistance of \((4)(5)/(4+5) = \frac{20}{9} \) \( \Omega \), as shown in part (b) of Figure 6.23. Then, using the voltage divider rule twice, we have in the steady state
\[
\frac{e_A}{9} = \frac{\frac{20}{9}}{\frac{3}{2} + \frac{9}{9}} (24) = 15 V
\]
and
\[
\frac{e_o}{3} = \frac{2}{\frac{3}{2} + 2} (15) = 6 V
\]
which agrees with the answer found from the input-output equation.

6.6 OBTAINING THE STATE-VARIABLE MODEL.

To obtain the model of a circuit in state-variable form, we define an appropriate set of state variables and then derive an equation for the derivative of each state variable in terms of only the state variables and inputs. The choice of state variables is not unique, but they are normally related to the energy in each of the circuit's energy-storing elements. Recalling that the energy stored in a capacitor is \( \frac{1}{2} C e^2 \) and for an inductor is \( \frac{1}{2} L i^2 \), we generally select the capacitor voltages and inductor currents as the state variables. For fixed linear circuits, exceptions occur only when there are capacitor voltages or inductor currents that are not independent of one another. This unusual situation will be illustrated in Example 6.10.

For each capacitor or inductor, we want to express \( \dot{e}_C \) or \( di_i/dt \) as an algebraic function of state variables and inputs. We do this by writing the capacitor and inductor element laws in their derivative forms as
\[
\dot{e}_C = \frac{1}{C} i_C
\]
\[
\frac{di_i}{dt} = \frac{1}{L} e_L
\]
For the second equation, we write the capacitor element law as \( e_c = (1/C)i_c \). To express the capacitor current \( i_c \) in terms of the state variables and input, we apply Kirchoff’s current law at the upper node, getting
\[
i_c + \frac{1}{R}e_c + i_L - i(t) = 0
\]
where we have written the resistor current in terms of the state variable \( e_c \). Solving (38) for \( i_c \) gives
\[
i_c = -i_L - \frac{1}{R}e_c + i(t)
\]
the right side of which is written entirely in terms of the state variables and input. Substituting (39) into the capacitor element law, we find the second state-variable equation to be
\[
e_c = \frac{1}{C} \left[ -i_L - \frac{1}{R}e_c + i(t) \right]
\]
Finally, we find the voltage \( e_o \) from the algebraic output equation
\[
e_o = e_c
\]
To solve the state-variable equations in (37) and (40), we must know the input and the initial values of the state variables. Note also that once we have the state-variable and output equations, we can always combine them into an input-output differential equation. In this example, we could differentiate (40), substitute (37) into it, and finally use (41) to replace \( e_c \) by \( e_o \). The result would be the answer to Example 6.2.

There are several ways of summarizing a general procedure for constructing a state-variable model. We assume here that each capacitor voltage and inductor current is chosen to be a state variable. The unusual case where the number of state variables is less than the number of energy-storing elements will be treated later in this section.

1. We show the positive senses for each \( e_c \) and \( i_L \) on the circuit diagram, and then label \( i_c \) and \( e_o \), so that each current reference arrow enters the capacitor or inductor at the positive end of the voltage reference. Insofar as possible, we label the voltages of the nodes with respect to ground in terms of the state variables and inputs. We can then use additional symbols for any remaining node voltages, but we try to minimize the use of new variables.

2. We need to find algebraic expressions for each capacitor current \( i_c \) and each inductor voltage \( e_L \). We make use of Kirchoff’s laws and Ohm’s law, but at this time do not use the element laws for the capacitors and inductors. We require algebraic equations in this step, and the laws for the energy-storing elements will be used in the next step. In general, we may have to solve a set of simultaneous algebraic equations in order to get individual equations for each \( i_c \) and \( e_L \) in terms of the state variables and inputs.

3. For the state-variable equations, we substitute the expressions for \( i_c \) and \( e_L \) into the capacitor and inductor element laws, as given by (36). Finally, for each output that is not a state variable, we write an algebraic expression in terms of state variables and inputs.
The next two examples illustrate this general procedure for circuits of moderate complexity.

**EXAMPLE 6.8**

Derive the state-variable model for the circuit shown in Figure 6.26. The outputs of interest are $e_B$, $i_C$, and $i_1$.

**SOLUTION** We choose as state variables the inductor current $i_L$ and the capacitor voltages $e_A$ and $e_B$. We need algebraic equations for the voltage across the inductor and the current through each capacitor. The inductor voltage is identical to the state variable $e_B$. Thus, by the element law for the inductor, one of the state-variable equations is

$$\frac{di_L}{dt} = \frac{1}{L} e_B$$

The current $i_C_1$ will appear in a Kirchhoff current-law equation for node A, namely

$$i_C_1 = \frac{1}{R_1} e_A(t) - e_A - \frac{1}{R_2} (e_A - e_B)$$

(42)

For $i_C_2$, we consider node B, getting

$$i_C_2 = \frac{1}{R_2} (e_A - e_B) - i_L - \frac{1}{R_3} e_B$$

(43)

Substituting (42) and (43) into the respective element-law equations gives the final two state-variable equations. The complete set of three equations is

$$\frac{di_L}{dt} = \frac{1}{L} e_B$$

$$e_A = \frac{1}{C_1} \left[ \left( \frac{1}{R_1} + \frac{1}{R_2} \right) e_A + \frac{1}{R_2} e_B + \frac{1}{R_1} e_A(t) \right]$$

(44)

$$e_B = \frac{1}{C_2} \left[ -i_L + \frac{1}{R_2} e_A - \left( \frac{1}{R_2} + \frac{1}{R_3} \right) e_B \right]$$

As required, we have expressed the derivative of each of the state variables as an algebraic function of the state variables and the input $e_A(t)$. The output voltage $e_B$ is the same as one of the state variables, and the output current $i_C$ is given by (43). The output equation for $i_1$ is

$$i_1 = \frac{1}{R_1} [e_A(t) - e_A]$$

(45)

For a MATLAB exercise using this example, see Problem 6.42 at the end of this chapter.

The next example has only two energy-storing elements and hence only two state variables. However, not all of the node voltages can be immediately expressed in terms of state variables and inputs.

**EXAMPLE 6.9**

Find the state-variable model for the circuit shown in Figure 6.27(a), when $e_0$ is the output.

**SOLUTION** We make the usual choice of $e_C$ and $i_L$ as state variables, with the positive senses shown on the diagram. The voltage at node $D$ is $e_D = \frac{1}{2} i_L$, and node $B$ corresponds to the output voltage $e_0$. Because we want to retain $i_C$ in our equations, we use Kirchhoff’s voltage law to express the voltage at node $A$ as $e_A + e_C$. These symbols, as well as the reference directions for $i_C$ and $e_C$, are added to the diagram in Figure 6.27(b). Applying Kirchhoff’s current law to nodes $B$ and $A$, we have

$$2 e_0 + i_A - i_C = 0$$

$$[e_A + e_C - e_0(t)] + 2 (e_0 + e_C) + i_C = 0$$

(46)

The quantity inside the brackets is the current through the 1-Ω resistor—that is, it is the voltage at node $A$ minus the source voltage divided by 1 Ω. We now solve (46) simultaneously for $e_0$ and $i_C$ in terms of the state variables and the input. Doing this gives the algebraic equations

$$e_0 = \frac{1}{5} [-i_L - 3e_C + e_A(t)]$$

(47a)

$$i_C = \frac{1}{5} [3i_L - 6e_C + 2e_A(t)]$$

(47b)
A state-variable model for Figure 6.28(a) is developed in Example 6.10. The treatment of Figure 6.28(b) is left for a problem at the end of the chapter. In such cases, we may need to redefine a state variable in order to avoid having the derivative of the input appear on the right-hand side of the state-variable equation. For certain outputs, however, it may not always be possible to avoid an input derivative on the right-hand side of the output equation.

**EXAMPLE 6.10**

Find the state-variable model for the circuit shown in Figure 6.28(a). Take the outputs to be $e_A$, $e_B$, and $i_C$.

**SOLUTION**

Because $e_A$ and $e_B$ cannot be chosen as state variables, suppose we select $e_A$ as the single state variable. Then we need an equation for $e_A$ in terms of $e_A$ and $e_i(t)$. First we write the element law for $C_1$ as

$$ e_A = \frac{1}{C_1} \frac{d}{dt} i_C $$

where the positive sense of $i_C$ is downward. Then we apply Kirchhoff’s current law at node $A$, obtaining

$$ C_2 (e_A - e_i(t)) + \frac{1}{R_1} [e_A - e_i(t)] + \frac{1}{R_1} e_A + i_C = 0 $$

Solving (52) for $i_C$, and substituting the result into (51), we find

$$ e_A = C_1 \left[ -C_2 e_A - \left( \frac{1}{R_1} + \frac{1}{R_2} \right) e_A + C_2 e_i(t) + \frac{1}{R_2} e_i(t) \right] $$

which can be rearranged to yield

$$ e_A = \frac{1}{C_1 + C_2} \left[ \frac{1}{R_1} + \frac{1}{R_2} \right] e_A + \frac{1}{R_2} e_i(t) $$

Equation (53) would be in state-variable form were it not for the term involving $i_C$ on the right side. The derivative of the input should not appear in the final equation, so we define a new state variable, denoted by $x$, using the same procedure as for the mechanical system in Example 3.8. Transferring the term involving $i_C$ to the left side of (53), we have

$$ \frac{d}{dt} \left[ e_A - \left( \frac{C_2}{C_1 + C_2} \right) e_i(t) \right] = \frac{1}{C_1 + C_2} \left[ \frac{1}{R_1} + \frac{1}{R_2} \right] e_A + \frac{1}{R_2} e_i(t) $$

We define the bracketed term on the left to be the new state variable

$$ x = e_A - \left( \frac{C_2}{C_1 + C_2} \right) e_i(t) $$

Then $e_A$ is given by the output equation

$$ e_A = x + \left( \frac{C_2}{C_1 + C_2} \right) e_i(t) $$

and, when we substitute (55) and (56) into (54), the state-variable equation becomes

$$ \dot{x} = \left( \frac{1}{C_1 + C_2} \right) \left[ \frac{1}{R_1} + \frac{1}{R_2} \right] e_i(t) $$

**Figure 6.28** Circuits having fewer state variables than energy-storing elements.
Note that the circuit is first-order and can be modeled by a single state-variable equation. However, we did find that the state variable had to be a linear combination of the voltage across \( C_1 \) and the input. We can obtain the capacitor voltage \( e_A \) from the algebraic output equation (56) after solving the state-variable equation. For the output equation for \( e_B \), we combine (50) and (56) to obtain

\[
e_B = \left( \frac{C_1}{C_1 + C_2} \right) e(t) - x
\]

For the final output, we substitute (56) into the element law for \( C_1 \):

\[
e_{C_1} = C_1 e_A = C_1 \left[ \hat{x} + \left( \frac{C_2}{C_1 + C_2} \right) \hat{e}_t \right]
\]

and then, by (57), we write

\[
e_{C_1} = \left( \frac{C_1}{C_1 + C_2} \right) \left\{ -\left( \frac{1}{R_1} + \frac{1}{R_2} \right) x + \left[ \frac{1}{R_2} - \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \left( \frac{C_2}{C_1 + C_2} \right) \right] e(t) + C_2 \hat{e}_t \right\}
\]

The output equations in a state-variable model should be purely algebraic whenever possible. This is an example of the unusual case where we cannot avoid a derivative of the input on the right-hand side.

### 6.7 OPERATIONAL AMPLIFIERS

Some important types of electrical elements, unlike those in earlier sections, have more than two terminals to which external connections can be made. Controlled sources arise in the models of transistors and other electronic devices. Rather than being independently specified, the values of such sources are proportional to the voltage or current somewhere else in the circuit. One purpose for which such devices are used is to amplify electrical signals, giving them sufficient power, for example, to drive loudspeakers, instrumentation, or various electromechanical systems. Ideal voltage and current amplifiers are shown in parts (a) and (b), respectively, of Figure 6.29.

![Figure 6.29](image)

(a) Ideal voltage and current amplifiers. (c), (d) Amplifiers with internal resistances causing nonideal behavior.

The models for many common devices have the two bottom terminals connected together. They may also include the added resistors shown in parts (c) and (d) of the figure in order to represent some of the imperfections of the device. For part (c) of the figure to approach part (a), \( R_4 \) must be very large and \( R_6 \) very small. In order for part (d) of the figure to approach part (b), \( R_8 \) must be very small and \( R_7 \) very large.

As is the case with any source in a circuit diagram, we assume that a controlled source can supply as much power as is required by the passive elements connected to it. Although we do not discuss in detail the internal mechanism responsible for the behavior of any element, a brief comment here should be helpful for those who encounter devices that are represented by controlled sources. The drawings in Figure 6.29 do not show all of the external connections to the physical device. In addition to the time-varying input \( e(t) \) or \( i(t) \), there are constant voltage sources that are normally not explicitly shown. These additional sources, which are called supply voltages, provide needed power for the time-varying output signal. The supply voltages that are needed are usually given in the specifications for the device to be used. Also specified are the maximum values of input voltage or current for which the operation of the device can be expected to remain in its linear region.

We shall assume that the values of our controlled sources are directly proportional to the signals controlling them. In addition to the sources presented in Figure 6.29, we can also have a voltage source controlled by a current somewhere else in the circuit, as well as a current source controlled by a voltage. However, we shall emphasize the voltage-controlled voltage source, because that will lead to the concept of the operational amplifier.

Electronic devices can do much more than simply amplify an input signal. In order to accomplish other objectives, additional passive elements are connected around the controlled source. In the following example, two external resistors are added to a controlled source. The controlled source is modeled by the simple circuit in Figure 6.29(a), with the constant \( \alpha \) replaced by \(-A\).

### EXAMPLE 6.11

Find \( e_0 \) for the circuit shown in Figure 6.30.

**SOLUTION** Summing the currents at node \( A \) gives

\[
\frac{1}{R_1} [e_A - e(t)] + \frac{1}{R_2} [e_A - e_0] = 0
\]

Because \( e_0 = -AE_A \),

\[
\left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_2} \right) e_A = \frac{1}{R_1} e(t)
\]

\[
e_A = \frac{R_1}{R_1 + R_2} e(t)
\]

![Figure 6.30](image)

Figure 6.30 Circuit for Example 6.11.