Topics covered so far:

- Analogies between mechanical (translation and rotation), fluid, and electrical systems
  - Review of domain-specific elements and constitutive laws
  - Review of transformer and gyrator elements (power-conserving transformations)
  - Review of interconnection laws (energy flow conduits among elements)

- State equation generation
  - State variables associated with energy storage elements
  - Generated equations for mechanical, electrical and hydraulic systems
  - Explicit ODE’s
    - first order; state variable derivatives are algebraic functions of state variables and inputs
  - Implicit ODE’s
    - can arise when energy storage elements are dependent
    - state variable derivatives on left and right-hand sides of equations
  - Differential-algebraic equations
    - can arise when an algebraic relation exists among state variables, instead of a differential equation (recall massless connection)

Shortcomings in the traditional equation formulation method (gathering all the constitutive laws and attempting to combine them into a coherent set of state equations):

- Selection of state variables is not always obvious
- Sometimes not clear which of two possible equations to use to define a variable (recall fluid system example with two possible ways to express fluid inertia flow rate)
- Can be difficult to anticipate or recognize state variable dependencies

These shortcomings will be addressed by expressing models from mechanical, electrical and fluid domains in terms of generalized elements. All domains will look identical. Formal algorithmic procedures will be applied to generate equations.

1. Generalized Variables and Power Ports

Recall we can express power as the product of generalized effort and flow \( P = e(t)f(t) \) and we can define “generalized” quantities applicable to all energy domains.

- effort \( e \) and flow \( f \)
- momentum \( p \) (integral of effort)
- displacement \( q \) (integral of flow)

For a system that contains multiple components joined together

- Components are connected at ports
- When two components are joined, the two complementary power variables are simultaneously constrained to be equal for both components at the port
  - e.g., from recent example, torque and speed at the end of the motor output shaft have the same magnitudes as torque and speed at the pump input shaft
2. List of Generalized Elements

Any dynamic lumped-parameter system can be modeled using the following set of generalized elements:

- Kirchoff’s loop (all elements in loop have same flow, efforts sum to zero)
- Kirchoff’s node (all elements at node have same effort, flows sum to zero)
- Dissipator (dumps energy to environment)
- Potential energy storage device (energy is a function of displacement)
- Kinetic energy storage device (energy is a function of momentum)
- Source (of effort or flow, prescribed from the environment)
- Transformer (relates effort to effort, flow to flow; either within or between two different energy domains)
- Gyrator (relates effort to flow, flow to effort)

3. The Power Bond – Half-Arrows and Causal Strokes

In bond graphs, the system elements are connected by “power bonds”. Each bond contains an effort and flow signal. In block diagrams, effort and flow would be shown as separate, individual signals.

Each bond has the following:

- **half-arrow** to indicate direction of algebraically positive power flow

\[
\text{A} \xrightarrow{\epsilon} \text{B}
\]

Instantaneously, power may flow from element A to B, or from B to A. However, if power is algebraically positive, then it is flowing from A to B.

Half-arrow is related to sign convention.

- **causal stroke** to indicate the input to the element’s constitutive law equation

\[
f = \Phi_A(e) \quad \text{A} \xrightarrow{\epsilon} \text{B} \quad \epsilon = \Phi_B(f) \quad e = \Phi_A^{-1}(f) \quad \text{A} \xrightarrow{\epsilon} \text{B} \quad f = \Phi_B^{-1}(e)
\]

In the left figure, the equation for element A has \(f\) on the left-hand side (LHS). Flow is the output. Element B takes that flow, plugs it into the right-hand side (RHS) of its equation, and computes effort \(e\). \(e\) is therefore on the LHS of element B’s equation. \(e\) is plugged into the RHS of element A’s equation, to generate \(f\).

Causal strokes are completely independent of half-arrows. Moving the causal stroke, as in the right figure, means we rearrange the equations, but power is still flowing from A to B when it is positive.

Causal stroke is related to input-output form of the equations
4. Generalized Bond Graph Elements

The following pages describe the specific elements and their generalized constitutive laws and energy equations.

1. Generalized Inertia (Kinetic Energy storage)
   - \( I \) element: static relation between flow and momentum
   - electrical – relation between flux and current typically nonlinear
   - mechanical – typically linear except for relativistic mechanics, where momentum is nonlinear function of velocity
   - preferred “integral” causality – effort in, flow out

<table>
<thead>
<tr>
<th>Generalized Variables</th>
<th>Generalized Relation</th>
<th>Linear Relation</th>
<th>SI Units for Linear Inertance Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mechanical translation</td>
<td>( p = \Phi_f(V) )</td>
<td>( p = mV )</td>
<td>( m = N \cdot s^2/m = kg )</td>
</tr>
<tr>
<td>Mechanical rotation</td>
<td>( \omega = \Phi_f^{-1}(p_r) )</td>
<td>( \omega = p_r/J )</td>
<td></td>
</tr>
<tr>
<td>Hydraulic systems</td>
<td>( p_p = \Phi_f(Q) )</td>
<td>( p_p = I \Omega )</td>
<td>( I = N \cdot s^2/m^5 )</td>
</tr>
<tr>
<td>Electrical systems</td>
<td>( i = \Phi_f^{-1}(\lambda) )</td>
<td>( i = \lambda/L )</td>
<td>( L = V/s/A )</td>
</tr>
</tbody>
</table>

- half-arrow always goes into I element
  - define positive force and positive velocity in the same direction
  - if force and velocity are in the same direction, the mass will accelerate because power is flowing into it and being converted to kinetic energy

a) system applying positive force, velocity in positive direction (positive power flow)

Power flows from system into mass, increasing its kinetic energy.

b) system applying negative force, velocity in negative direction (positive power flow)

Power flows from system into mass, increasing its kinetic energy.
c) instantaneously negative power flow (positive force, negative velocity)

System is slowing mass down. Power flows from mass into system. The mass transfers its kinetic energy back to the system.

This doesn’t change the half-arrow direction. The half-arrow direction is our sign convention, stating simply that if power were positive, it would be flowing into the mass. That’s still true in this case. Power is negative right now, so it’s not flowing into the mass.

Energy Storage

Causality

Integral (preferred)  
- numerical integration is stable and good

Derivative  
- numerical differentiation can be unstable
2. Generalized Capacitor (Potential Energy storage)

- C element: static relation between effort and displacement
- linear or nonlinear
- constitutive law can be offset from origin: spring deflection defined as zero for nonzero force (mass hanging from spring in gravity field)
- preferred "integral" causality – flow in, effort out

<table>
<thead>
<tr>
<th>General Relation</th>
<th>Linear Relation</th>
<th>SI Units for Linear Capacitance Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized</td>
<td>( q = \Phi_C(e) )</td>
<td>( q = Ce )</td>
</tr>
<tr>
<td>Mechanical</td>
<td>( e = \Phi_C^{-1}(q) )</td>
<td>( e = q/C )</td>
</tr>
<tr>
<td>translation</td>
<td>( X = \Phi_C(F) )</td>
<td>( X = CF )</td>
</tr>
<tr>
<td>Mechanical</td>
<td>( F = \Phi_C^{-1}(X) )</td>
<td>( F = kX )</td>
</tr>
<tr>
<td>rotation</td>
<td>( \theta = \Phi_C^{-1}(\tau) )</td>
<td>( \theta = C\tau )</td>
</tr>
<tr>
<td>Hydraulic systems</td>
<td>( V = \Phi_C(P) )</td>
<td>( V = CP )</td>
</tr>
<tr>
<td>Electrical systems</td>
<td>( P = \Phi_C^{-1}(V) )</td>
<td>( P = V/C )</td>
</tr>
<tr>
<td></td>
<td>( e = \Phi_C^{-1}(q) )</td>
<td>( e = q/C )</td>
</tr>
</tbody>
</table>

- half-arrow always into C element
  - if a tensile force is acting on the spring, and it is getting longer, then power is flowing into the spring from the system, and its potential energy is increasing
  - if a tensile force is acting, but the spring is shortening, then the spring is returning some of its potential energy to the system
  - in either case, if extension and force are both positive in tension (or compression), then power will be flowing into the spring if power is positive
Energy Storage

Causality

Integral (preferred)  Derivative
- numerical integration is stable and good  - numerical differentiation can be unstable
3. Generalized Dissipator
- static relation (linear or nonlinear) between effort and flow
- **R** element constitutive law must be in first and third quadrant to be passive
- power must always flow out of the system and into a passive resistor
- causality does not matter if the constitutive law is invertible

<table>
<thead>
<tr>
<th>Generalized variables</th>
<th>General Relation</th>
<th>Linear Relation</th>
<th>SI Units for Linear Resistance Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>e = \Phi_R(f)</td>
<td>e = Rf</td>
<td>R = e/f</td>
<td></td>
</tr>
<tr>
<td>f = \Phi_R^{-1}(e)</td>
<td>f = Ge = e/R</td>
<td></td>
<td></td>
</tr>
<tr>
<td>F = \Phi_R(V)</td>
<td>F = bV</td>
<td>b = N - s/m</td>
<td></td>
</tr>
<tr>
<td>V = \Phi_R^{-1}(F)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mechanical rotation</td>
<td>\tau = c\omega</td>
<td>c = N - m - s</td>
<td></td>
</tr>
<tr>
<td>\omega = \Phi_R^{-1}(\tau)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hydraulic systems</td>
<td>P = RQ</td>
<td>R = N - s/m^5</td>
<td></td>
</tr>
<tr>
<td>Q = \Phi_R^{-1}(P)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Electrical systems</td>
<td>e = \Phi_R(i)</td>
<td>R = V/\Lambda = \Omega (ohm)</td>
<td></td>
</tr>
<tr>
<td>i = \Phi_R^{-1}(e)</td>
<td>i = Ge</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- half arrow always into R element (energy always flows from the system into the environment. Power is always positive.)
4. **Sources** (non-dissipative energy exchange with environment)

- power flow can be instantaneously into or out of source element, but algebraically positive power direction will be fixed
- half-arrow typically out of source, but not always
- bond graph element $\text{Se}$, $\text{Sf}$
- ideal vs. non-ideal sources (battery in Figure 3.7)
- causality is fixed – $\text{Se}$ has effort out to system, $\text{Sf}$ has flow out to system

**TABLE 3.4. The 1-Port Source Elements**

<table>
<thead>
<tr>
<th>Bond Graph Symbol</th>
<th>Defining Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Generalized variables</td>
<td>$\text{Se} \rightarrow$</td>
</tr>
<tr>
<td>Mechanical translation</td>
<td>$\text{Sf} \rightarrow$</td>
</tr>
<tr>
<td>Mechanical rotation</td>
<td>$\text{SF} \rightarrow$</td>
</tr>
<tr>
<td>Mechanical rotation</td>
<td>$\text{SV} \rightarrow$</td>
</tr>
<tr>
<td>Hydraulic systems</td>
<td>$\text{SP} \rightarrow$</td>
</tr>
<tr>
<td>Electrical systems</td>
<td>$\text{Sf} \rightarrow$</td>
</tr>
<tr>
<td>Electrical systems</td>
<td>$\text{Si} \rightarrow$</td>
</tr>
</tbody>
</table>

**FIGURE 3.7.** Study of a battery connected to a variable resistance. (a) Electric circuit diagram; (b) bond graph; (c) plot of source, real battery, and resistance characteristics.
Example of ideal source which often has half-arrow into the source element: gravity

A. Force and velocity positive downward – downward gravity force, if mass is moving downward, is increasing kinetic energy of mass. Thus, power flows out of source (Earth’s gravitational field) and into the mass when positive.

B. Force and velocity positive upward – if mass is moving up (positive velocity), and gravity force “mg” is positive, the gravity source is slowing the mass down. Kinetic energy flows out of the mass and into the Earth’s gravitational field.

You can change the half-arrow direction by considering gravity to be “-mg upward” and velocity positive upward.

In this case, with the mass moving upwards, “-mg times v” will be negative, and power will be instantaneously flowing out of the mass, into the source. If v is negative, then “-mg times v” is positive and the falling mass is receiving power from the gravitational source.
5. Generalized Kirchoff Loops and Nodes

- **0-junction**: all connected elements have same effort, flows sum to zero (e.g., node in electrical circuit)
- **1-junction**: all connected elements have same flow, efforts sum to zero (e.g., Newton’s law for a mass, loop in electrical circuit)
- algebraic signs in effort and flow summations are determined by half-arrows
- causal strokes determine which variables are outputs and inputs
- causality restriction: there can be only one flow input to a 1-junction, and only one effort input to a 0-junction
6. Power-Conserving Transformations

- **transformer TF**: relates effort to effort, and flow to flow – either within a single energy domain, or between two different energy domains
- **gyrator GY**: relates effort to flow, and flow to effort (but is power-conserving, unlike an R element)
- transformer and gyrator “modulus” (lever ratio, piston area, etc.) may be constant, or may be varying. If the modulus is generated somewhere else in the system, or externally, it must be sent to the TF or GY element by a signal. A TF or GY element with a modulating signal is called a “modulated transformer” (MTF), or “modulated gyrator” (MGY).
- causality restrictions in Table 3.7
### TABLE 3.6. Causal Forms for Basic 1-Ports

<table>
<thead>
<tr>
<th>Element</th>
<th>Acausal Form</th>
<th>Causal Form</th>
<th>Causal Relation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort source</td>
<td>$S_e \rightarrow$</td>
<td>$S_e \rightarrow$</td>
<td>$e(t) = E(t)$</td>
</tr>
<tr>
<td>Flow source</td>
<td>$S_f \rightarrow$</td>
<td>$S_f \rightarrow$</td>
<td>$f(t) = F(t)$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$R \leftarrow$</td>
<td>$R \leftarrow$</td>
<td>$e = \Phi_R(f)$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$C \leftarrow$</td>
<td>$C \leftarrow$</td>
<td>$e = \Phi_C^{-1} \left( \int f , dt \right)$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$C \leftarrow$</td>
<td>$C \leftarrow$</td>
<td>$f = \frac{d}{dt} \Phi_C(e)$</td>
</tr>
<tr>
<td>Inertia</td>
<td>$I \leftarrow$</td>
<td>$I \leftarrow$</td>
<td>$f = \Phi_I^{-1} \left( \int e , dt \right)$</td>
</tr>
<tr>
<td>Inertia</td>
<td>$I \leftarrow$</td>
<td>$I \leftarrow$</td>
<td>$e = \frac{d}{dt} \Phi_I(f)$</td>
</tr>
</tbody>
</table>

### TABLE 3.7. Causal Forms for Basic 2-Ports and 3-Ports

<table>
<thead>
<tr>
<th>Element</th>
<th>Acausal Graph</th>
<th>Causal Graph</th>
<th>Causal Relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Transformer</td>
<td>$\frac{1}{1} \frac{T}{F} \frac{2}{2}$</td>
<td>$\frac{1}{1} \frac{T}{F} \frac{2}{2}$</td>
<td>$e_1 = me_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{1}{1} \frac{T}{F} \frac{2}{2}$</td>
<td>$f_2 = mf_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{1}{1} \frac{T}{F} \frac{2}{2}$</td>
<td>$f_1 = f_2/m$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$e_2 = e_1/m$</td>
</tr>
<tr>
<td>Gyrator</td>
<td>$\frac{1}{1} \frac{G}{Y} \frac{2}{2}$</td>
<td>$\frac{1}{1} \frac{G}{Y} \frac{2}{2}$</td>
<td>$e_1 = rf_2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$e_2 = rf_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$f_1 = e_2/r$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$f_2 = e_1/r$</td>
</tr>
<tr>
<td>0-Junction</td>
<td>$\frac{1}{3} \frac{0}{2}$</td>
<td>$\frac{1}{3} \frac{0}{2}$</td>
<td>$e_2 = e_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$e_3 = e_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$f_1 = -(f_2 + f_3)$</td>
</tr>
<tr>
<td>1-Junction</td>
<td>$\frac{1}{3} \frac{1}{2}$</td>
<td>$\frac{1}{3} \frac{1}{2}$</td>
<td>$f_2 = f_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$f_3 = f_1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$e_1 = -(e_2 + e_3)$</td>
</tr>
</tbody>
</table>
5. Examples

1. Electrical System

2. Mechanical System (Translational)

3. Mechanical System (Planar motion)

4. Hydraulic System
6. Procedures for Assembling Bond Graphs for Different Types of Systems
(from Karnopp et al., *System Dynamics*, 4th Edition)

*Construction Procedure for Mechanical Translation*

1. On a schematic of the physical system, use arrows and symbols to indicate the positive direction of “distinct” absolute velocity components. These would include all individual mass elements, all prescribed input velocities, and the velocities of any other physical locations that may prove useful in establishing useful relative velocities. State whether force-generating elements, springs and dampers, are positive in compression or tension.

2. Use 1-junctions to represent each distinct velocity from 1. Label the 1-junction with the velocity symbol from the schematic. This will help to remind you which junction is associated with which velocity component. You might use a 1-junction to represent the reference of zero absolute velocity. This will later be eliminated, but it might be helpful for establishing relative velocities.

3. Attach to each 1-junction any element that relates to the absolute velocity represented by the junction. In general, distinct masses are inertias associated with distinct velocities represented by some of the 1-junctions. Remember, positive power is always directed into an element — $I$, $-R$, or $-C$ element.

4. Use 0-junctions to establish proper relative velocities across the remaining elements so that the elements are positive in compression or tension as was assumed in the schematic.

5. Eliminate the bonds with zero velocity and reduce to the final model.
**Construction Procedure for Fixed-Axis Rotation**

1. On a schematic of the physical system, use arrows and symbols to indicate the positive direction of "distinct" absolute angular velocity components. These would include all individual rotational inertial elements, all prescribed input angular velocities, and the angular velocity of any other physical locations that may prove useful in establishing useful relative angular velocities. State whether torque-generating elements, rotational springs and dampers, are positive when twisted clockwise or counterclockwise.

2. Use 1-junctions to represent each distinct angular velocity from 1. Label the 1-junction with the angular velocity symbol from the schematic. This will help to remind you which junction is associated with which angular velocity component. You might use a 1-junction to represent the reference of zero angular velocity. This will later be eliminated, but it might be helpful for establishing relative angular velocities.

3. Attach to each 1-junction any element that has that angular velocity. In general, distinct rotational inertias are \(-I\) elements associated with distinct angular velocities represented by some of the 1-junctions. Remember, positive power is always directed into an \(-I\), \(-R\), or \(-C\) element.

4. Use 0-junctions to establish proper relative angular velocities across the remaining elements so that the elements are positive when twisted one way or the other, as was assumed in the schematic.

5. Eliminate the bonds with zero angular velocity and reduce to the final model.
4.3.4 Fluid Circuit Construction

Hydraulic and acoustic circuits have a lot in common with electrical circuits. Remember that for electrical circuits, it was easy to establish a 0-junction for each location in a circuit at which an “absolute” voltage could be defined. The passive 1-port elements were then inserted between the appropriate 0-junctions using 1-junctions to establish the voltage differences acting on the elements. Finally, one of the 0-junctions was picked as representing a reference voltage and was eliminated together with all the bonds emanating from it. For fluid circuits, pressures act like voltages and volume flow rates act like electrical currents, so a short version of the circuit construction procedure is as follows:

1. For each distinct “absolute” pressure, establish a 0-junction. If necessary, include a 0-junction for atmospheric pressure.
2. Insert $R$, $C$, and $l$ elements between appropriate 0-junctions using 1-junctions.
3. Assign power sign convention half arrows using a “through” scheme so that the elements react to pressure differences.
4. Attach any pressure or flow sources.
5. The 0-junction for atmospheric pressure can be eliminated if “gage” pressures are to be used.
6. The graph can be simplified if there are any 2-port 0- or 1-junctions with “through” sign convention half arrows.
See text for figures corresponding to the following electrical circuit procedure.

**Circuit Construction Procedure**

1. **Assign a power convention to the circuit schematic.**
   This step must always be done regardless of the modeling procedure being used. If the ultimate goal of the model is to derive circuit dynamic equations or to simulate response, we must have a power convention. On the circuit, this is done by showing the positive voltage drop and current directions. For the \(-I\), \(-R\), and \(-C\) elements, the positive voltage drop (+ to −) is shown in the same direction as the positive current. This ensures that power is directed inward on the corresponding bond graph element. For the source elements (\(S_f\) for current source and \(S_v\) for voltage source), it is not critical which directions are chosen for positive voltage drop and positive current. If positive current is defined such that the current moves “uphill” against the positive voltage as is done in Figure 4.2a, then positive power will come from the source into the rest of the circuit. If either the positive voltage direction or current direction is chosen in the opposite direction, then positive power will be absorbed by the source. There is absolutely nothing wrong with this, and, in fact, real sources may sometimes absorb power from the attached system and sometimes supply power to the attached system.

2. **Label each node voltage on the circuit schematic and use a 0-junction to represent each node voltage as shown in Figure 4.2b.**
   A node voltage is the voltage above and below or to the left and right of each circuit element. In Figure 4.2a, the node voltages are labeled using letters. For convenience, the ground voltage, \(e\), is repeated several times. Remember, every bond that touches a particular 0-junction has an identical voltage.

3. **Establish the positive voltage drops across the elements using 1-junctions.**
   Remember that 1-junctions add efforts (voltages) according to the power convention. By properly directing the half arrows on 1-junctions, the proper voltage drop can be established across each bond graph element. Figure 4.2c shows this construction. For example, for the \(-R\) element representing the resistor, \(R_1\), the “effort” on the bond is \(e_a - e_b\), which is the positive voltage drop defined in the circuit schematic. Notice that positive power is out of the voltage source element and the voltage on the source bond is \(e_a - e_c\), as was
defined in the schematic. Also, the output voltage, $e_{out}$, is exposed using a flow source, $S_f$, of zero current. The voltage drop across this flow source is $e_{out} = e_c - e_e = e_c - 0$. The reader should check the other elements and ensure that all have their defined positive voltages.

4. **Remove all bonds that have zero power.**

Before the bond graph can be used for equation derivation or simulation, the reference voltage must be established. Our reference is $e_e$, and it is zero since it is the ground voltage. Since every bond that touches a 0-junction has the identical voltage, all the bonds inside the curve of Figure 4.2c have zero voltage and each of those bonds carries no power. We can either append an effort source of zero voltage to one of the 0-junctions representing $e_e$, or we can simply erase all the bonds that carry no power, with the result shown in Figure 4.2d.

5. **Simplify the bond graph by using the bond graph identities defined in Chapter 3.**