ENGI9496 Lecture Notes – Multiport Models in Mechanics
(New text Section 4.2.3; Section 9.1 generalizes to 3D motion)

Definitions

**Generalized coordinates** – any set of coordinates that can be used to completely describe the “configuration” of a system. A set of generalized coordinates is typically not unique. Generalized coordinates can include positions, angles, velocities, electrical fluxes, pressures, etc.

**Degrees of freedom (DOF)** – minimum number of generalized coordinates needed to completely describe the configuration of a system. For example if you know the crank angle of a slider crank, you can compute all other link positions in terms of that angle.

**Inertial (primitive) coordinate** – a coordinate describing the absolute position of a body’s centre of mass, or its absolute position.

\( \bar{q}_k \): vector of the minimal (smallest possible) set of generalized coordinates

- \( \text{dim}(\bar{q}_k) = q \), where \( q \) = number of DOF

\( \bar{q}_i \): vector of the inertial coordinates

- \( \text{dim}(\bar{q}_i) = p = 3n \) for 2D systems, or \( 6n \) for 3D systems, where \( n \) = number of bodies
- typically a maximal (largest possible) set of generalized coordinates

\( \bar{v}_i = \dot{\bar{q}}_i \): vector of inertial (primitive) velocities

**Constraint** – an algebraic equation relating generalized coordinates. For 2D systems described by inertial coordinates, there should be \( 3n-q \) constraints, and \( 6n-q \) constraints for 3D systems.

**Approach to Mechanical System Modeling**

One approach to modeling multibody mechanical systems, which is easy to understand and implement in bond graphs, is to define a set of maximal inertial coordinates (sometimes called “primitive coordinates”), and constrain them using velocity constraint equations. As we’ve seen previously, if you constrain the velocity nodes in a bond graph, then the force or torque equations (Newton’s Laws) will be automatically satisfied.

We will restrict ourselves to planar motion (2D), and consider absolute coordinates as well as body-fixed coordinates, which require gyrator elements to capture inner product terms of Euler’s Equations as described in Section 9.1)
Summary of Newton’s Laws for Individual Links

\[ \begin{align*}
\text{Translation} & \quad \begin{cases}
\vec{F} & = m \vec{a} \\
\vec{F}_x & = ma_x \\
\vec{F}_y & = ma_y 
\end{cases} \\
\text{Rotation} & \quad \begin{cases}
\vec{M} & = I \vec{\omega} \\
\vec{M}_x & = I_{cg} \alpha_x \\
\vec{M}_y & = I_{cg} \alpha_y \\
\vec{M}_\theta & = I_{cg} \ddot{\theta}
\end{cases}
\end{align*} \]

Absolute Coordinates

Consider a body with mass and rotational inertia, subject to external forces at points A and B. Centre of gravity is G. For multibody systems, the easiest formulation is one where all bodies contribute three inertial coordinates (x,y,θ). To convert this to a pendulum, make velocity of point A equal to zero (through flow sources, or approximately zero using parasitic springs). This will create pin forces at A. For such a pendulum:

- 1 degree of freedom (DOF); knowing θ, you can compute x and y; should be 3 – 1 = 2 position constraint equations
- there will be much derivative causality, which can be removed using parasitic elements
- differentiate position constraints to get velocity constraints (implement in bond graphs using TF or MTF elements)
- reversing causality of MTF elements may create the risk of singularities (division by zero)
Using two of the above bodies, you can easily create a continuous double pendulum
- two bodies give $2 \times 3 = 6$ inertial coordinates (all measured with respect to a single origin)
- two degrees of freedom (system configuration can be described by the angles $\theta_2$ and $\theta_3$, therefore there must be $6 - 2 = 4$ position constraint equations

Develop a bond graph of the double-pendulum. Then, turn it into a slider-crank by affixing $D$ to a mass that can only slide.
Double-Pendulum Bond Graph
Extension to Slider-Crank
Body-Fixed Coordinates

We now consider an x-y axis system that is fixed to, and rotates with, the body. We must now distinguish between absolute, or inertial coordinates (resolving vectors in fixed x-y directions); and body-fixed coordinates (resolving vectors in directions relative to the body).

When we draw bond graphs, given that bond graphs are a velocity-based formulation (as opposed to position-based), we will find ourselves expressing absolute velocity vectors in body-fixed coordinates. Body-fixed coordinates introduce a complication, as will be described below.

Generic Rigid Body

In the Appendix to this document, some review of coordinate transformations, and derivative of a vector resolved into a rotating coordinate frame, is given.

Development of Bond Graph
Development of Rigid Body Bond Graph, Body-Fixed Coordinates (cont’d)
Simulation Exercise: Shop Crane

Consider a shop crane such as those sold at Princess Auto:

To develop a dynamic, multi-body simulation of this, we will take advantage of both methods of rigid body bond graphs – absolute and body-fixed coordinates.
Shop Crane Bond Graph, Body-Fixed Coordinates
Appendix - Body Fixed (Rotating) Coordinate Frames

To this point, we have expressed vectors in x and y components - an arbitrary vector \( \mathbf{A} \) is the sum of an x component multiplied by a fixed unit vector \( \mathbf{i} \) and a y component times a fixed unit vector \( \mathbf{j} \).

\[
\mathbf{A} = x \mathbf{i} + y \mathbf{j}
\]

To differentiate the vector, we technically need to use the product rule of calculus. However, since \( \mathbf{i} \) and \( \mathbf{j} \) are fixed, they have no derivative and the derivative of \( \mathbf{A} \) is simply:

\[
\frac{d}{dt} \mathbf{A} = \dot{x} \mathbf{i} + \dot{y} \mathbf{j}
\]

In robotics and advanced dynamics, especially 3D kinematics, it is customary to define vector components along reference frames that are affixed to a body and rotate with that body, instead of defining components in fixed \( \mathbf{i} \) and \( \mathbf{j} \) directions. In Fig. 1 below, the relative position of \( \mathbf{B} \) with respect to \( \mathbf{A} \), can be expressed in either absolute or body-fixed coordinates:

![Fig. 1](image)

\( \hat{i}, \hat{j} = \text{fixed unit vectors in } x, y \text{ directions} \)

\( \hat{2} \mathbf{i}, \hat{2} \mathbf{j} = \text{unit vectors fixed to link, rotating with angular velocity } \omega_2 \)

\[
\mathbf{r}_{B/A} = AB(\cos \theta_2 \hat{i} + \sin \theta_2 \hat{j}) = AB(\hat{2} \mathbf{i}) \quad (1)
\]

We can differentiate \( \mathbf{r}_{B/A} \) to get tangential velocity easily by differentiating \( \mathbf{i} \)-\( \mathbf{j} \) components. We can also, by inspection, write \( \mathbf{v}_{B/A} \) in body-fixed components (tangential velocity has magnitude \( AB \omega_2 \), direction perpendicular to line AB).

\[
\mathbf{v}_{B/A} = \frac{d}{dt} \mathbf{r}_{B/A} = -AB \sin \theta_2 \dot{\theta}_2 \hat{i} + AB \cos \theta_2 \dot{\theta}_2 \hat{j} \quad (2)
\]

\[
= AB \omega_2 (\sin \theta_2 \hat{i} + \cos \theta_2 \hat{j}) = AB \omega_2 (\hat{2} \mathbf{j})
\]

From Eq'ns (1), (2), we can see that

\[
\begin{align*}
\hat{2} \mathbf{i} &= \cos \theta_2 \hat{i} + \sin \theta_2 \hat{j} \\
\hat{2} \mathbf{j} &= -\sin \theta_2 \hat{i} + \cos \theta_2 \hat{j} \quad (3)
\end{align*}
\]

In matrix form:

\[
\begin{bmatrix} \hat{2} \mathbf{i} \\ \hat{2} \mathbf{j} \end{bmatrix} = \begin{bmatrix} \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix}
\]

\[
\begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} \hat{2} \mathbf{i} \\ \hat{2} \mathbf{j} \end{bmatrix}
\]

\[
\begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = R_{12} \begin{bmatrix} \hat{2} \mathbf{i} \\ \hat{2} \mathbf{j} \end{bmatrix}
\]

\[
\begin{bmatrix} \hat{2} \mathbf{i} \\ \hat{2} \mathbf{j} \end{bmatrix} = R_{21} \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix}
\]
The matrices $R_{21}$ and $R_{12}$ are called "rotation matrices", because they allow us to convert a vector expressed in coordinates of one frame, into a vector expressed in coordinates of a second frame, where the first and second frames are rotated by an angle $\theta$.

Given a vector $A$ expressed in frame 1 (fixed $i-j$) or frame 2 components, as indicated by left superscript:

$$1^\perp A = R_{12} 2^\perp A$$
$$2^\perp A = R_{21} 1^\perp A$$

It is apparent from the structure of the matrix equations on page 1 that

$$R_{12}^{-1} = R_{21} = R_{12}^T$$

This is a property of rotation matrices, that they are orthogonal - their inverse is equal to their transpose. As an example of rotation matrices in action, transform the tangential velocity $v_{B/A}$ from frame 2 to frame 1 coordinates:

$$2v_{B/A} = AB\omega_2 \begin{bmatrix} 2\hat{j} \end{bmatrix} = \begin{bmatrix} 0 \\ AB\omega_2 \end{bmatrix}$$
$$1v_{B/A} = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} 0 \\ AB\omega_2 \end{bmatrix} = \begin{bmatrix} -AB\omega_2 \sin \theta_2 \\ AB\omega_2 \cos \theta_2 \end{bmatrix} = AB\omega_2 \left(-\sin \theta_2 \hat{i} + \cos \theta_2 \hat{j}\right)$$

Body fixed frames can make it easier to express position vectors. In Fig. 1, the position of B with respect to A is simply $AB$ in the $2i$ direction, regardless of the orientation of the body. When we differentiate vectors expressed in body-fixed frames, however, we have to take into account the fact that the unit vectors, because they change direction, now contribute a derivative. If we differentiate Eqn's (3), we see that

$$\frac{d}{dt} \left(2\hat{i}\right) = -\sin \theta_2 \dot{\theta}_2 \hat{i} + \cos \theta_2 \dot{\theta}_2 \hat{j} = \dot{\theta}_2 \left(-\sin \theta_2 \hat{i} + \cos \theta_2 \hat{j}\right) = \dot{\theta}_2 \hat{k} \times (2\hat{i}) = \vec{\omega}_2 \times (2\hat{i})$$
$$\frac{d}{dt} \left(2\hat{j}\right) = -\cos \theta_2 \dot{\theta}_2 \hat{i} - \sin \theta_2 \dot{\theta}_2 \hat{j} = -\dot{\theta}_2 \left(\cos \theta_2 \hat{i} + \sin \theta_2 \hat{j}\right) = -\dot{\theta}_2 \hat{k} \times (2\hat{j}) = \vec{\omega}_2 \times (2\hat{j})$$

You should be able to visualize the cross products using the right-hand rule. Thus, the derivative of a rotating unit vector is equal to the cross product of the frame's angular velocity and the unit vector itself. This means that the derivative of a vector expressed with respect to a rotating frame has a total derivative as given below. These results extend to the three dimensional case.

Given

$$\bar{A} = a(2\hat{i}) + b(2\hat{j})$$
$$\frac{d}{dt} \bar{A} = a\frac{d}{dt}(2\hat{i}) + b\frac{d}{dt}(2\hat{j}) + a\frac{d}{dt}(2\hat{i}) + b\frac{d}{dt}(2\hat{j})$$
$$\frac{d}{dt} \bar{A} = a\frac{d}{dt}(2\hat{i}) + a\vec{\omega}_2 \times (2\hat{i}) + b\vec{\omega}_2 \times (2\hat{j})$$
\[
\frac{d}{dt} \vec{A} = \dot{a}(\hat{2}\vec{i}) + \dot{b}(\hat{2}\vec{j}) + \vec{\omega}_2 \times [a(\hat{2}\vec{i}) + b(\hat{2}\vec{j})]
\]

\[
\frac{d}{dt} \vec{A} = \left( \frac{d\vec{A}}{dt} \right)_{\text{rel}} + \vec{\omega}_2 \times \vec{A}
\]

where the first term is the derivative as seen by an observer moving and rotating with the coordinate frame. The first term is the rate of change of the components. The second term, which has the original vector crossed with the angular velocity of the reference frame, is an additional rate of change arising from the rotation of the unit vectors. This term would be seen only by an external non-rotating observer.

This has important implications in 3D kinetics and the derivation of gyroscopic torques. As will be seen, there is considerable advantage in using reference frames with angular velocity equal to (or nearly equal to) the body to which they are attached.

\[
\frac{d}{dt} \vec{A} = \left( \frac{d\vec{A}}{dt} \right)_{\text{rel}} + \vec{\omega} \times \vec{A} \quad (4)
\]