SYSTEM DYNAMICS

Modeling, Simulation, and Control of Mechatronic Systems

Fifth Edition

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4.2.3 Plane Motion

For mechanical translation it was fine to imagine the mass elements as point masses, and for fixed-axis rotation it was fine to consider only the rotational inertia of a mass element. But, in general, mechanical bodies with finite mass both translate and rotate in real applications. Consider an automobile as it moves over roadway unevenness. The body of the vehicle moves forward, sideways, and vertically, and it rolls side to side, pitches front to rear, and rotates about a vertical axis as it turns. If we modeled the car body as rigid, we would have to characterize the body by its mass and the moments of inertia about three perpendicular axes. The motion of the vehicle would be quite complex and certainly not able to be described by mechanical translation alone or fixed-axis rotation alone. Instead, the dynamics are governed by a simultaneous combination of translation and rotation. To describe this motion in three dimensions is quite complicated, but is greatly facilitated by using bond graphs. Chapter 9 discusses this complex topic. When our attention is restricted to plane motion, model construction is straightforward.

Plane motion results when the inertial bodies of a physical system are constrained to translate in two dimensions and to rotate only about an axis perpendicular to the plane of motion. Figure 4.17a shows a rigid body of mass, \( m \), and moment of inertia, \( J \), about its center of mass. The \( X \) - and \( Y \) -axes are assumed to be inertial axes. The body is translating in the \( XY \) -plane and it is rotating about the inertial \( Z \) -axis, which points out of the page according to the right-hand-rule. Any forces acting on the body from attached devices are not shown in the figure.

It is useful to describe the motion of the body by describing the translation of the center of mass and the angular velocity of the body. The center of mass has an absolute velocity vector pointing in some direction in the \( XY \) -plane. This vector is resolved into two mutually perpendicular components, \( v_X \) and \( v_Y \), aligned in the \( XY \) -directions as shown on the figure. The angular velocity is the angular rate of change of any line fixed to the body with respect to the \( XY \) -axes.

The kinetic energy \( T \) of a rigid body is characterized in a particularly simple way by the velocity of the center of mass and the angular velocity of the body,

\[
T = \frac{1}{2}mv_X^2 + \frac{1}{2}mv_Y^2 + \frac{1}{2}J\omega^2 \tag{4.5}
\]

in which it is notable that by focusing on the motion of the center of mass and using the moment of inertia about the center of mass the energies of translation and rotation are uncoupled.

Since bond graphs bookkeep energy, to account for all the energy of a rigid body in plane motion we must represent the translation velocity of the body in two perpendicular directions as well as the angular velocity. This is done in the bond graph fragment of Figure 4.17b. The 1-junctions are used to represent each center of mass velocity component and the angular velocity. Attached to the velocity 1-junctions are \(-I\) elements with inertia parameter equal to the mass, \( m \), and attached to the 1-junction for the angular velocity is an \(-I\) element with the moment of inertia, \( J \), as the parameter. This fragment characterizes all the energy of a body in plane motion. Any devices, such as springs and dampers, that interact with the rigid body will end up communicating with the 1-junctions of Figure 4.17b when incorporated into a bond graph model.

In general, devices will be attached to rigid bodies at different places on the body. For example, the point \( p \) indicated in Figure 4.17a might be such a point. We will discover that we need the velocity components of the attachment points indicated as \( v_{px}, v_{py} \) in the figure. The attachment point is located with respect to the center of mass by the position vector \( \vec{r}_p \) composed of the components \( x_p, y_p \). The kinematic relationship (see Reference [2]) that allows determination of the velocity of any point on a rigid body is

\[
\vec{v}_p = \vec{v}_{cm} + \vec{\omega} \times \vec{r}_p \tag{4.6}
\]

where the cross-product term accounts for the contribution of \( \omega \) to the velocity of the point in question. This relationship gets used over and over when modeling rigid bodies in plane motion. In plane motion, the angular velocity vector is always perpendicular to the position vector, thus making the use of Eq. (4.6) particularly simple.
For the geometry of Figure 4.17a, application of Eq. (4.6) yields

\[ v_{p_x} = v_X - y_P \omega \quad \text{and} \quad v_{p_y} = v_Y + x_P \omega. \quad (4.7) \]

It should be noted that for large angular motions of the body, \( x_P \) and \( y_P \) will change as the body moves. In this case, if \( r_P \) is the length of the vector \( \vec{r}_P \), then \( x_P = r_P \cos \theta \) and \( y_P = r_P \sin \theta \), where is the angle between \( \vec{r}_P \) and the axis with \( \theta = \omega \). Equation (4.7) then becomes

\[ v_{p_x} = v_X - (r_P \sin \theta) \omega \quad \text{and} \quad v_{p_y} = v_Y + (r_P \cos \theta) \omega, \quad (4.7a) \]

and one can recognize that displacement modulated transformers as discussed in Section 3.2 and shown in Figure 3.10 will be involved in the bond graph of the system. This type of geometrically nonlinear mechanical system is discussed at length in Chapter 9.

As a simpler first example in which the angle of the rigid body is assumed not to experience large excursions, consider the heave–pitch vehicle model shown in Figure 4.18a. This is an extension of the quarter-car model done previously in Figure 4.10. This model is sometimes called the half-car model. We are looking at the side of the vehicle with the front at the right and the rear at the left. The front and rear suspensions each consist of a tire springs with stiffnesses \( k_{1f} \), \( k_3 \), unsprung masses \( m_{usf} \) , \( m_{usr} \), and suspension springs and dampers with parameters \( k_1, b_1 \), and \( k_3, b_3 \). The inputs to the system are the prescribed input velocities at the front and rear, \( v_{i_f}(t) \) and \( v_{i_r}(t) \). The body is modeled as a rigid body with center of mass located a distance \( a \) from the front suspension connection and a distance \( b \) from the rear suspension connection. The body has mass, \( m_\text{b} \), and centroidal moment of inertia, \( J \). Gravity is acting vertically downward.

For this model the vehicle is not moving forward but is mounted in a test rig that supplies the input velocities to the tire springs as if a bumpy road were passing under the vehicle. The body is assumed to move only vertically and to rotate, or pitch, through small angles. Thus, the rigid body is characterized by the vertical center of mass velocity, \( v_g \), and the pitch angular velocity, \( \omega \). Other important velocities are oriented and labeled in the figure, including the vertical velocity components at the front and rear of the body, \( v_f \) and \( v_r \). These components are introduced for convenience in anticipation of construction of the relative velocity across the front and rear suspension units. As indicated in Figure 4.18a, all spring and damper forces are assumed positive in compression.

The procedure for constructing a bond graph model of a system with rigid bodies in plane motion is basically identical to the procedure for mechanical translation and fixed-axis rotation, but with the added complexity of dealing with kinematics described by Eq. (4.6). Figure 4.18b shows the use of 1-junctions to represent all the distinct velocities and angular velocities of the system. Attached to these 1-junctions are bond graph elements that have these specific velocity and angular velocity components. The front and rear suspension unsprung masses and input velocities should look familiar because of the earlier quarter-car example.

**FIGURE 4.18.** Heave–pitch vehicle model. Example 1 of plane motion.
The rigid body is characterized by the center-of-mass velocity and angular velocity, so \( I \) elements for the mass, \( m \), and centroidal moment of inertia, \( J \), are attached to the appropriate 1-junctions. The weight of the body is an effort source attached to the center-of-mass vertical velocity 1-junction. Positive power is directed into the source owing to the velocity convention (positive upward) adopted for this example. The same is true for the effort sources representing the weights of the unsprung masses, front and rear.

In Figure 4.18c, 0-junctions have been used to establish the relative velocities across the springs and dampers. The power convention is such that all elements are positive in compression as indicated for this system. Some simplifications have already been done. Notice how the 1-junctions for \( v_f \) and \( v_r \) were used to help in constructing the relative velocities across the front and rear suspensions.

Also notice that for the suspension elements, the relative velocities were constructed only once, and then 1-junctions were used to enforce that the respective springs and dampers have the same relative velocity as in Figure 4.12. From Figure 4.18c it is obvious that the model is not complete, and yet all elements have been incorporated. This is where the kinematics plays a major role.

For small angular motions, using Eq. (4.6) and using the right-hand-rule for cross products, we can derive that

\[
v_f = v_g + a\omega \quad \text{and} \quad v_r = v_g - b\omega. \tag{4.8}
\]

(These relations may seem almost obvious but they are special cases of the vertical velocity relation in Eq. (4.7a) when remains near 0 for \( v_f \) and \( \pi \) for \( v_r \).)

These kinematic constraints must be enforced to complete the model. Until now, it has been recommended that positive directions be shown on a schematic and then transferred to the bond graph model. If this is not done then the model may still be all right. But in a simulation of the system response, we might not know whether a positive velocity was up or down, or whether a positive force was compressive or extensive. This is important information to have, so it is always a good idea to have a power convention in mind. However, if the kinematic constraints are not enforced correctly, then the model is just plain wrong.
Kinematic constraints are always relationships among flow variables. To add the flows according to the constraints we use 0-junctions. We just need to pay particular attention to the power convention when adding up the flows to enforce the constraints. In Figure 4.18d the constraints from Eqs. (4.8) are enforced with two different half-arrow patterns using 0-junctions. The reader should check that in both cases the velocity components add correctly to produce Eqs. (4.8). Transformers are used to convert the angular velocity into velocity components, that is, \( \omega \) into \( v_{ao} \) or \( b_{oo} \). On the transformers in the figure, the moduli are appended according to the definition of the transformer given in Chapter 3. Although either version in Figure 4.18d can be used, if the right-hand version of part (d) is used, as is done in Figure 4.18e, then a bond graph simplification can be done at the 1-junctions for \( v_f \) and \( v_r \).

A second example of plane motion is the system shown in Figure 4.19a. The system is a cart of mass, \( m \), acted upon by an input force, \( F(t) \). Attached to the cart through a spring and damper, \( k \) and \( b \), is a cylinder that rolls without slip on the cart. The cylinder is a rigid body of mass, \( m_c \), with center of mass in the center and moment of inertia about its center of mass, \( J_c \), and is of radius \( R \). We desire a model that, if solved analytically or simulated, will predict the motion–time history of this system for any prescribed input force.

The schematic shows positive velocity and angular velocity directions, and the spring and damper are assumed positive in tension. Figure 4.19b uses 1-junctions to represent the distinct velocities and angular velocities, and the elements that have these distinct flows have been attached to the 1-junctions. The rigid-body cylinder is a body in plane motion and is represented by its center-of-mass velocity, \( v_c \), and its angular velocity, \( \omega \), and thus has the \(-1\) elements for its mass and moment of inertia appropriately attached. In Figure 4.19c the relative velocity is established across the spring and damper. Perusal of this figure clearly indicates that the model is not complete, even though all energy has been accounted for.

Again, there is a kinematic relationship that must be derived before we can complete the model. Application of Eq. (4.6) to the velocity of the point at the bottom of the cylinder, \( v_p \), positive to the right, yields

\[
v_p = v_c - R\omega.
\]  

(4.9)

Since the cylinder rolls without slip, the velocity at the bottom of the cylinder must be the same as the velocity of the cart at the same point. Thus, the kinematic constraint is

\[
v_m = v_c - R\omega.
\]  

(4.10)

In Figure 4.19d this constraint is enforced using a 0-junction and the model is complete.

As a final example of systems with plane motion dynamics, consider the system in Figure 4.20a. An input velocity, \( v_{in}(t) \), is prescribed on one end of a spring, \( k_1 \), and the other end is attached to an inextensible rope that wraps around a floating cylinder of mass \( m_1 \), radius \( R_1 \), and moment of inertia \( J_1 \). The rope then passes over a second cylinder that is pinned at its center and has radius \( R_2 \) and moment of inertia \( J_2 \). The rope finally is attached to a mass, \( m_2 \), which is attached to ground through the spring and damper, \( k_2 \) and \( b \). The idea is to formulate a model that, if simulated, would predict the motion–time history of the system to a prescribed input velocity. Positive velocity and angular velocity directions are shown in the figure, and all springs and dampers are assumed positive in extension.

In Figure 4.20b, 1-junctions are used to represent the distinct velocities and angular velocities. The floating cylinder requires a 1-junction for its center-of-mass vertical velocity and a 1-junction for its angular velocity. Notice that the velocities at the left and right of the floating cylinder have been defined. The one at the left, \( v_{l1} \), has been introduced as a convenience for establishing the relative velocity across the spring with constant \( k_1 \). The velocity at the right of the cylinder, \( v_{r1} \), will be useful for relating to the angular velocity, \( \omega_{2} \), of the pinned cylinder. Also, we can see that, since the rope is inextensible, \( v_{r1} \) is equal to the mass velocity, \( v_m \). Attached to these 1-junctions are the bond graph elements that

have these distinct flows. The power arrow for the flow source, $-S_f$, is directed out of the source and into the system, because if the spring with constant $k_1$ is defined as positive in tension and if the top end of the spring is moving upward (defined as positive), then power would be flowing into the system.

In Figure 4.20c, 0-junctions are used to establish the proper relative velocities across the springs and dampers so as to put these elements positive in tension. The spring and damper attached to $m_2$ have been modeled using the "reducible loop" simplification of Figure 4.12. The reference velocity $v_{ref} = 0$, can be erased in the final bond graph. Clearly, the model in Figure 4.20c is not complete.

For cylinder 1, application of Eq. (4.6) yields the kinematic constraints

$$v_{r_1} = v_1 + R_1 \omega_1 \quad \text{and} \quad v_{r_1} = -v_1 + R_1 \omega_1.$$
Moreover, since the rope is inextensible, the velocity at the rim of cylinder 2 is the same as \( v_r \), thus,

\[ v_r = R_2 \omega_2. \]

(4.12)

Equations (4.11) are enforced using 0-junctions as shown in Figure 4.20d. The velocity \( v_r \) is related to \( \omega_2 \), as required by Eq. (4.12). And \( v_r \) is set equal to \( v_{m2} \) by simply connecting the 1-junctions with a bond. The model in Figure 4.20d is now complete. Some simplifications are possible, but this is left to the reader.

As has been demonstrated in all examples of mechanical systems with elements in plane motion, bond graph models are constructed by first establishing distinct velocities and angular velocities using 1-junctions that assign the same flow to each attached bond and add the efforts according to the power convention. This is followed by establishing the proper relative motions across compliances and resistances, using 0-junctions that add flows according to the power convention and assign the same effort to each attached bond. This part of the modeling procedure is straightforward and invariant. With some practice these steps will become automatic. The most difficult part of model construction is deriving and enforcing the kinematic constraints. There is no easy set of rules for recognizing and deriving these relationships. The constraints will always involve flow variables and will always involve application of the vector relationship, Eq. (4.6). Practice is the only way to become comfortable with this aspect of modeling.

A major advantage to the use of bond graphs is that if the velocities are properly constrained using power-conservative elements, then the force and torque relationships will automatically be correctly represented.

**Body-fixed Coordinates.** The concepts of plane motion were developed using the general motion of a rigid body, shown in Figure 4.17. In order to characterize all the energy of a rigid body in plane motion it is necessary to know the velocity of the center of mass and the angular velocity of the body. In Figure 4.17, the velocity vector of the center of mass was resolved into two mutually perpendicular components aligned in the inertial \( XY \)-directions. When there are several places where a rigid body interacts with other parts of a system and when the rigid body can execute large angular motions, it is convenient to introduce the concept of *body-fixed coordinates*. This is a coordinate frame that remains attached to the body at the center of mass and moves with the body as it translates and rotates under the action of whatever is attached to it. The virtue of such a frame is that the attachment points remain at fixed locations relative to this frame, and the inertial properties, that is, moments and products of inertia, remain invariant with respect to this frame. These properties are very important for three-dimensional rigid-body motion, which is discussed fully in Chapter 9. We introduce the concept here for plane motion.

The general rigid body of Figure 4.17 is shown again in Figure 4.21a with body-fixed coordinates. The \( xy \)-coordinate frame is attached to the body at the center of mass, and it is aligned in principal directions, although this is not essential. The instantaneous velocity vector that is pointing in some direction in the plane of motion is resolved into two mutually perpendicular directions aligned along the body-fixed axes. These components, \( v_x \) and \( v_y \), are different from the inertial components in Figure 4.17, since they change direction as the body rotates. This change in direction of velocity vector components results in acceleration components that must be accounted for. Also shown in Figure 4.21a

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*This section can be omitted without any loss of generality.*
is the attachment point, \( P \), that is located with respect to the \( xy \)-coordinates by the fixed lengths \( x_p, y_p \).

In Reference \([2]\), it is shown that for any vector, say the velocity vector, \( \vec{v} \), referred to a rotating frame, the absolute time rate of change of the vector, or acceleration in this case, is

\[
\frac{d}{dt} \vec{v} = \frac{\partial}{\partial t} \vec{v}_{\text{rel}} + \vec{\omega} \times \vec{v},
\]

(4.13)

where the first term on the right is the acceleration as observed relative to the moving frame, and the second term accounts for the contribution of the frame rotation to the absolute acceleration. Applying this relationship to the rigid body of Figure 4.21a yields the center of mass absolute acceleration, \( a_x, a_y \), as

\[
a_x = \dot{v}_x - \omega v_y \quad \text{and} \quad a_y = \dot{v}_y + \omega v_x,
\]

(4.14)

where the components due to rotation are derived by carrying out the cross products using the right-hand-rule. For example, the \( \omega \)-vector pointing out of the page, when crossed into the \( x \)-direction velocity component, \( v_x \), yields a vector of length \( \omega v_x \) pointing in the \( y \)-direction.

Newton’s laws apply to the absolute acceleration of the center of mass, and become

\[
\sum F_x = m a_x \quad \text{and} \quad \sum F_y = m a_y,
\]

(4.15)

where the sum of forces in the \( x \)- and \( y \)-directions come from any attached systems to the body, and the rotational contributions to the acceleration have been brought to the force side of the equation after multiplying by the body mass, \( m \).

Returning now to Figure 4.21b, it is tempting to represent the plane motion of the rigid body as shown, where the body-fixed velocity components, \( v_x, v_y \), are used rather than the inertial components, \( v_x, v_y \) from Figure 4.17. The kinetic energy of the body is accounted for, but the absolute acceleration is not. In other words, if the forces from external elements were added using the 1-junctions of Figure 4.21b, then the resulting equations of motion would not be correct. The fix to this problem lies in Eqs. (4.15).

The cross-product terms of Eqs. (4.15) have a remarkable symmetry. In the first equation, there is a force component in the \( x \)-direction that is equal to \( m \omega \) times the velocity in the \( y \)-direction, and in the second equation, there is a force component in the \( y \)-direction equal to \( m \omega \) times the velocity in the \( x \)-direction. Since gyrotrons relate efforts to flows across them, consider Figure 4.21c where a modulated gyrotrons has been inserted between the 1-junctions representing \( v_x \) and \( v_y \). A modulated gyrotrons is used because the modulus, \( m \omega \), is not constant but varies as the body moves and changes. Convincing yourself that the forces on the \(-1 \) elements exactly duplicate the left-hand side of Eqs. (4.15). When body-fixed coordinates are used, the \(-MGY\) — must be part of the bond graph.

For the attachment point in Figure 4.21a, the velocity components in the body-fixed directions are related to the center of mass velocity components using Eq. (4.6), with the result

\[
v_{p_x} = v_x - v_p \omega \quad \text{and} \quad v_{p_y} = v_y + v_p \omega.
\]

(4.16)

As an example of the use of body-fixed coordinates, consider the system in Figure 4.22a. The rigid body of mass, \( m \), and centroidal moment of inertia, \( J \), is attached to ground at points 1 and 2 through springs and dampers as shown. The ground side ends of the springs and dampers are on frictionless carts that ensure that horizontal elements remain horizontal regardless of the motion of the body. Horizontal spring 1 has a velocity input at its ground side end. The body is shown displaced from its starting orientation, and body-fixed coordinates are attached at the center of mass. The attachment points are located with respect to the body-fixed coordinate frame by the fixed distances \( x_1, y_1 \) for point 1 and \( x_2, y_2 \) for point 2, where \( x_2 \) is a negative quantity. In Figure 4.22b the body-fixed velocity components at the attachment points are indicated, and in Figure 4.22c

![Figure 4.22](image-url)
1-junctions are used to represent all distinct velocities. Attached to these 1-junctions are elements that have these absolute velocities. Notice the representation of the body when using body-fixed coordinates. The body-fixed velocity components at the attachment points are indicated using 1-junctions. These are defined as a convenience for ultimately determining the horizontal velocity at the attachment points for use as inputs to the horizontal spring–damper elements. As noted in Figure 4.22b, the body-fixed velocities at the attachments are

\[ v_{x1} = v_x - y_1 \omega, \]
\[ v_{y1} = v_y - x_1 \omega, \]
\[ v_{x2} = v_x - y_2 \omega, \]
\[ v_{y2} = v_y + x_2 \omega, \]  

(4.17)

and the horizontal velocity at the attachments are

\[ v_{h1} = v_x \cos \theta - v_1 \sin \theta, \]
\[ v_{h2} = v_x \cos \theta - v_2 \sin \theta. \]  

(4.18)

These kinematic relationships are enforced using 0-junctions as shown in Figure 4.22d. The transformers used to enforce constraints from Eqs. (4.18) are modulated transformers, \( -MTF \), because the moduli of these elements vary as the body moves. Figure 4.22d is the final bond graph for this example. The bond graph does appear a bit busy, but it contains a lot of information. In Chapter 9, some shorthand notation is introduced for complex systems that allows the bond graph to be constructed while showing fewer bonds.

### 4.3 HYDRAULIC AND ACOUSTIC CIRCUITS

In this section, special but important classes of fluid-flow systems will be modeled using bond graphs. The models to be used exhibit a close analogy to the mechanical and electrical systems studied in the previous section of this chapter. The variables to be used here have been discussed in Chapter 2 (Table 2.4), and the basic fluid system elements have been discussed in Chapter 3. Tables 3.1, 3.2, 3.3, and 3.4 and Figures 3.1, 3.2, and 3.4 show the type of 1-port fluid elements to be used in this section. Figure 3.8 shows a 2-port transformer with a fluid port, and Figure 3.11 shows 0- and 1-junction fluid elements. So far the discussion has been fairly general, but it is now time to be more specific about the modeling of fluid systems.

One category of systems that is important in engineering and can be modeled using the elements shown previously is commonly called **hydrostatic**. These are systems composed of pumps, motors, pipes, pistons, valves, filters, and accumulators that use nearly incompressible fluids, such as water or hydraulic oil. Such systems are found in machine tools, earth-moving equipment, power