Analysis and Simulation of Planar Mechanism Systems Using Bond Graphs

A method is presented for incorporating planar mechanisms into dynamic system models using bond graphs. Through the use of stiff coupling springs at the mechanism joints, the nonlinear geometrical relationships are uniformly and simply described by displacement modulated transformers and the system state equations can be written with no algebraic complications. In contrast to the more elegant kinematic techniques for describing mechanism dynamics, the present method results in higher order systems of equations but the equations themselves are simpler and not densely coupled. In addition, coupling forces are available at the joints. An example demonstrates that the extra eigenvalues associated with the coupling springs can readily be found for any configuration so that the spring constants can be chosen to minimize computation time.

Introduction

There are a large number of analytical techniques and computer-oriented methods for the study of mechanisms. Commonly a mechanism is first studied kinematically and arranged such that a useful type of motion can be achieved, and then subsequently the effects of flexibility and dynamic loads are evaluated for high speed operation. While dynamic analyses based on the laws of mechanics typically assume motion or force time histories for inputs to the mechanism, it is in fact quite common for electrical, hydraulic, pneumatic, or other systems to interact dynamically with the mechanism. It is therefore of interest to include mechanism dynamics within a broad modeling concept which can describe all types of physical systems.

One candidate method for dynamic analysis is based on bond graphs [1]. Although bond graphs are of relatively recent origin, they have proved useful in a great variety of dynamic modeling contexts [2], [3], [4]. That bond graphs are capable of handling rigid body dynamics for both large and small motions has been clear for some time [2], [5], but in practice there can exist major differences in case of formulation depending upon how generalized coordinates are chosen and how the physical constraints are applied.

The general approach to rigid body systems presented in [2] is often effective, but sometimes the geometric constraints among rigid bodies linked together can result in difficult algebraic relations among the state variables. In bond graph terms, these are problems of differential causality and they become difficult because of the nonlinearities associated with large angular motion. Reference [1] presents a sophisticated scheme for handling the geometric constraints using matrix transformations. Here we present essentially the opposite extreme, namely an approach in which finite compliance is assumed to exist at linkage connection points. This approach simplifies equation formulation since the links all are treated in integral causality—that is, the various link momenta are independent—and the motions of connection points are always computed using a standard type of multiport transformation [5]. With this approach, the bond graph for the mechanism is easily combined with a bond graph for electrical, hydraulic or other components in the normal manner.

Because of the compliant interconnections, constraint forces automatically are available in the scheme presented here. In other techniques it may be necessary to set up auxiliary relations to compute forces within the system. When bearing stiffnesses are fairly low the present method is certainly justified; however, when it is physically justified to assume that the linkage connections are essentially infinitely stiff, then the compliances used must be carefully chosen. If the coupling springs are too soft, the relative motion at the connection points will be unrealistically large while if the springs are too stiff, spurious high frequency vibrations will be present which will require very short step sizes for numerical stability in digital simulation. Fortunately it turns out to be quite easy to set up a linearized version of the system equations and to optimize the coupling spring stiffnesses as well as the integration time step on the basis of eigenvalues.

Using the bond graph approach, the unforced state equations for the mechanism are of the form

\[
\begin{bmatrix}
\dot{\mathbf{x}} \\
\dot{\theta}
\end{bmatrix} =
\begin{bmatrix}
A_1(\theta) & 0 \\
A_4(\theta) & 0
\end{bmatrix}
\begin{bmatrix}
\mathbf{x} \\
\theta
\end{bmatrix}
\]

(1)

where \(\mathbf{x}\) is a vector of state variables (linear and angular momenta and displacements), \(\theta\) is a vector of angles indicating the positions of the links, and \(A_1\) and \(A_4\) are submatrices which contain trigonometric functions of the components of \(\theta\). If an equilibrium
position for the mechanism is found such that all springs are relaxed and there is no motion, then \( g \) will vanish and \( \theta = \theta_0 \), a set of equilibrium link angles. Because of the form of equation (1), vibrations around the equilibrium position are described by equation (1) with \( \theta_0 \) substituted for \( \theta \). Derivatives of \( A_1 \) and \( A_2 \) with respect to \( \theta \) play no role because the equilibrium value of \( g \) is zero. Thus linearization of the mechanism dynamic equations is immediate and one can see quickly how the system eigenvalues depend upon the choice of coupling spring stiffnesses. For most mechanisms, the dependence of the eigenvalues upon the equilibrium angles \( \theta_0 \) is not strong and for the purpose of choosing appropriate coupling spring stiffnesses and time steps only a few sets of eigenvalues need be computed. Often the remaining parts of the system are either linear or linearizable relatively simply so that equation (1) may be augmented to include the entire state vector and system eigenvalues can be studied in order to check upon the reasonableness of the model and its parameters. This is one advantage of having a uniform modeling scheme for the entire system.

A Bond Graph for Plane Motion

In Fig. 1 a link in plane motion is shown together with a universal bond graph segment and some coupling springs that connect to other elements of the mechanism. The basic simplicity of the method to be outlined below is that all links are treated in the same manner with the possible exception of those rotating about a fixed pivot which, if desired, may be treated in a simplified manner. (This simplification will be illustrated in the example.)

All variables in the system are indicated by subscripts indicating the bond number and the following standard bond graph scheme:

\[
\begin{align*}
\text{e} &= \text{forces and torques} \\
\text{f} &= \text{velocities and angular velocities}
\end{align*}
\]

In all cases, \( e_i = dp_i/dt \) and \( f_i = dq_i/dt \) and all displacements are with respect to inertial space.

Referring to the bond graph of Fig. 1(b), the inertial properties of the link are as follows:

\[
\begin{align*}
f_1 &= p_1/I_1 \\
f_2 &= p_2/I_2 \\
f_3 &= p_3/I_3
\end{align*}
\]

where \( I_1 = I_2 \) is the mass of the link and \( I_3 \) is the centroidal moment of inertia. The force-momentum and torque-angular momentum relations are

\[
\begin{align*}
p_1 &= e_1 - e_11 \\
p_2 &= e_2 - e_22 \\
p_3 &= e_3 - e_33
\end{align*}
\]

In order to find the various forces and torques in equation (3), the coefficients in the two multiport transformers must be evaluated. These transformers not only transform the coupling spring forces into the efforts on the inertial element, but also transform the flows from the inertia elements into velocities of the coupling spring ends. Since the transformations are power-conserving the flow transformation matrix is simply the transpose of the effort transformation with the "through" sign convention shown in Fig. 1 of [5]. The actual matrices are readily derived by differentiating the displacement relations. For example, the \( x \) and \( y \) displacements of the upper right pivot point in terms of the center of mass coordinates are

\[
q_x = q_x + a \cos q_1 \\
q_y = q_y + a \sin q_1
\]

Noting from the bond graph that \( f_1 = f_1, f_2 = f_2, f_3 = f_3 \), the equations for the velocity transformations follow from equation (4) directly by time differentiation.

\[
\begin{bmatrix}
f_1' \\
f_2' \\
f_3'
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -a \sin q_1 \\
0 & 1 & a \cos q_1
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}
\]

The required force transformation uses the transpose of the matrix in equation (5).

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & a \sin q_1 \\
0 & 1 & -a \cos q_1
\end{bmatrix}
\begin{bmatrix}
e_1' \\
e_2' \\
e_3'
\end{bmatrix}
\]

This is then the complete characterization of the right-hand MTF in Fig. 1(b), and since integral causality will be maintained in this formulation scheme, equations (5) and (6) are in the correct form for immediate use in writing the final system equations.

From Fig. 1(a), it may be seen that the lower attachment point is at an angle of \( q_1 + \gamma \) from the mass center. The displacement, velocity and force relations follow as above.

\[
q_x = q_x - b \cos (q_1 + \gamma) \\
q_y = q_y - b \sin (q_1 + \gamma)
\]

\[
\begin{bmatrix}
f_1' \\
f_2' \\
f_3'
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & b \sin (q_1 + \gamma) \\
0 & 1 & -b \cos (q_1 + \gamma)
\end{bmatrix}
\begin{bmatrix}
f_1 \\
f_2 \\
f_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
e_1 \\
e_2 \\
e_3
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & -b \sin (q_1 + \gamma) \\
0 & 1 & b \cos (q_1 + \gamma)
\end{bmatrix}
\begin{bmatrix}
e_1' \\
e_2' \\
e_3'
\end{bmatrix}
\]

where the facts that \( f_s = f_1, f_s = f_2, \) and \( f_{ss} = f_3 \) have been used.
Any body in plane motion can be described as shown in Fig. 1 and modulated transformers can be found to express the motion of any point on the body. If forces are then generated by relative motion through springs as shown or dampers, then the transformers automatically properly apply the forces to the body as illustrated. In the example, we represent pin joints by equal spring constant elastic elements in the z and y directions, but other types of elastic and damping elements could be used.

For a link which pivots about a fixed axis, one may achieve a simpler representation than the one shown in Fig. 1 by representing the link as a single inertia element with the moment of inertia about the pivot as the parameter. In this case the pivot bearing forces will not automatically appear as forces in elastic constraint elements, but the simplification results in fewer state equations.

As the example illustrates, the motions of points on a link in fixed-axis rotation are also computed using MTF elements as above.

An Example System

The example chosen is identical in basic configuration to the example of reference [1]. A schematic diagram of the system is shown in Fig. 2. Since the mechanism itself consists of several links and a slider, which represents a punch used to form metal cups, and yet is only a single degree-of-freedom system, this example is a fairly test of the proposed method and is inherently better suited to the approach of reference [1]. Nonetheless it is a useful test case for both methods.

Fig. 3 is a diagram of the mechanism showing coupling springs and defining important quantities. The numbering scheme is based on the bond graph of Fig. 4. Note that the angles of \( \theta_1, \theta_2, \theta_3, \theta_4, \) and \( \theta_5 \) are needed for the seven modulated transformers. Also, \( \theta_6 \) is needed if SE 62 is to be used to apply forces as a function of the displacement of the punch. These five variables may be computed by integrating \( f_{a1}, f_{a2}, f_{a3}, f_{a4}, \) and \( f_{a5} \) which are functions of the 20 state variables associated with I- and C-elements in integral causality. In Fig. 4 the only bond numbers shown are those associated with state variables, although in writing the state equations it is convenient to number all bonds in order to have labels for intermediate quantities.

The state variables for the motor, which is modeled as in reference [2], are \( p_r \) (flux linkage) and \( p_i \) (angular momentum). The twist in the drive shaft is \( \theta_6 \) and all other state variables are associated with the mechanism. The coupling springs are \( C_{19}, C_{20}, C_{30}, C_{40}, C_{41}, C_{42}, C_{58}, \) and \( C_{59} \). The remaining dynamic elements are \( I_{14} \) and \( I_{50} \) which represent fixed axis rotation of the flywheel and fixed link, and \( I_{29}, I_{27}, I_{28}, I_{51}, \) and \( I_{53} \) which represent translational and rotational inertias of the triangular and floating links.

The seven MTF's are characterized as described above and, as the causality indicates, the writing of the state equations proceeds in a straightforward standard manner [2], i.e., no algebraic loops and no derivative causality is found. For simplicity, we have not shown several types of bearing friction which were included in reference [1]. These dissipative effects are modeled by appending R-elements reacting to relative angular velocities as appropriate. The addition of these R-elements does not affect the basic causality shown and the state equations are only slightly more complicated. In several of the computer runs bearing friction was included.

For this rather difficult example, 25 equations must be integrated using the present method while the method of reference [1] requires integration of only nine equations. Also, if the real bearing stiffnesses are very high, we must choose artificially low coupling spring constants which keep the relative motion low but do not lead to excessively high vibrational frequencies and short integration step sizes. On the other hand, the information generated by the 25 equations is more complete than that provided by the nine, and the simplicity of the formulation is paid for by having the computer work harder. Many human beings would be willing to trade off in this direction.

Simulation Results

The 25 first-order governing equations were derived for the example system of Fig. 2 directly from the bond graph of Fig. 4. These equations were solved numerically on a digital computer using the system parameters listed below.

- driving voltage, SE1, = 24.0 volts
- motor inductance, \( I_3 \), = 0.5 Henry
- motor gyrator parameter = 0.0409 N-m/A

![Fig. 4 Bond graph for example system](image-url)
motor inertia, \( I_7 = 3.52 \times 10^{-4} \text{ kg-m}^2 \)
coupling shaft compliance, \( C_9 = 54.0 \text{ m/N} \)
gear reduction ratio = 100:1
flywheel inertia, \( I_{14} = 9.0 \times 10^{-4} \text{ kg-m}^2 \)
flywheel radius, \( a_i = 0.036 \text{ m} \)
triangular link mass, \( I_{26}, I_{27} = 5.0 \text{ kg} \)
triangular link rotary inertia, \( I_{28} = 4.5 \times 10^{-3} \text{ kg-m}^2 \)
triangular link dimensions
\[
\begin{align*}
  b_1 &= 9.1 \times 10^{-2} \text{ m} & r_1 &= 13.5^\circ \\
  b_2 &= 4.9 \times 10^{-2} \text{ m} & r_2 &= 26.5^\circ \\
  b_3 &= 5.9 \times 10^{-2} \text{ m} & r_3 &= 54.0^\circ 
\end{align*}
\]
fixed link inertia, \( I_{50} = 5.6 \times 10^{-3} \text{ kg-m}^2 \)
fixed link length, \( c_i = 0.1075 \text{ m} \)
floating link mass, \( I_{51}, I_{52} = 8.00 \text{ kg} \)
floating link rotary inertia, \( I_{53} = 3.2 \times 10^{-3} \text{ kg-m}^2 \)
floating link lengths, \( \rho_i = \rho_2 = 4.1 \times 10^{-2} \text{ m} \)
cutter mass, \( I_{61} = 10 \text{ kg} \)

An eigenvalue analysis was first performed in order to determine the effect of coupling spring stiffness on system frequencies. This type of analysis is easily and inexpensively performed on a digital computer for any desired system configuration, and yields valuable information with respect to numerical problems for transient response simulation.

Fig. 5 shows the effect of coupling spring stiffness on system frequencies for the example problem for one particular position. All coupling spring stiffneses were maintained equal for all stiffnesses tested. For very soft springs, all links are virtually decoupled from the flywheel motion. The single remaining frequency (25 Hz) is that associated with flywheel and motor inertias interacting with the shaft compliance. This is the actual system natural frequency. The frequency that would normally be associated with the gyroscopic coupling between the motor inductance and rotary inertia has been overdamped by the interaction of the motor electrical and mechanical resistances.

As the coupling spring stiffnesses are increased, the inertial loading on the flywheel also increases resulting in an increase of all system frequencies except a decrease of the one associated with the shaft compliance. As the stiffness-generated frequencies approach the actual system frequency (stiffnesses of \( \approx 10^4 \) to \( 10^6 \text{ N/m} \)), strong interaction occurs and the dynamic behavior becomes complicated. In this stiffness range, the coupling spring constants should be interpreted as actual bearing stiffnesses as opposed to artificial element parameters since the coupling springs are interacting with the torsional spring of the shaft.

As the lowest coupling-spring-generated frequency exceeds the actual system frequency, the system modes become again decoupled and the system behavior approaches the infinite stiffness case. The computational price one pays for the savings in algebraic manipulation afforded by the coupling springs is indicated by the highest frequency associated with any particular stiffness chosen. For instance, if the coupling spring stiffness is \( 10^6 \text{ N/m} \) then the lowest spring-generated frequency is 45 Hz or approximately 2.4 times the actual system frequency. The highest spring-generated frequency is 995 Hz and dictates the integration step size for transient response simulation. Thus, if differential causality were permitted from the model inception then only the 20 Hz actual system frequency would have resulted for the single degree-of-freedom system. By including the coupling springs, significantly less algebra resulted during model formulation; however, the resulting model requires an integration step size of perhaps 50 times smaller than the single degree-of-freedom model.

Transient response simulations were carried out for coupling spring stiffnesses of \( 10^7 \text{ N/m} \). At first the cutter force was set to zero (in Fig. 4, \( SE_{52} = 0 \)) and the system motion was simulated for several revolutions of steady-state operation. These results are shown in Fig. 6 using the notation of Fig. 2. This demonstrates that the coupling springs are capable of enforcing the kinematic constraints of the system.

To demonstrate deviations from the steady state, a step change in the cutter force was provided as a system input and a simulation was carried out for 0.25 sec. Fig. 7 shows the response of the motor RPM during the 0.25 sec interval. The RPM has been.
nondimensionalized with respect to the RPM just prior to cutter force application. The corresponding motor current response is shown in Fig. 8. As expected, the motor speed decreases in response to the sudden load on the cutter while the motor current increases. The ripples in both curves are indicative of the oscillatory response of the linkage.

The tool "chatter" associated with this sudden loading is indicated in Fig. 9. Here the horizontal force component of the coupling spring attached to the cutter is shown for a short interval of time. This force corresponds to the effort on bond 59 in Fig. 4. This force oscillation will ultimately decay while its average value approaches that of the step force input.

**Conclusions**

A general development was presented for using bond graphs to model the dynamics of planar mechanisms and to incorporate these models into overall dynamic system models including hydraulic, pneumatic, electric, or other actuators.

To avoid complicated algebraic problems during formulation of the system equations, the concept of a coupling spring was introduced. These are compliance elements which are located at all points where geometric constraints would normally be imposed on the motion of the mechanism. By selecting the stiffness of these elements sufficiently high, all kinematic constraints are enforced while integral causality is maintained throughout the bond graph. In this way, the formulation of system equations is simplified. The price paid for the ease in formulation is the increased system order as well as the artificial high frequencies introduced by the stiff coupling springs. Thus significantly smaller integration time steps are required than would be if the coupling springs were not used.

The procedure is demonstrated for an example system composed of a DC-motor-driven, five-bar linkage. For this system 25 first-order equations must be integrated as opposed to nine equations if differential causality were permitted. Also, the required time step is approximately 50 times smaller than would be necessary if differential causality were permitted. However, the ease in formulation plus the fact that the joint forces are yielded directly helps to compensate for the increased computer expense. In practice it may also happen that the effective bearing stiffnesses are low enough that relatively low vibrational frequencies do occur. In this case the dynamic behavior of the linkage would be modeled by using the actual bearing stiffness values. Finally it should be noted that a significantly simpler model can be made if some links have so little mass that they can be modeled as purely spring-like elements. In the example system if both the floating link and the fixed link were so modeled then six dynamic equations would be eliminated and a very significant saving of computational time would result since some of the highest frequency eigenvalues would disappear.

**References**