CHAPTER FIVE

TIME RESPONSE OF LOW-ORDER LINEAR SYSTEMS

5-1 INTRODUCTION

In this chapter we consider the dynamic response of low-order linear constant-coefficient systems. The models are described by a set of state-space equations together with a set of initial conditions. Our dual tasks are to extract insight and to obtain formal results concerning the dynamic behavior of the models in question.

Our principal method will be the use of Laplace transforms, which enable us to convert a problem in linear differential equations expressed in the time domain into a set of linear algebraic equations expressed in the complex frequency domain. The transformation is well worth knowing and using because it yields both insight and analytical results in an efficient manner.

Example 5-1 To motivate our discussion of the technical details that follow in this chapter, we first consider a familiar example. A mass-spring configuration is shown in Fig. 5-1a. The system bond graph is given in Fig. 5-1b, where the $C$, $R$, and $I$ elements represent the spring, damper, and mass, respectively. The $S_e$ element represents the gravity force on the mass. Here are some questions we wish to address:

1. Under what conditions will the system oscillate?
2. What are the steady-state values of the system variables?
3. Exactly how does the system move from its initial state to its final state?

We shall see that the amount of work required to answer each of these questions differs, and our transform method allows us direct control over the effort invested.
Do not try to understand each step in detail now. Just try to follow the pattern. Later you can return to this example when you understand the details of the operations.

We begin by writing the state equations for the system from the bond graph of Fig. 5-1b

\[
p(t) = -\frac{b}{m} p(t) - kx(t) + mg \quad \text{(5-1a)}
\]

\[
x(t) = \frac{1}{m} p(t) \quad \text{(5-1b)}
\]

where \( b \) = damping constant

\( m \) = mass

\( k \) = spring constant

\( g \) = acceleration of gravity

and we have indicated which variables are functions of time. Next, we obtain transformed equations

\[
sP(s) - p(0) = -\frac{b}{m} P(s) - kX(s) + \frac{mg}{s} \quad \text{(5-2a)}
\]

\[
sX(s) - x(0) = \frac{1}{m} P(s) \quad \text{(5-2b)}
\]

Each equation above is the transform of one of the state equations. The unknowns \( P(s) \) and \( X(s) \) are the Laplace transforms of \( p(t) \) and \( x(t) \), respectively. For now, it is sufficient to notice that Eqs. (5-2) are linear algebraic equations and the independent variable is not \( t \) (time) but \( s \) (generalized frequency). The initial conditions are incorporated into Eqs. (5-2) in the terms \( p(0) \) and \( x(0) \), which are values of \( p \) and \( x \) at \( t = 0 \).

Next we solve for the particular unknown of interest. In this case, let us choose \( X(s) \), corresponding to both spring deflection and mass position. From Eqs. (5-2) we obtain
\[ X(s) = \frac{(s + b/m)x(0) + 1/mp(0) + g/s}{s^2 + (b/m)s + k/m} \]  \hspace{1cm} (5-3)

This is a transformed solution for \( x(t) \) obtained by solving the simultaneous algebraic equations (5-2a) and (5-2b).

Our first equation can be answered from a study of the denominator of Eq. (5-3). We calculate the roots of the denominator polynomial; i.e., we solve

\[ s^2 + \frac{b}{m}s + \frac{k}{m} = 0 \]  \hspace{1cm} (5-4)

to get

\[ s_{1,2} = -\frac{b}{2m} \pm \frac{1}{2} \sqrt{\left(\frac{b}{m}\right)^2 - 4 \frac{k}{m}} \]  \hspace{1cm} (5-5)

If

\[ \left(\frac{b}{m}\right)^2 - 4 \frac{k}{m} > 0 \]  \hspace{1cm} (5-6)

the roots are real and the system will not oscillate. If

\[ \left(\frac{b}{m}\right)^2 - 4 \frac{k}{m} < 0 \]  \hspace{1cm} (5-7)

the roots are complex and the system will execute damped oscillations.

In physical terms, we see that as the damping \( b \) increases relative to the stiffness \( k \), there is less tendency for oscillation. This type of insight is very useful in the preliminary stages of design and analysis.

To answer the second question, we can apply the final value theorem, (FVT), which is derived in appendix B, to the Laplace solution for \( X(s) \), provided the roots \( s_1 \) and \( s_2 \) both have negative real parts (which is the case if \( b, k, \) and \( m \) are all positive). Applying the FVT yields

\[ x(t \to \infty) = \lim_{s \to 0} sX(s) = \frac{s [(s + b/m)x(0) + 1/mp(0) + g/s]}{s^2 + (b/m)s + k/m}, \quad s \to 0 \]

or

\[ x(t \to \infty) = \frac{mg}{k} \]  \hspace{1cm} (5-8)

This result tells us that the spring stretches enough to generate a force \( kx(t \to \infty) \) to counterbalance the weight \( mg \). That is reasonable.

We have answered two of our original questions from the transformed solution \( X(s) \) rather than from the complete time solution \( x(t) \), and the computational effort was quite small. To obtain the complete time response, we must transform back from the frequency \( s \) domain to the time \( t \) domain. This is done most effectively by the use of partial fractions and a table of Laplace transforms.
Suppose \( m = 1 \text{ kg}, \ b = 5 \text{ N}\cdot\text{s/m}, \) and \( k = 4\text{N/m} \). Let \( p(0) = 0 \) and \( x(0) = 0 \). Find \( x(t) \). Observe that

\[
X(s) = \frac{g}{s^2 + 5s + 4}s = \frac{g}{(s + 4)(s + 1)s}
\]  

This can be expressed as

\[
X(s) = \frac{g/12}{s + 4} - \frac{g/3}{s + 1} + \frac{g/4}{s}
\]

which yields the inverse

\[
x(t) = g/12 \ e^{-4t} - g/3 \ e^{-3t} + g/4
\]

Check that the initial condition is correct: \( x(0) = 0 \). Observe that the steady-state \( x(t \to \infty) = g/4 \), which corresponds to Eq. (5-8). Finally, note that the solution does not oscillate, i.e., contains no sine or cosine terms.

The Laplace Procedure

The next sections develop the details for each of the steps in the Laplace-transform approach to state-equation study and solution. Now we summarize the major steps for future reference:

1. Laplace-transform the state equations.
2. Solve for the unknown(s) of interest.
3. Find the roots of the transform denominator.
4. Separate the transform solution by partial fractions.
5. Invert to get the complete time solution.

5-2 THE LAPLACE TRANSFORM

Transform methods were introduced by Oliver Heaviside in the late 1800s in the form of an operational calculus and have been the subject of much mathematical study since. We shall restrict our attention to those features of the Laplace transform which make it useful for our problems. More complete discussions are available.\(^1\sim3\)

**Definition** The Laplace transform of a given time function \( x(t) \) is denoted by \( X(s) \) and written

\[
X(s) = \mathcal{L}\{x(t)\}
\]

The defining relation is

\[
\mathcal{L}\{x(t)\} = \int_0^\infty x(t) e^{-st} \ dt \quad (5-12)
\]

\(^1\) Numbered references appear at the end of the chapter.
This transformation converts a function of time \( t \) into a function of generalized frequency \( s \), where the units of \( s \) are the reciprocal of the units of time. The variable \( s \) is a complex quantity.

Although it is of limited practical use to us, the inverse transform is defined by the complex inversion integral

\[
x(t) = \begin{cases} 
\lim_{T \to \infty} \frac{1}{2\pi j} \int_{s=jT}^{s=-jT} e^{st} X(s) \, ds & t > 0 \\
0 & t < 0
\end{cases} \tag{5-13}
\]

Our principal approach is to learn about a number of common transform pairs, \( x(t) \) matched with \( X(s) \), and to use the pairs to solve our problems. Although our working table of pairs (Table 5-1) is small, it covers a large percentage of typical linear engineering problems. Furthermore, extensive tables of transform pairs are available if

<table>
<thead>
<tr>
<th>Entry</th>
<th>( X(s) )</th>
<th>( x(t), , t &gt; 0 )</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{s} )</td>
<td>1</td>
<td>Unit step</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{s^2} )</td>
<td>( t )</td>
<td>Unit ramp</td>
</tr>
<tr>
<td>3</td>
<td>( \frac{1}{s + \sigma} )</td>
<td>( e^{-\sigma t} )</td>
<td>Exponential</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{1}{(s + \sigma)^2} )</td>
<td>( te^{-\sigma t} )</td>
<td>Repeated root</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{\omega}{s^2 + \omega^2} )</td>
<td>( \sin \omega t )</td>
<td>Sine</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{s}{s^2 + \omega^2} )</td>
<td>( \cos \omega t )</td>
<td>Cosine</td>
</tr>
<tr>
<td>7</td>
<td>( \frac{\omega}{(s + \sigma)^2 + \omega^2} )</td>
<td>( e^{-\sigma t} \sin \omega t )</td>
<td>Damped sine</td>
</tr>
<tr>
<td>8</td>
<td>( \frac{s + \sigma}{(s + \sigma)^2 + \omega^2} )</td>
<td>( e^{-\sigma t} \cos \omega t )</td>
<td>Damped cosine</td>
</tr>
<tr>
<td>9</td>
<td>( sX(s) - x(0) )</td>
<td>( \frac{dx(t)}{dt} )</td>
<td>First derivative</td>
</tr>
<tr>
<td>10</td>
<td>( \frac{1}{s} X(s) )</td>
<td>( \int_0^t f(t) , dt )</td>
<td>Integral</td>
</tr>
<tr>
<td>11</td>
<td>( X(s + \sigma) )</td>
<td>( e^{-\sigma t} x(t) )</td>
<td>Frequency shift</td>
</tr>
<tr>
<td>12</td>
<td>( e^{-\sigma t} X(s) )</td>
<td>( x(t - t) )</td>
<td>Time shift</td>
</tr>
<tr>
<td>13</td>
<td>( \frac{1}{a} X \left( \frac{s}{a} \right) )</td>
<td>( x(at) )</td>
<td>Scale change</td>
</tr>
</tbody>
</table>
the need arises. Thus, our objective is to use the defining relation (5-12) only enough to become familiar with some common pairs.

**Some Common Transform Pairs**

Consider the time function \( x_1(t) = 1 \), a constant. Then, from Eq. (5-12).

\[
X_1(s) = \int_0^\infty (1) e^{-st} \, dt = \left. -\frac{1}{s} e^{-st} \right|_0^\infty = \left. -\frac{1}{s} (e^{-st} |_{t=\infty} - e^{st}) \right|_0^\infty = \frac{1}{s}
\]

provided that \( e^{-st} |_{t=\infty} \). We shall assume that \( s \) is restricted to ensure this situation. Then we have computed the first entry in Table 5-1.

Now try \( x_2(t) = e^{-at} \). Again from Eq. (5-12) we get

\[
X_2(s) = \int_0^\infty e^{-at} e^{-st} \, dt = \int_0^\infty e^{-(a+s)t} \, dt = \left. -\frac{1}{s+a} e^{-(a+s)t} \right|_0^\infty = \frac{1}{s+a}
\]

provided that \( s \) is restricted to make the first limit \( (t \to \infty) \) converge to zero. This is entry 2 in the transform table.

One more pair should suffice at this point. Suppose \( x_3(t) = \sin \omega t \). Then

\[
X_3(s) = \int_0^\infty (\sin \omega t) e^{-st} \, dt
\]

\[
= \int_0^\infty \frac{1}{2j} (e^{j\omega t} - e^{-j\omega t}) e^{-st} \, dt
\]

\[
= \frac{1}{2j} \left[ \int_0^\infty e^{-(s-j\omega)t} \, dt - \int_0^\infty e^{-(s+j\omega)t} \, dt \right]
\]

\[
= \frac{1}{2j} \left( \frac{1}{s-j\omega} - \frac{1}{s+j\omega} \right)
\]

\[
= \frac{1}{2j} \frac{2j\omega}{s^2 + \omega^2} = \frac{\omega}{s^2 + \omega^2}
\]

This pair is listed as entry 3 in the table. In most cases of interest, \( X(s) \) is a polynomial ratio; \( X(s)/D(s) \).

Clearly, a large table of transform pairs could be built up by patiently computing the transforms of a series of time functions. Before considering the problem of inverting transforms, we investigate some useful properties of the Laplace transform.

**Some Transform Properties**

In the study of constant-coefficient state equations, several operations occur repeatedly. The transform properties we introduce in this section are related to those operations.

**Superposition**

Let \( X(s) \) be the Laplace transform of \( x(t) \). Let us investigate the Laplace transform of \( Cx(t) \), where \( C \) is a constant
\[ \mathcal{L} \{ Cx(t) \} = \int_0^\infty Cx(t)e^{-st} \, dt = C \int_0^\infty x(t)e^{-st} \, dt \]

or
\[ \mathcal{L} \{ Cx(t) \} = CX(s) \tag{5-14} \]

That is, multiplying a time function by a constant multiplies its transform by the same constant. Now consider that we have two transform pairs \( x_1(t), X_1(s) \) and \( x_2(t), X_2(s) \). Suppose we wish to evaluate the transform of \( x_3(t) \), where
\[ x_3(t) = C_1x_1(t) + C_2x_2(t) \]
is an arbitrary linear combination. We have
\[ \mathcal{L} \{ x_3(t) \} = \mathcal{L}\{C_1x_1(t) + C_2x_2(t)\} \]
\[ = \int_0^\infty [C_1x_1(t) + C_2x_2(t)]e^{-st} \, dt \]
\[ = \int_0^\infty [C_1x_1(t)]e^{-st} \, dt + \int_0^\infty [C_2x_2(t)]e^{-st} \, dt \]
\[ = C_1 \int_0^\infty x_1(t)e^{-st} \, dt + C_2 \int_0^\infty x_2(t)e^{-st} \, dt \]
or
\[ \mathcal{L} \{ C_1x_1(t) + C_2x_2(t) \} = C_1X_1(s) + C_2X_2(s) \tag{5-15} \]

Thus, we say that the Laplace transform obeys the superposition property.

**Differentiation** Suppose that we know the pair \( x(t) \) and \( X(s) \), its transform. We wish to evaluate the transform of \( \dot{x}(t) \), the first derivative. Using integration by parts, we get
\[ \mathcal{L}\left\{ \frac{dx(t)}{dt} \right\} = \int_0^\infty \frac{dx(t)}{dt} e^{-st} \, dt = x(t)e^{-st} \bigg|_0^\infty - \int_0^\infty x(t)(-s)e^{-st} \, dt \]
or
\[ \mathcal{L} \left\{ \frac{dx}{dt} \right\} = sX(s) - x(0) \tag{5-16} \]

provided \( s \) is restricted to make the first limit \( (t \to \infty) \) vanish. Thus, the first derivative can be obtained directly from the original transform by multiplying by \( s \) and subtracting the initial condition.

For example, let \( x(t) = 2 \sin \omega t \). Then \( X(s) = 2\omega/(s^2 + \omega^2) \). To find the transform of \( \dot{x}(t) \) let
\[ \mathcal{L} \{ x(t) \} = sX(s) - x(0) = \frac{2\omega s}{s^2 + \omega^2} - 2 \sin (0) = \frac{2\omega s}{s^2 + \omega^2} \]

On the other hand, \( \dot{x}(t) \) is given directly by \( \dot{x}(t) = 2\omega \cos \omega t \). From Table 5-1 the transform is found to be
\[ \mathcal{L} \{ \dot{x} \} = \frac{2\omega s}{s^2 + \omega^2} \]

when we include the constant \( 2\omega \). So the formula yields the correct result.
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**Initial-value theorem (IVT)** It is often useful to check a solution by investigating the initial value associated with the transform solution. We state here the initial-value theorem (IVT) for that purpose.

\[ x(t \to 0) = \lim_{s \to \infty} sX(s) \]  \hspace{1cm} (5-17)

For example, let \( x(t) = 6 \cos 2t + 3e^{-4t} \). Then

\[ X(s) = \frac{6s}{s^2 + 4} + \frac{3}{s + 4} \]

and

\[ x(t \to 0) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \left( \frac{6s^2}{s^2 + 4} + \frac{3s}{s + 4} \right) = 6 + 3 = 9 \]

Clearly, this is what \( x(t = 0) \) yields directly.

**Final-value theorem (FVT)** A related theorem with considerable usefulness is the final-value theorem (FVT), derived in Appendix B,

\[ x(t \to \infty) = \lim_{s \to \infty} sX(s) \]  \hspace{1cm} (5-18)

This theorem must be applied with caution since it will yield the correct value only if there is a unique finite limit for \( x(t \to \infty) \). More precisely, all roots of the denominator of \( X(s) \) must have negative real parts except for a single root at the origin.

For example, let \( X(s) = 2/(s + 4)(s + 1) \). The roots are \(-4\) and \(-1\); the FVT applies

\[ x(t \to \infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{2s}{(s+4)(s+1)} = 0 \]

Since the time function is

\[ x(t) = -\frac{3}{2}e^{-4t} + \frac{3}{2}e^{-t} \]

and \( x(t \to \infty) = 0 \). Now suppose \( X(s) = 6/(s + 7)s \). The roots are \(-7\) and 0. The FVT applies. Thus

\[ x(t \to \infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{6s}{(s+7)s} = \frac{6}{7} \]

Since the time function is

\[ x(t) = -\frac{3}{2}e^{-7t} + \frac{3}{2} \]

we see that the FVT predicted correctly.

To see why it is important to check first the applicability of the FVT, consider
this case. Let \( x(t) = \cos \omega t \). Then

\[
x(s) = \frac{s}{s^2 + \omega^2}
\]

Now the FVT predicts

\[
x(t \to \infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} s^2 \frac{s^2}{s^2 + \omega^2} = 0
\]

But the roots are \( \pm j\omega \), which do not have negative real parts, and the time function \( x(t) \) does not have a unique "final value."

5-3 LAPLACE SOLUTION OF STATE EQUATIONS

In this section we show how to transform constant-coefficient state equations in a systematic manner. Then we discuss the insight into the system dynamics obtainable from a Laplace solution by means of the characteristic polynomial. Finally, we offer a formal solution method based on determinants.

**Transformed State Equations**

Begin with the state equations in general form for a second-order system

\[
\begin{align*}
\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + b_1u(t) \quad (5-19a) \\
\dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + b_2u(t) \quad (5-19b)
\end{align*}
\]

and

\[
x_1(0) = x_{10}, \quad x_2(0) = x_{20} \quad (5-19c)
\]

Now apply the Laplace transform to each side of Eq. (5-19a)

\[
\mathcal{L}\{x_1(t)\} = \mathcal{L}\{a_{11}x_1(t) + a_{12}x_2(t) + b_1u(t)\}
\]

Denoting the transforms of \( x_1(t) \), \( x_2(t) \), and \( u(t) \) by \( X_1(s) \), \( X_2(s) \), and \( U(s) \), respectively, we get

\[
sX_1(s) - x_1(0) = a_{11}X_1(s) + a_{12}X_2(s) + b_1U(s)
\]

If the unknowns are collected on the left and the known data are collected on the right, the equation becomes

\[
(s - a_{11})X_1(s) - a_{12}X_2(s) = x_1(0) + b_1U(s) \quad (5-20a)
\]

Apply the same procedure to Eq. (5-19b). The result is

\[
-a_{21}X_1(s) + (s - a_{22})X_2(s) = x_2(0) + b_2U(s) \quad (5-20b)
\]

We now have two linear algebraic equations (5-20a) and (5-20b) in two unknowns \( X_1(s) \) and \( X_2(s) \). The inputs \( U(s) \) and initial conditions \( x_1(0) \) and \( x_2(0) \) are incorporated into the equations.
This procedure can be applied to a set of state equations of any order, giving
\[(s - a_{11})X_1(s) - a_{12}X_2(s) - \cdots - a_{1n}X_n(s)\]
\[= X_1(0) + b_{11}U_1(s) + b_{12}U_2(s) + \cdots + b_{1m}U_m(s) \quad (5-21a)\]
\[\]
\[-a_{n1}X_1(s) - a_{n2}X_2(s) \cdots + (s - a_{nn})X_n(s)\]
\[= x_n(0) + b_{n1}U_1(s) + \cdots + b_{nm}U_m(s) \quad (5-21b)\]

There are \(n\) equations. The coefficients on the left are the \textit{negatives} of the \(A\) array. Added to each diagonal entry, i.e., the negative of the coefficient of \(X_i(s)\) in the \(i\)th equation is \(s\). On the right appear the initial conditions, one for each equation, plus the transformed inputs weighted by the \(B\) array. Chapter 15 examines these equations based on array notation, which is a very efficient way to study such systems.

Let us apply the Laplace transformation to a specific set of equations. Suppose we have derived
\[x_1 = -2x_1 + 4x_2 + 7u(t) \quad x_2 = -3x_1 + 2u(t) \quad (5-22)\]
and
\[x_1(0) = -2 \quad x_2(0) = 0\]

Applying the Laplace-transform procedure one step at a time, we get
\[sX_1(s) - x_1(0) = -2X_1(s) + 4X_2(s) + 7U(s)\]
\[sX_2(s) - x_2(0) = -3X_1(s) + 0X_2(s) + 2U(s)\]

A zero entry has been written in for element \(a_{22}\). Now collect unknowns on the left
\[(s + 2)X_1(s) - 4X_2(s) = x_1(0) + 7U(s) = -2 + 7U(s) \quad (5-23a)\]
\[3X_1(s) + (s - 0)X_2(s) = x_2(0) + 2U(s) = 0 + 2U(s) \quad (5-23b)\]

Close examination of the equations shows that \(-A\) components appear on the left, plus \(s\) on the diagonal. Initial conditions and inputs appear on the right with the inputs weighted by the components of \(B\).

**Solution for \(X_i(s)\)**

The next step is to solve for a particular unknown of interest. Say we wish to find \(X_2(s)\) from Eqs. (5-23). We can solve Eq. (5-23b) for \(X_1(s)\) in terms of \(X_2(s)\)
\[X_1(s) = \frac{1}{4}[-sX_2(s) + 2U(s)] \quad (5-24)\]

Substitute this result into Eq. (5-23a) to get
\[(s + 2)(\frac{1}{4})[-sX_2(s) + 2U(s)] - 4X_2(s) = -2 + 7U(s)\]
or, after the necessary algebra has been carried out

\[ X_2(s) = \frac{6 + (2s - 17)U(s)}{s^2 + 2s + 12} \]  \hspace{1cm} (5-25)

The result is in the form of a polynomial ratio, with \( U(s) \) yet to be specified. The numerator includes a contribution from the initial conditions (the 6 term) and the input.

Now let us solve for \( X_1(s) \), the other unknown. Referring to Eq. (5-24) and substituting, we get

\[ X_1(s) = \frac{1}{3} - \frac{6 + (2s - 17)U(s)}{s^2 + 2s + 12} + 2U(s) \]

or

\[ X_1(s) = \frac{-2s + (7s + 8)U(s)}{s^2 + 2s + 12} \]  \hspace{1cm} (5-26)

Behold, the same denominator appears! This denominator, called the characteristic polynomial of the system, contains the necessary information to gain much insight into the system dynamics. Notice that the characteristic polynomial does not depend upon either the initial conditions or the inputs. As we shall see next, it can be derived directly from the \( A \) array by a suitable determinant operation.

**Cramer's rule** Consider again the general second-order system in the transformed \( s \) domain

\[ (s - a_{11})X_1(s) - a_{12}X_2(s) = x_1(0) + b_1U(s) \]  \hspace{1cm} (5-20a)

\[ -a_{21}X_1(s) + (s - a_{22})X_2(s) = x_2(0) + b_2U(s) \]  \hspace{1cm} (5-20b)

Using Cramer's rule, we can find any unknown by forming a ratio of determinants. The denominator is the determinant of the array of coefficients of all the unknowns. Here we have a \( 2 \times 2 \) array. The numerator array is obtained from the denominator array by replacing the column corresponding to the unknown of interest by the (entire) right-hand side, which is a column itself. For example, if we want \( X_j(s) \), we put the right-hand-side column into the \( j \)th column in the coefficient array.

Thus, to solve for \( X_1(s) \) from Eqs. (5-20)

\[ X_1(s) = \begin{vmatrix} x_1(0) + b_1U(s) & -a_{12} \\ x_2(0) + b_2U(s) & s - a_{22} \\ s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix} \]  \hspace{1cm} (5-27)

The vertical lines indicate the determinant of the array contained within. The denominator determinant will yield a polynomial in \( s \). In fact, it will yield a second-order polynomial in this case but an \( n \)th-order polynomial for an \( n \)th-order system. This polynomial is the characteristic polynomial of the system. Notice from the Cramer's-rule solution that it will be the same for all system unknowns. Only the numerator changes, as different columns are replaced.
In particular, the second-order characteristic polynomial (CP) from Eq. (5-27) is
\[ \text{CP}(s) = (s - a_{11})(s - a_{22}) - a_{12}a_{21} \]
or
\[ \text{CP}(s) = s^2 + (-a_{11} - a_{22})s + (a_{11}a_{22} - a_{12}a_{21}) \] (5-28)
This is an extremely useful result, as we shall see.

If Eq. (5-28) is applied directly to the state equations in (5-22), we get
\[ \text{CP}(s) = s^2 + (2 + 0)s + (-2)(0) - 4 \cdot (-3) \]
or
\[ \text{CP}(s) = s^2 + 2s + 12 \]
This agrees with Eq. (5-25).

The general \( n \)-th order CP(\( s \)) is given by
\[ \text{CP}(s) = \begin{vmatrix} s - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s - a_{nn} \end{vmatrix} \] (5-29)
which yields an \( n \)-th order polynomial
\[ \text{CP}(s) = s^n + (-a_{11} - a_{22} \cdots - a_{nn})s^{n-1} + \cdots + (-A) \] (5-30)
The first coefficient is unity; the second is called the trace of \(-A\); the constant term is the determinant of \(-A\).

While the Cramer's rule approach is useful for low-order systems, it is also valuable for higher-order problems because typical engineering problems of higher order have many zeros in the \( A \) array. Hence the determinant of the modified \( A \) array in Eq. (5-29), as well as the derived numerator array, can be expanded about a sparse row or column and reduced by that method. Appendix B explains the procedure.

We have shown two ways to obtain the Laplace solution for any unknown \( X_i(s) \). The specific denominator obtained from either Cramer's rule or ad hoc reduction before the \( U(s) \) functions are substituted in is the characteristic polynomial, which plays a crucial role in subsequent analysis. It is time to consider the time solution and other information that can be derived from such a solution.

### 5-4 TIME RESPONSE

The key steps in obtaining a time response from a Laplace-transform solution are:

1. Find the roots of the denominator polynomial.
2. Separate the Laplace solution by partial fractions.
3. Invert each fraction by table look-up.
Each of these steps is discussed and illustrated in a series of examples. The final part of this section shows how valuable information can be developed from the final-value theorem and how the initial-value theorem can be used to check the Laplace solution.

The Roots of the Denominator $D(s)$ Consider $X(s)$ in the form

$$X(s) = \frac{N(s)}{D(s)} = \frac{n_1s^m + n_2s^{m-1} + \cdots + n_ms^1 + n_{m+1}}{s^n + d_1s^{n-1} + d_2s^{n-2} + \cdots + d_n} \quad (5-31)$$

where $m$ is the degree of the numerator, $n$ is the degree of the denominator, and $n > m$. We can rewrite Eq. (5-31) in factored form as

$$X(s) = \frac{N(s)}{(s + p_1)(s + p_2) \cdots (s + p_n)} \quad (5-32)$$

where $-p_1, -p_2, \ldots, -p_n$ are called the poles of $X(s)$. The poles are found by solving the equation $D(s) = 0$. They are values of $s$ which make the polynomial vanish and expressions such as Eq. (5-32) blow up. Note that there are exactly $n$ roots of $D(s)$. In general, factoring an $n$th-order polynomial, i.e., finding its roots, is difficult manually. Computers are very helpful for large problems, and the structure of a problem frequently simplifies the factorization.

The part of $D(s)$ that is obtained from the system itself before the specific inputs are entered is, of course, the characteristic polynomial $\text{CP}(s)$. The roots of this polynomial are called the eigenvalues of the system. They are values of $s$ for which the characteristic equation $\text{CP}(s) = 0$ is satisfied. These roots offer great insight into the system's dynamic-response tendencies or the natural or unforced dynamics of the system. In Eq. (5-32) some poles are eigenvalues and some come from the specific forcing functions $U(s)$.

In defining factors we want to keep real coefficients. Hence, if a quadratic factor has complex-conjugate roots, we do not reduce it to two first-order factors but preserve it in $\sigma, \omega$ form. The motivation comes from the entries in Table 5-1, specifically the denominator forms. Notice that they include first-order forms $s + \sigma$ and second-order forms $(s + \sigma)^2 + \omega^2$ but no higher forms.

**Example 5-2**

$$D(s) = s^2 + 6s + 8$$

By inspection, $D(s) = (s + 4)(s + 2)$, and the roots are $-4$ and $-2$.

**Example 5-3**

$$D(s) = s^2 + 6s + 10$$

Solve for the roots

$$s_{1,2} = \frac{-6 \pm \sqrt{6^2 - 4(10)}}{2}$$
or $s_{1,2} = -3 \pm j1$, a complex-conjugate pair. In this case, we shall not factor the quadratic form further.

**Example 5-4**

$$D(s) = s^3 - 1s^2 - 2s$$

Since $D(s)$ is a cubic, it has three roots. Write $D(s)$ as $s(s^2 - 1s - 2)$, from which it can be written by inspection as $(s+0)(s+1)(s-2)$. The roots are 0, -1, and +2.

**Example 5-5**

$$D(s) = (s^3 + 4s^2 + 9s + 10)(s^2 + 4)$$

This polynomial is already partially factored. By guessing at a (real) root for the cubic, we find that $D(s) = 0$ for $s = -2$. (Check it.) Hence, we can remove the factor $s + 2$ from the cubic by division

$$\begin{align*}
\frac{s^2 + 2s + 5}{s + 2} & \frac{s^3 + 4s^2 + 9s + 10}{s^2 + 2s} \\
2s^2 + 9s & 5s + 10 \\
2s^2 + 4s & 5s + 10
\end{align*}$$

Now we have $D(s) = (s + 2)(s^2 + 2s + 5)(s^2 + 4)$. Study the middle factor; its roots are $-1 \pm j2$. The roots of the last factor are $\pm j2$. Now we have the complete story on $D(s)$; all five of its roots are known. Do not reduce $D(s)$ further at this point.

**Partial-fraction expansion**

Partial-fraction expansion is a key step in preparing a Laplace-transform function $X(s)$ for inversion from $s$ to $t$. Separate $X(s)$ into a set of fractions, i.e., polynomial ratios, each of whose denominators is either a first- or a second-order factor, depending upon the roots. Construct a numerator that is a polynomial one degree less than the denominator, with constants to be determined. Now comes the work. Determine the constants by cross-multiplying the separate factors to reconstruct the proper numerator. By equating the coefficients of like powers of $s$ in the numerator a set of equations from which to find the constants can be derived.

**Example 5-6**

$$X(s) = \frac{4}{s^2 + 6s + 8} = \frac{4}{(s + 4)(s + 2)}$$

Then

$$X(s) = \frac{C_1}{s + 4} + \frac{C_2}{s + 2}$$
or
\[ X(s) = \frac{C_1(s + 2) + C_2(s + 4)}{(s + 4)(s + 2)} \]

but \( C_1(s + 2) + C_2(s + 4) = 4 \), which leads to
\[ (C_1 + C_2)s + (2C_1 + 4C_2) = 4 \]
or, by equating powers of \( s \),
\[
\begin{align*}
C_1 + C_2 &= 0 \\
2C_1 + 4C_2 &= 4
\end{align*}
\]

Since \( C_1 = -2 \) and \( C_2 = 2 \) from above,
\[ X(s) = \frac{-2}{s + 4} + \frac{2}{s + 2} \]

**Example 5-7**

\[ X(s) = \frac{4s^2 + 5}{s(s^2 + 4)} \]

Then
\[ X(s) = \frac{C_1}{s} + \frac{C_2s + C_3}{s^2 + 4} \]
or
\[ X(s) = \frac{C_1(s^2 + 4) + s(C_2s + C_3)}{s(s^2 + 4)} \]

which leads to the polynomial equation
\[ (C_1 + C_2)s^2 + (C_3)s + (4C_1) = 4s^2 + 5 \]
Then
\[
\begin{align*}
C_1 + C_2 &= 4 \\
C_3 &= 0 \\
4C_1 &= 5
\end{align*}
\]

Solving these equations and substituting yields
\[ X(s) = \frac{1.25}{s} + \frac{2.75s}{s^2 + 4} \]

**Example 5-8** An important special case arises when a root is repeated in \( D(s) \). We illustrate the treatment for a repeated root here. Let
\[ X(s) = \frac{s + 3}{(s + 4)^2} \]

Then
\[ X(s) = \frac{C_1}{(s + 4)^2} + \frac{C_2}{s + 4} \]

\[
\begin{align*}
&= \frac{1}{(s + 4)^2} \\
&\quad \cdot \left( \frac{s + 4}{(s + 4)^2} \right) \\
&\quad \cdot \left( \frac{(s + 4)^2}{(s + 4)^2} \right) \\
&\quad \cdot \left( \frac{(s + 4)^2}{(s + 4)^2} \right)
\end{align*}
\]
Note the special form of the first numerator. Recombine to find
\[ C_1 + C_2(s + 4) = s + 3 \]
or
\[ C_2 = 1 \\
C_1 + 4C_2 = 3 \]
Thus
\[ X(s) = \frac{-1}{(s + 4)^2} + \frac{1}{s + 4} \]

More general information on treatment of repeated quadratic factors and factors repeated more than twice is available. The case shown in the example, a real root of multiplicity 2, is the most common practical case by far.

We have illustrated only the most basic techniques, but there are some useful tricks. For example, we can find \( C_1 \) in the first example with little effort as follows:

1. Multiply the original \( X(s) \) and the partial-fraction expansion by \( C_1 \)'s denominator \( s + 4 \).
2. Cancel \( s + 4 \) out of \( X(s) \), leaving
\[ \frac{4}{s + 2} = C_1 + \frac{C_2(s + 4)}{s + 2} \]
3. Now make \( s + 4 = 0 \) by setting \( s = -4 \). This makes \( C_2 \)'s factor vanish and
\[ C_1 = \frac{4}{-4 + 2} = -2 \]

A generalization of this idea can often be used to find constants one at a time. You might try the idea on the other examples.

**Inversion from \( X(s) \) to \( x(t) \)**

Our approach to inversion from \( X(s) \) to \( x(t) \) is based on knowing the table entries for a very common and important set of transform pairs. The last step in obtaining \( x(t) \) from \( X(s) \) is to put the separate fractions of \( X(s) \) into table format.

**Example 5-9**

\[ X(s) = \frac{-2}{s + 4} + \frac{2}{s - 2} \]

By direct table look-up we find
\[ x(t) = -2e^{-4t} + 2e^{2t} \]

**Example 5-10**

\[ X(s) = \frac{1.25}{s} + \frac{2.75 s}{s^2 + 4} \]
By direct table look-up we get
\[ x(t) = 1.25 + 2.75 \cos 2t \]

**Example 5-11**

\[ X(s) = \frac{s + 5}{s^2 + 6s + 10} \]

The roots of \( D(s) \) are \(-3 \pm j1\) (from a previous example), so we write \( X(s) \) in the form

\[ X(s) = \frac{s + 5}{(s + 3)^2 + 1^2} \]

Now we reformat the numerator as

\[ X(s) = \frac{(s + 3) + 2(1)}{(s + 3)^2 + 1^2} \]

or

\[ X(s) = \frac{s + 3}{(s + 3)^2 + 1^2} + \frac{2(1)}{(s + 3)^2 + 1^2} \]

The inverse is
\[ x(t) = e^{-3t} \cos 1t + 2e^{-3t} \sin 1t \]

**Example 5-12**

\[ X(s) = \frac{-3}{s + 2} + \frac{2(s + 6)}{s^2 + 2s + 5} + \frac{-(s - 8)}{s^2 + 4} \]

The second and third terms need to be reformatted as

\[ X(s) = \frac{-3}{s + 2} + \frac{2[(s + 1) + 2.5(2)]}{(s + 1)^2 + 2^2} + \frac{-1s + 4(2)}{s^2 + 2^2} \]

Then, inverting on a term-by-term basis, we get
\[ x(t) = -3e^{-2t} + 2e^{-1t} \cos 2t + 5e^{-1t} \sin 2t - 1 \cos 2t + 4 \sin 2t \]

**Remark on a Useful Equivalence**

The inversion procedure we have been using often generates sine and cosine terms of the same frequency. It is convenient to combine such terms into a single term as follows. Let

\[ x(t) = C_1 \sin \omega t + C_2 \cos \omega t \]

where \( C_1, C_2, \) and \( \omega \) are given. But
\[ x(t) = M \sin (\omega t + \phi) = M \sin \omega t \cos \phi + M \cos \omega t \sin \phi = (M \cos \phi) \sin \omega t + (M \sin \phi) \cos \omega t \]
Then, since $C_1$ and $C_2$ are known, we have

$$M = \sqrt{C_1^2 + C_2^2} \quad \text{and} \quad \tan \phi = \frac{C_2}{C_1}$$

The quantity $M$ is the magnitude of the signal $x(t)$, and $\phi$ is the phase angle.

**Example 5-13**

$$x(t) = -1 \cos 2t + 4 \sin 2t$$

Then $C_1 = 4$ and $C_2 = -1$, so that

$$M = \sqrt{4^2 + (-1)^2} = \sqrt{17} = 4.123$$

and $\tan \phi = -\frac{1}{4}$ so $\phi = -0.245$ rad. Thus

$$x(t) = 4.123 \sin (2t - 0.245)$$

**Example 5-14**

$$x(t) = 5e^{-1t} \sin 2t + 2e^{-1t} \cos 2t$$

Then $C_1 = 5$, $C_2 = 2$, and $e^{-1t}$ can be factored out. We find

$$M = \sqrt{5^2 + 2^2} = \sqrt{29} = 5.385$$

and $\tan \phi = \frac{2}{5}$, so $\phi = 0.381$ rad. Then

$$x(t) = 5.385e^{-1t} \sin (2t + 0.381)$$

**Checking the Solution**

One way to check the time solution for a problem is to apply the initial-value theorem to the Laplace transform. We see whether $x(t)$ for $t = 0$ and $X(s)$ subject to the IVT agree.

**Example 5-15**

$$X(s) = \frac{-2s^4 + 14s^3 + 28s^2 + 102s + 116}{(s + 2) (s^2 + 25 + 5) (s^2 + 4)}$$

By the IVT,

$$x(t = 0) = \lim_{s \to \infty} sX(s) = -2$$

The solution to this $X(s)$ is given in Example 5-12 as

$$x(t) = -3e^{-2t} + 2e^{-1t} \cos 2t + 5e^{-1t} \sin 2t - 1 \cos 2t + 4 \sin 2t$$

Clearly,

$$x(0) = -3 + 2 - 1 = -2$$
In essence, we have checked the leading numerator coefficient of \( X(s) \) and our time solution. In a problem derived from physical conditions, the initial condition will be known and both \( X(s) \) and \( x(t) \) can be checked independently.

A second important check that can sometimes be made is for a constant steady-state value. This check should be applied only when the roots of the denominator have negative real parts or there is at most one root at zero. It involves the use of the final-value theorem.

**Example 5-16**

\[
X(s) = \frac{4s + 1}{s(s + 2)(s + 4)}
\]

Predict \( x(t \to \infty) \) by \( \lim_{s \to 0} sX(s) \):

\[
x(t \to \infty) = \frac{1}{2(4)} = 0.125
\]

The time solution is

\[
x(t) = 0.125 + 1.75e^{-2t} - 1.875e^{-4t}
\]

from which we see that \( x(t \to \infty) \to 0.125 \). (Also notice that the initial condition checks at zero.)

**Nature of the Response**

The partial-fraction expansion shows what types of time functions are involved in a response. For example, Table 5-1 shows that poles (or eigenvalues) which are real and negative correspond to decaying exponentials, while only complex poles (or eigenvalues) lead to damped sinusoidal solution terms. A system which has purely sinusoidal response terms must have a factor \( s^2 + \omega^2 \) in its \( D(s) \) polynomial, while one which oscillates in a decaying manner will have a factor \( (s + \sigma)^2 + \omega^2 \). These correspond to poles at \( s = \pm j\omega \) and \( s = -\sigma \pm j\omega \), respectively. Remember that \( D(s) \) comes partly from the characteristic polynomial and partly from the forcing functions. Therefore, a system with complex eigenvalues will have damped oscillatory parts in its response even if the forcing does not contribute any oscillatory response through its own transform. Thus we say that the eigenvalues determine the natural response dynamics of the system independent of the particular forcing.

**Eigenvalue Interpretation**

The eigenvalues are the roots of the characteristic polynomial \( CP(s) \), as previously described. The actual response then depends upon the initial conditions. In fact, the initial conditions determine only the numerator polynomial \( N(s) \). Furthermore, all system variables have the same denominator, except in certain special cases when one or more common factors are canceled from both \( N(s) \) and \( CP(s) \). In this section
we exploit the information and insight contained in the eigenvalues of first- and second-order systems. Since all higher-order system responses are composed of sums of first- and second-order response, our considerations are general enough to be very useful in practice.

**First-order systems**

Any first-order system without forcing can be put in the form

\[ x(t) = -ax(t) \quad x(t_0) = x_0 \]  \hspace{1cm} (5-33)

The characteristic polynomial is \( s + a \), the eigenvalue is \(-a\), and the time response has the factor \( e^{-at} \) in it. The initial condition provides the particular weighting of the time factor. The solution is

\[ x(t) = x_0 e^{-at} \]  \hspace{1cm} (5-34)

Provided the initial condition is not zero and \( a \) is finite, we can nondimensionalize this result as

\[ \frac{x}{x_0} = e^{-t/\tau} \]  \hspace{1cm} (5-35)

where \( x/x_0 \) and \( t/\tau \) are dimensionless and \( \tau \) is defined as \( 1/|a| \), the time constant. Only one curve is needed to represent the nondimensionalized first-order response of Eq. (5-35), provided that \( a \) is positive. That curve has initial value 1 for \( t/\tau = 0 \) and decays exponentially to 0 as \( t/\tau \to \infty \).

The exponential response curve in dimensionless form is shown in Fig. 5-2. Several key points from the graph are labeled, and their values are displayed, in Table 5-2. Notice that the curve starts at 1.0 and drops off by a factor of \( 1/e \) for each unit of \( t/\tau \). Thus \( \tau \) is the critical constant in assessing first-order transient response. Recall that \( \tau \) is given as \( 1/a \) (\( a \) positive), minus where \(-a\) is the eigenvalue. Hence, any first-order response can easily be estimated from its equation by inspecting the coefficient \( a \).

**Example 5-17** Let \( x = -4x \). When does \( x \) fall to within 1 percent of its final value? From the dimensionless response curve of Fig. 5-2 we see that \( x/x_0 \) falls to within 1
Table 5-2 Values for Fig. 5-2

<table>
<thead>
<tr>
<th>Point</th>
<th>Time $t/\tau$</th>
<th>$x/x_0$ Exact</th>
<th>Approximate</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0</td>
<td>$e^0$</td>
<td>1.000 ...</td>
<td>100</td>
</tr>
<tr>
<td>$b$</td>
<td>1</td>
<td>$e^{-1}$</td>
<td>0.368</td>
<td>&lt;37</td>
</tr>
<tr>
<td>$c$</td>
<td>2</td>
<td>$e^{-2}$</td>
<td>0.135</td>
<td>&lt;14</td>
</tr>
<tr>
<td>$d$</td>
<td>3</td>
<td>$e^{-3}$</td>
<td>0.050</td>
<td>5</td>
</tr>
<tr>
<td>$e$</td>
<td>4</td>
<td>$e^{-4}$</td>
<td>0.018</td>
<td>&lt;2</td>
</tr>
<tr>
<td>$f$</td>
<td>5</td>
<td>$e^{-5}$</td>
<td>0.007</td>
<td>&lt;1</td>
</tr>
</tbody>
</table>

percent (of zero) by 5 time constants; i.e., when $t/\tau$ is 5, the ratio is down to 0.007. So the quick and easy answer is $5\tau$. The given equation tells us that since $a = 4$, $\tau = 0.25$. Therefore, by $t$ equal to 1.25 the response is within 1 percent of final value.

Another property of the exponential responses is also depicted in Fig. 5-2. Observe the dashed lines. Curve 1 goes from unity to zero in a 1-time-constant interval. The response curve is tangent to this line at $t/\tau = 0$. The same property holds for curve 2 and in fact is true for every point on the curve. Can you prove that? At least try it for integer values of $t/\tau$.

By knowing that an exponential curve approaches its final value by a factor of $1/e$ for every interval $\tau$ and by using the tangent property at key points (for example, $\tau$, $2\tau$, $3\tau$, ...) it should be easy for you to sketch an exponential response of the form $x_0e^{-t/\tau}$ (or $x_0e^{-at}$) given the data $x_0$ and $a$.

Second-order systems

A second-order system has a characteristic polynomial of the form

$$\text{CP}(s) = s^2 + Bs + C$$ (5-36)

which yields two roots, i.e., eigenvalues. Two distinct cases can arise. Consider the general solution expression

$$s_{1,2} = \frac{-B \pm \sqrt{B^2 - 4C}}{2}$$ (5-37)

where we assume that $B$ and $C$ are both positive. If $B^2 - 4C > 0$, there are two real roots. The solution can be written as

$$x(t) = C_1e^{-\sigma_1t} + C_2e^{-\sigma_2t}$$ (5-38)

where $\sigma_1$ and $\sigma_2$ are positive real constants. The solution is the sum of two exponentials. If $B^2 - 4C < 0$, there are complex-conjugate roots. One way to write the roots is

$$s_{1,2} = -\sigma \pm j\omega$$ (5-39)
where \( \sigma \) and \( \omega \) are positive real constants. This leads to a time response of the form

\[
x(t) = C_1 e^{-\sigma t} \sin (\omega t + \phi)
\]

(5-40)

This is called an exponentially damped sinusoidal response; \( \sigma \) defines the decay envelope, while \( \omega \) defines the oscillation frequency. \( C_1 \) determines the amplitude, while \( \phi \) determines the phase; these are found from the initial conditions and do not concern us here. The fundamental dynamic nature of the response is governed by \( \sigma \) and \( \omega \), which are derived in turn from \( B \) and \( C \) in \( CP(s) \).

A second way to write the roots for complex conjugates is derived from \( CP(s) \) as follows:

\[
CP(s) = s^2 + 2\xi \omega_n s + \omega_n^2
\]

(5-41)

Then

\[
s_{1,2} = -\xi \omega_n \pm j \omega_n \sqrt{1 - \xi^2}
\]

(5-42)

where \( \xi \) is the damping ratio (a positive constant) and \( \omega_n \) is the natural frequency (a positive constant).

In Fig. 5-3 we plot a series of response curves for a given second-order system. These curves are the solution to the equation \( \ddot{x} + 2\xi \omega_n \dot{x} + \omega_n^2 x = 0 \) with the initial conditions \( x(0) = x_0 \) and \( \dot{x}(0) = 0 \). Start with the curve labeled by \( \xi = 0 \). It corresponds to an undamped oscillation of period \( 2\pi/\omega_n \). This oscillation persists indefinitely with its peak amplitude undiminished. As the damping ratio is increased, each curve has a more rapid decay of amplitude and a longer period. The amplitude-decay envelope is in fact governed by \( e^{-\omega_n t} \), while the frequency is \( \omega_n \sqrt{1 - \xi^2} \). Thus the equivalence between \( \sigma \) and \( \omega \), on the one hand, and \( \xi \) and \( \omega_n \), on the other, is

\[
\sigma = \xi \omega_n \quad \text{and} \quad \omega = \omega_n \sqrt{1 - \xi^2}
\]

(5-43)

In engineering we normally speak of a system with \( \xi = 0 \) as undamped, with \( \xi < 1 \) as underdamped, with \( \xi = 1 \) as critically damped (two real repeated roots), and with \( \xi > 1 \) as overdamped.

In summary, we note that two parameters control the natural time response of

![Figure 5-3 Characteristic response curves for the damped sinusoidal case.](image)
a second-order system; we can convert the CP(s) coefficients $B$ and $C$ into either $\sigma$, $\omega$ or $\zeta,\omega_n$ form if the system has complex-conjugate roots or into $\sigma_1,\sigma_2$ form if the system has two real roots. All this information is nicely represented by a correlation of typical time responses with the s plane, i.e., the complex plane showing eigenvalue locations (see Fig. 5-4). We remind you that all linear-system models with real constant coefficients lead to eigenvalues that are either real or paired as complex conjugates. Consequently, all natural time responses are sums of exponential curves and (damped) sinusoidal curves. The eigenvalues tell you what the time factors of the natural responses are, independent of initial conditions and input functions. We shall learn later how to exploit such information in analysis and design.
5-5 EXAMPLES

Now that we have developed tools sufficient to the task of predicting system response, we apply them to several examples.

Example 5-18 Refer to Fig. 5-1. Suppose the mass is 1 kg, the spring stiffness is 100 N/m, and the damper value is 20 N/m-s. The acceleration of gravity is 9.8 m/s². Find the position of the mass if the initial conditions on \( x \) and \( p \) are both zero. Does the system oscillate?

The state equations for the system are given by Eq. (5-1); with the parameter values given above for \( m \), \( k \), and \( b \) they become

\[
p(t) = -20p(t) - 100x(t) + 9.8 \quad (5-44a)
\]

and

\[
x(t) = 1p(t) \quad (5-44b)
\]

When Eq. (5-44) are Laplace-transformed and unknowns collected on the left-hand side, we get

\[
(s + 20)P(s) + 100X(s) = p(0) + \frac{9.8}{s} \quad (5-45a)
\]

\[-1P(s) + sX(s) = x(0) \quad (5-45b)
\]

Note that the quantity \( \frac{9.8}{s} \) is the transform of the constant term (9.8). Letting the initial conditions be zero and solving for the transform of \( x \) yields

\[
X(s) = \frac{s + 20}{s + 10} \quad (5-46)
\]

\[
\begin{vmatrix}
 s + 20 & \frac{9.8}{s} \\
 -1 & 0 \\
 s + 20 & 100 \\
 -1 & s
\end{vmatrix}
= \frac{9.8}{s^2 + 20s + 100}
\]

The denominator in Eq. (5-46) is the characteristic polynomial. To determine whether the system will oscillate, we calculate the eigenvalues by solving the characteristic equation \( CP(s) = 0 \).

\[
s_{1,2} = \frac{-20 \pm \sqrt{(20)^2 - 4(100)}}{2} = -10 \pm 0 \quad (5-47)
\]

This means that there are repeated roots. The system does not oscillate because the roots are real. Now write \( X(s) \) in the form

\[
X(s) = \frac{9.8}{s(s + 10)^2} \quad (5-48)
\]

expand by partial fractions

\[
X(s) = \frac{C_1}{s} + \frac{C_2}{(s + 10)^2} + \frac{C_3}{s + 10}
\]

and solve for \( C_1 \), \( C_2 \), and \( C_3 \). The equations are

\[
C_1 + C_3 = 0
\]
\[ 20C_1 + C_2 + 10C_3 = 0 \]
\[ 100C_1 = 9.8 \]

Thus
\[ X(s) = \frac{0.098}{s} + \frac{-0.98}{(s + 10)^2} + \frac{-0.098}{s + 10} \quad (5-49) \]

Inverting term by term yields
\[ x(t) = 0.098 - 0.98te^{-10t} - 0.098e^{-10t} \quad (5-50) \]

Check the solution by the initial condition \( x(t = 0) = 0.098 - 0.098 = 0 \). Apply the FVT to Eq. (5-48). This yields
\[ x(t \to \infty) = \lim_{s \to 0} sx(s) = \frac{9.8}{(10)^2} = 0.098 \]

This result is the same as that derived from Eq. (5-50) with \( t \to \infty \). (Observe that the limit of \( te^{-10t} \) as \( t \to \infty \) is zero.)

We conclude that the system does not oscillate. Suppose we wished to induce oscillations at, say, a frequency of 5 rad/s. How should the spring stiffness \( k \) be adjusted to ensure this?

Return to Eq. (5-44a) and (5-46). The value of 100 represents \( k \). This could be shown more directly by retaining \( k \) in the state equations and working it through the transform procedure to the point of Eq. (5-47). If we did so, we should find the roots as
\[ s_{1,2} = \frac{-20 \pm \sqrt{(20)^2 - 4k}}{2} \quad (5-51) \]

The root form we seek is \( \sigma \pm j\omega \) with \( \omega = 5 \). To achieve this, set
\[ \sqrt{(20)^2 - 4k} = j10 \quad \text{or} \quad k = 125 \text{ N m} \]

The resulting roots are \( -10 \pm j5 \); the system will oscillate at a frequency of 5 rad/s, as desired, but it will also be damped. You may wish to verify that the new \( x(t) \) solution is
\[ x(t) = 0.0784 - 0.0784e^{-10t} \cos 5t - 0.1568e^{-10t} \sin 5t \]

or
\[ x(t) = 0.0784 + 0.1753e^{-10t} \sin (5t - 2.678) \]

where the phase angle is in radians.

**Example 5-18** Consider the electric-circuit diagram and bond graph in Fig. 5-5. We wish to find \( v_{out}(t) \) as it depends upon \( v_{in}(t) \). The state equation describing the circuit is
\[ q_4 = -0.15q_4 + 0.1v_{in} \quad (5-52) \]

and the output equation is
\[ v_{out} = 0.59q_4 \quad (5-53) \]
If we use Eq. (5-53) in Eq. (5-52), we arrive directly at the input-output equation

\[ v_{\text{out}} = -0.15v_{\text{out}} + 0.05v_{\text{in}} \]  \hspace{1cm} (5-54)

We wish to find the output in response to an input of the form

\[ v_{\text{in}}(t) = 110 \sin 0.1t \]  \hspace{1cm} (5-55)

In particular, we are interested in the part of the response that persists after the influence of the initial condition effect is small. Let

\[ V_{\text{in}}(s) = \mathcal{L}\{v_{\text{in}}(t)\} \quad \text{and} \quad V_{\text{out}}(s) = \mathcal{L}\{v_{\text{out}}(t)\} \]

Then Eq. (5-54) can be transformed to yield

\[ (s + 0.15)V_{\text{out}}(s) = v_{\text{out}}(0) + 0.05V_{\text{in}}(s) \]  \hspace{1cm} (5-56)

where \( v_{\text{out}}(0) \) is the initial condition, derivable from Eq. (5-53) in terms of \( q_4(0) \).

The general Laplace solution for \( V_{\text{out}}(s) \) is

\[ V_{\text{out}}(s) = \frac{v_{\text{out}}(0)}{s + 0.15} + \frac{0.05V_{\text{in}}(s)}{s + 0.15} \]  \hspace{1cm} (5-57)

The first term on the right can be inverted by inspection to yield \( v_{\text{out}}(0)e^{-0.15t} \). We shall concentrate on the second term for now

\[ V_2(s) = \frac{0.05}{s + 0.15} \frac{110(0.1)}{s^2 + (0.1)^2} \]  \hspace{1cm} (5-58)

or

\[ V_2(s) = \frac{C_1}{s + 0.15} + \frac{C_2s + C_3}{s^2 + (0.1)^2} \]  \hspace{1cm} (5-59)

where \( V_2(s) \) is the second part of the solution and we have used the transform of Eq. (5-55). Solving for the constants by cross-multiplying, we get

\[ V_2(s) = \frac{16.923}{s + 0.15} + \frac{-16.923s + 2.538}{s^2 + (0.1)^2} \]

or

\[ V_2(s) = \frac{16.923}{s + 0.15} + \frac{-16.923s}{s^2 + (0.1)^2} + \frac{25.38(0.1)}{s^2 + (0.1)^2} \]
This inverts to yield
\[ v_2(t) = 16.923e^{-0.15t} - 16.923(\cos 0.1)t + 25.38(\sin 0.1)t \]  
(5-60)

As a partial check, we note that \( v_2(0) = 0 \), which corresponds to the result obtained by applying the IVT to Eq. (5-58).

Let us combine the sine and cosine terms of Eq. (5-60), obtaining
\[ v_2(t) = 16.923e^{-0.15t} + 30.505 \sin (0.1t - 0.588) \]

The complete solution for \( v_{\text{out}}(t) \) is
\[ v_{\text{out}}(t) = (v_{\text{out}}(0) + 16.923e^{-0.15t} + 30.505 \sin (0.1t - 0.588) \]  
(5-61)

After a certain period of time, the first term, involving the initial condition and the transient part of the solution, will become small compared with the persistent sinusoid. The magnitude of the sinusoidal response is 30.505, compared with an input magnitude of 110. Hence the circuit attenuates the sinusoidal input signal and delays it in phase relative to the input signal. If we studied input voltages at various forcing frequencies, we would conclude that the circuit is a low-pass filter which attenuates high frequencies and passes low ones with little attenuation and phase shift.

5-6 SUMMARY

In this chapter we have defined and applied the Laplace transform to the solution of state equations with constant coefficients. The method converts a set of differential equations in the time domain into a set of algebraic equations in the (complex) frequency domain.

Useful information was extracted directly from the Laplace transform solution. In particular the roots of the characteristic polynomial, called eigenvalues, determine what types of time functions will be found in the response no matter what input is involved.

The time solution was obtained by using partial-fraction expansion and comparison of terms to a standard table of Laplace transforms.

PROBLEMS

5-1 Show that the Laplace transform of a unit impulse at \( t = 0 \) is 1 by first finding the Laplace transform of \( x(t) \) (Fig. P5-1). Then set \( h\Delta = 1 \), and let \( h \rightarrow \infty \) as \( \Delta \rightarrow 0 \).

![Figure P5-1](image_url)
5-2 Derive entry 8 in Table 5-1 from entry 6 by using the frequency-shift relation.

5-3 (a) Derive entry 5 in Table 5-1 from entry 6 by using entry 10.

(b) Derive entry 5 in Table 5-1 from entry 6 by using entry 9.

5-4 Prove entry 13 in Table 5-1 directly from the defining Laplace-transform integral.

5-5 Find the Laplace transform of the following functions:

(a) \( x(t) = 3e^{-3t} \)  
(b) \( x(t) = 4e^{-t} + 7e^{-3t} \)  
(c) \( x(t) = 3e^{-5t} \cos 2t - 1e^{-3t} \sin 2t \)

(d) \( x(t) = 8 \)  \( t \geq 0 \)  
(e) \( x(t) = \begin{cases} 0 & t < 3 \\ 4 & t \geq 3 \end{cases} \)

(f) \( x(t) = 5 \sin (2t + \frac{\pi}{4}) \)

(g) \( x(t) = ct^2 \)  
(h) \( x(t) = -2te^{-3t} \)

5-6 Find \( x(t) \) for the following Laplace transforms:

(a) \( X(s) = \frac{0.01}{2s + 1} \)

(b) \( X(s) = \frac{3}{s^2 + 3s + 2} \)

(c) \( X(s) = \frac{2s + 5}{s + 3} \)  

Hint: Divide to obtain a proper fraction. Then see the result of Prob. 5-1.

(d) \( X(s) = \frac{7s + 1}{s^2 + 13s + 36} \)

(e) \( X(s) = \frac{2s^3 + 5}{s^3 + 10s^2 + 33s + 36} \)  

Hint: Investigate a root near 4.

(f) \( X(s) = \frac{4e^{-s}}{s^2} \)

(g) \( X(s) = \frac{6s + 1}{s^2 + 9} \)  

(h) \( X(s) = \frac{6s + 1}{s^2 - 9} \)

5-7 Solve the following state equations. Check your answers.

(a) Find \( x_1(t) \) and \( x_2(t) \) for

\[
\begin{align*}
x_1 &= -2x_1 + 4x_2 \\
x_2 &= -1x_1 - 1x_2
\end{align*}
\]

with \( x_1(0) = -2 \)

\( x_2(0) = 0 \)

(b) Find \( k \) to give an oscillation frequency of 4

\[
\begin{align*}
x_1 &= -kx_2 \\
x_2 &= 2x_1
\end{align*}
\]

(c) Find \( b \) to yield repeated roots (note that \( b > 0 \)):

\[
\begin{align*}
x_1 &= -x_3 \\
x_2 &= -2x_2 + 1x_3 \\
x_3 &= 2x_1 - bx_3
\end{align*}
\]

(d) Find \( x(t) \) if

\[
\begin{align*}
u(t) &= 6 \sin 2t \\
x &= -5x + 10u(t)
\end{align*}
\]

with \( x(0) = 1 \)

(e) Repeat part (d) for

\[
\begin{align*}
u(t) &= \begin{cases} 0 & t < 2 \\ 5 & t \geq 2 \end{cases} \\
x(0) &= 0
\end{align*}
\]

What if \( x(0) = 1? \)

5-8 A classic mechanical system is shown in the figure. The parameters are \( m_1 = m_4 = 1 \) kg, \( k_2 = k_3 = 1 \) N/m, and \( g = 9.8 \text{ m/s}^2 \).

(a) If \( b_2 = b_4 = 0 \) (no damping), what are the oscillation frequencies? Hint: The resulting quartic equation is a special form that you can solve.

(b) Now let \( b_2 = b_4 = 1 \text{ N·m/s}. \) What is the steady-state configuration of the system? In particular, what is \( x_2(t \to \infty) \), the spring deflection? Hint: You can apply the FVT to \( X_2(s) \) since the roots all have negative real parts. It is not necessary to find the roots.

(c) If \( b_2 = 0 \), can \( b_4 \) be chosen so that the system does not oscillate? Explain.
5-9 The circuit in Fig. 5-9 acts as a filter. We are interested in the output voltage \( v_o(t) \) in response to the input voltage \( v_i(t) \) under no-load conditions.

(a) Make a bond-graph model of the circuit.
(b) Derive state equations and an output equation for \( v_o(t) \).
(c) Laplace-transform the state and output equations and derive \( V_o(s) \) in terms of \( V_i(s) \). Assume that the initial conditions are zero.
(d) Let \( R_1C_1 = 0.5 \text{ s}^{-1} \) and \( R_4C_2 = 0.25 \text{ s}^{-1} \). Also let \( v_i(t) = 10 \sin 3t \). Find \( v_o(t) \).

5-10 A rotational mechanical load is driven by a motor that sets the driving speed (Fig. P5-10a). A bond-graph model is shown in Fig. P5-10b, and the parameters are

\[
\begin{align*}
 k &= 1 \text{ N\cdot m} \\
g &= 20 (\omega_3 = 20\omega_4) \\
 J &= 2 \text{ kg\cdot m}^2 \\
b &= 20 \text{ N\cdot m\cdot s}
\end{align*}
\]

with \( T_L = 10 \text{ N\cdot m} \). The input \( \Omega_1(t) \) is shown in Fig. P5-10c

\[
\Omega_1 = \begin{cases} 
100 \text{ rad/s} & 0 \leq t < 1 \\
0 & 1 \leq t < \infty
\end{cases}
\]
Find $\omega_3(t)$, the angular velocity of the load inertia. Initial conditions are zero. *Hint:* Divide the problem into two periods. Solve for $\omega_3$ from $0 \leq t < 1$. Then use the system state at $t = 1$ as new initial conditions. Solve for $\omega_3$ from $1 \leq t < \infty$.

REFERENCES