Part II – Circuit Analysis

1. Circuits with Multiple Sources

- so far, our analyses have focused on simple circuits with one fixed source

→ we have often simplified circuits by finding equivalent resistance

- consider now circuits with multiple independent sources
  ⇒ can be solved using Ohm's Law, KVL, and KCL

Example 1.1

Consider the following circuit. Determine the power delivered to the circuit by the voltage sources.
Example 1.2

Recall that a current source provides a fixed amount of current, while the voltage across the current source depends on the circuit.

For the following circuit, determine $I$. 

![Circuit Diagram](image)
Example 1.3

Find \( v_0 \).

- solving for \( v_0 \) not so trivial as we need to solve
  3 equations simultaneously for 3 unknowns (currents)

- clearly circuit analysis can get complicated when circuits get large ⇒ we need systematic methods to solve large circuits

- we will consider 3 fundamental methods to analyze complex circuits (i.e., typically circuits with more than one source):
  
  (1) node-voltage analysis
  
  (2) mesh-current analysis
  
  (3) superposition

  (Other methods also exist to solve complex circuits.)

All methods are based on the fundamental laws:
  
  *Ohm’s Law, KVL and KCL.*
2. Controlled Sources

- to this point, we have only considered circuits with fixed, also called “independent” sources (voltage or current)

- in some circuits, voltage and current sources may be dependent or controlled by a parameter (eg. a voltage or current) of the circuit

- 4 possibilities:

- parameters $\mu$, $\rho$, $\alpha$, and $\beta$ are constants with the following dimensions:

  VCVS: $\mu$ V/V (dimensionless)
  CCVS: $\rho$ V/A
  VCCS: $\alpha$ A/V
  CCCS: $\beta$ A/A (dimensionless)

- when solving circuits with dependent sources, all the techniques we have learned so far (e.g., Ohm’s Law, KVL, KCL, voltage divider, current divider,...) apply
**Example 2.1**

The following circuit represents an amplifier with a gain defined to be \( v_{out}/v_{in} \). Determine the gain of the amplifier and the output voltage, \( v_{out} \).

![Circuit Diagram]
Example 2.2

For the following circuit, determine $i_0$. 

\[ v_s = 10 \text{ V} \]

\[ 2 \Omega \]

\[ 4 \Omega \]

\[ 2v_1 \]

\[ 1 \Omega \]

\[ i_0 \]
Example 2.3

For the following multistage amplifier circuit, determine an expression for the gain, $v_L/v_S$. 

\begin{align*}
\text{multistage amplifier}
\end{align*}
3. Node-Voltage Method

3.1. General Method

- systematic method that uses KCL to setup equations involving voltages → equations are then solved for voltages

- consider the following circuit

- circuit has \( n = 4 \) nodes and \( n_e = 3 \) “essential” nodes, where an essential node is a node with 3 or more branches

→ we need \( n_e - 1 \) equations to describe circuit so that it can be solved

- node-voltage method:
  → pick a reference node (often the one with the most branches or connected to negative side of a power supply)
  → label node voltages relative to reference node
  → write equations for currents leaving non-reference nodes in terms of node voltages (\( n_e - 1 \) equations total)
  → solve for node voltages using linear algebra techniques
  → once voltages are known, currents can be determined using Ohm’s Law
Example 3.1

Find the voltages and currents associated with the resistors in the following circuit.
Method 1:

Method 2:

Method 3 (Cramer’s Rule) ⇒ See “Aside” on p. 11.

Consider

\[ \begin{align*}
    a v_A + b v_B &= k_1 \\
    c v_A + d v_B &= k_2
\end{align*} \]

where \( a, b, c, d, k_1 \) and \( k_2 \) are all constants and \( v_A \) and \( v_B \) are unknowns to be determined.

We can represent the coefficients of the unknowns in a matrix \( \lambda \):

\[ \lambda = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

The determinant of \( \lambda \) is a single constant defined to be

\[ \det(\lambda) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a \cdot d - b \cdot c \]

Using Cramer's rule, we can solve for unknown \( v_A \) as follows:

\[ v_A = \frac{\det(\lambda_A)}{\det(\lambda)} \quad \text{where} \quad \lambda_A = \begin{bmatrix} k_1 & b \\ k_2 & d \end{bmatrix}. \]

That is, \( \lambda_A \) is 1st column replaced by independent constants in original equations.

Similarly,

\[ v_B = \frac{\det(\lambda_B)}{\det(\lambda)} \quad \text{where} \quad \lambda_B = \begin{bmatrix} a & k_1 \\ c & k_2 \end{bmatrix}. \]

That is, \( \lambda_B \) is 2nd column replaced by independent constants in original equations.

Of course, after finding \( v_A \) (or \( v_B \)), \( v_B \) (or \( v_A \)) can be found from either of the original two equations.

Cramer's rule can be generalized to \( n \) equations with \( n \) unknowns (i.e., \( n \times n \) matrix). For example, Cramer's rule is useful to solve for \( n = 3 \) unknowns.

In this course, we will not need to solve for more than 3 unknowns and we can typically turn 3 equations with 3 unknowns into 2 equations with 2 unknowns by a simple substitution and then apply above.
Example 3.2

Determine the currents and voltages of the resistors in the following circuit.
For practice:
⇒ Revisit Example 1.3 solving for $v_0$ using node-voltage analysis.

*See other examples in the text!*
3.2. Node-Voltage Method for Circuits with Dependent Sources

- in this case, node-voltage equations must be supplemented with constraint equations imposed by dependent sources

**Example 3.3**

Determine the gain, $V_{out}/V_{in}$, of the following amplifier circuit.

![Circuit Diagram](image-url)
3.3. Applying the Concept of a Supernode

- when a fixed or dependent voltage source is applied between 2 essential nodes, the nodes can be combined to form a "supernode" and one of the node equations can be eliminated

→ KCL still holds for supernode

**Example 3.4**

Determine the voltage across the 80 Ω resistor in the following circuit (a) first using previous techniques and (b) then using the supernode concept.

(a) not using supernode
(b) using supernode

- to simplify the analysis, look for supernodes before writing equations
Example 3.5

Determine the voltages at nodes $A$, $B$, and $C$ (relative to the negative side of the 120 V source) using the supernode technique.
4. Mesh-Current Method

4.1. General Method

- a little terminology:
  - path → a trace of adjoining basic elements with no elements included more than once
  - branch → a path that connects two nodes
  - loop → a path whose last node is the same as the starting node
  - mesh → a loop that does not enclose any other loops

- consider

- **mesh-current method**:
  → label all mesh currents on the circuit
  → write equations for voltages around each mesh using KVL in terms of the mesh currents
  → solve for mesh currents using linear algebra techniques
  → once currents are known, voltages can be determined using Ohm’s Law

- total number of equations to solve is equal to the number of mesh currents
- method can only be applied to planar circuits (i.e., circuits where branches do not cross) 
→ often non-planar circuits can be redrawn to be planar

**Example 4.1**

For the following circuit, determine all branch currents, \( i_{B1}, i_{B2}, \) and \( i_{B3} \).

(a) one approach determining branch currents directly:

→ can use KCL at node A and KVL around left and write mesh to produce 3 equations involving 3 branch currents

→ solving equations for branch currents

Note: Finding branch currents directly is not mesh current analysis!
(b) a better approach using mesh-current analysis allows us to write fewer equations to solve for mesh currents, which then can be simply used to determine any required branch currents (and, inevitably, any required voltages).

→ branch current $i_{B3}$ is simply derived from mesh currents, $i_1$ and $i_2$:

$$i_{B3} = i_2 - i_1$$
Example 4.2

For the following circuit, determine the power delivered by the current source and the voltage source.
Example 4.3

For the following circuit, determine the currents in all branches.
4.2. Mesh-Current Method for Circuits with Dependent Sources

- in this case, mesh-current equations must be supplemented with constraint equations imposed by dependent sources

*Example 4.4*

For the following circuit, determine $i_X$.
4.3. Applying the Concept of a Supermesh

- when a fixed (or dependent) current source connects 2 essential nodes, the branch can be ignored thereby creating a “supermesh” and eliminating one of the mesh equations

*Example 4.5*

Determine the currents and voltages of the following circuit using the supermesh concept.
- to simplify the analysis, look for supermesh before writing equations

**Example 4.6**

For the following circuit, determine the three mesh currents.
4.4. Node-Voltage Method vs. Mesh-Current Method

- Which method is preferred?
  ⇒ depends on circuit but not necessarily trivial to see

- questions to ask:
  → Which method results in fewer simultaneous equations?
  → Does the circuit contain supernodes?
    Yes ⇒ node-voltage method
  → Does the circuit contain supermeshes?
    Yes ⇒ mesh-current
  → Does only a part of the circuit need to be solved?
    ⇒ Which method suits best this partial solution?

Example 4.7

Solve for all branch currents in the following circuit using an appropriate method.
**Example 4.8**

Solve for $v_X$ using an appropriate method.

Another general method – “Superposition” - to come...
5. Thévenin and Norton Equivalents

5.1. Thévenin Equivalent

- often, we are only interested in the behaviour of the voltage and/or current across the terminals of a circuit

- if the circuit consists of only resistors and sources, either fixed or dependent (with dependency inside circuit), we can simply represent or model the circuit as one ideal voltage source, $V_{TH}$, with a series resistor, $R_{TH}$
- How can we find $V_{TH}$ and $R_{TH}$?
  → if terminals $a$ and $b$ are left open, then $V_{ab} = V_{TH}$, since there is no current through $R_{TH}$

$$V_{TH} = V_{ab}^{\text{(open cct)}}$$

→ now, if terminals $a$ and $b$ are shorted (so $V_{ab} = 0$):

⇒ current flowing from $a$ to $b$, $i_{SC}$, can be used to compute $R_{TH}$, since

$$V_{TH} = i_{SC} R_{TH}$$

To compute Thévenin parameters:

(1) open circuit $a - b$, compute $V_{ab}$ and this is $V_{TH}$
(2) short circuit $a - b$, compute $i_{SC}$ and then $R_{TH} = V_{TH}/i_{SC}$

- another simple approach to determine $R_{TH}$ is to
  (2a) remove sources (short circuit voltage sources so $v_S = 0$ and open circuit current sources so $i_S = 0$) and
  (2b) compute equivalent resistance looking in terminals $a - b$
- this works for any circuit! → we can also follow this process in the lab, measuring $V_{ab}$ and $i_{SC}$ and then deriving $V_{TH}$ and $R_{TH}$ for a completely unknown circuit

- to any circuit looking at $a-b$ from the right, the Thévenin equivalent model will behave the same as the original circuit

- converting a circuit or a portion of a circuit to its Thévenin equivalent can be an effective tool for analysis

**Example 5.1**

Let $R_L = 4 \ \Omega$ and compute $i_L$ for the following circuit.
5.2. Norton Equivalent

- similarly to the Thévenin equivalent, we can look at a circuit to determine its behaviour in terms of current delivered

- we can model any circuit consisting of only resistors and sources, either fixed or dependent (with dependency inside circuit), as one ideal current source, $I_{NO}$, with a parallel resistor, $R_{NO}$
- if we short terminals \(a-b\), we can see that

\[
I_{NO} = i_{SC}
\]

⇒ Norton equivalent current is short circuit current of circuit under consideration!

and if we open circuit terminals \(a-b\), we can see that

\[
R_{NO} = V_{ab}^{\text{(open cct)}} / i_{SC}
\]

- hence,

\[
R_{NO} = R_{TH} \quad I_{NO} = V_{TH}/R_{TH}
\]

⇒ (1) Norton and Thévenin resistances are the same and 
(2) the Norton current can be easily derived from the 
Thévenin voltage and vice versa

- similar to the Thévenin equivalent, converting a circuit or a 
portion of a circuit to its Norton equivalent can be an 
effective tool for analysis
Example 5.2

For previous example, already solved for \( i_L \) using Thévenin. Solve for \( i_L \) now using Norton equivalent assuming \( R_L = 4 \, \Omega \).

\[
-v_L + \frac{v_L R_L}{R_L + 2 \Omega} + 2 A = 0
\]

\[
\frac{-v_L}{R_L + 2 \Omega} = \frac{24 V - 2 \Omega 2 A}{R_L + 2 \Omega}
\]

\[
R_L = 4 \, \Omega
\]
5.3. Maximum Power Transfer

- in the lab, you experimentally investigated the concept of maximum power transfer

- for this simple series circuit, you (hopefully) found experimentally that the maximum power delivered to load occurred when $R_L = R_S$

- this can be proven analytically using a little calculus:

\[ P = \frac{V_s^2}{4R_L} \]

\[ \rightarrow \text{to find } R_L \text{ for maximum } P, \text{ set } \frac{dP}{dR_L} = 0 \]

\[ \frac{dP}{dR_L} = 0 = \]

\[ \Rightarrow R_L = R_S \text{ and resulting maximum power is:} \]

\[ P_{\text{max}} = \]
- but this circuit could be the Thévenin model of a general circuit (with $V_S = V_{TH}$ and $R_S = R_{TH}$) with a load resistor applied across terminals $a-b$

$\Rightarrow$ for a general circuit, maximum power transfer occurs for a load of $R_L = R_{TH}$, where $R_{TH}$ is the Thévenin resistance of the circuit

**Example 5.3**

Determine the maximum possible power delivered to the load for the following circuit.
5.4. Practical Voltage and Current Sources

- so far, we have focused on voltage sources as energy sources in circuits

symbols:

$$v_S$$

$$v_S$$

$$v_S$$

$$v_S$$

**Ideal Voltage Source**

→ fixed voltage level, regardless of the characteristics of the circuit

**Practical Voltage Source**

→ may not be able to hold a fixed voltage level, due to large current drawn by the circuit

- consider a Thévenin model of a practical voltage source:

where $V_{OC}$ is the "open circuit voltage" produced by an ideal source and $R_S$ is the "internal resistance" of the source

→ $V_{OC}/R_S$ is also the Thévenin voltage/resistance

→ $R_S$ is in series with ideal source and is typically small
- if $R_L$ is large (and, hence, $i_L$ is small) $\Rightarrow R_S << R_L$ and

\[ v_L = \]

\[ \approx \]

- if $R_L$ is small (and, hence, $i_L$ is large)

\[ v_L = \]

\[ << \]

$\Rightarrow$ voltage is not maintained according to voltage source specification!

**Ideal Current Source**

$\rightarrow$ provides a fixed amount of current to the circuit, regardless of the characteristics of the circuit

symbol:

\[ I \]

$\rightarrow$ just as the amount of current provided to a circuit by an ideal voltage source is not fixed but depends on the circuit, the voltage across a current source is not fixed but depends on the circuit
Example 5.4:

For the following circuit, resistor $R$ can be varied from 1 kΩ to 3 kΩ. Determine the maximum and minimum power that may be delivered by the current source.
**Practical Current Source**

→ may not be able to provide its fixed current if requirements from circuit result in large voltage across source

- consider a Norton model of a practical current source:

![Norton model diagram](image)

where \( I_{SC} \) is the "short circuit current" produced by an ideal source and \( R_P \) is the "internal resistance" of the source

→ \( I_{SC}/R_P \) is also the Norton current/resistance

→ \( R_P \) is in parallel with ideal source and is typically large

- if \( R_L \) is small (and, hence, \( i_L \) is large) \( \Rightarrow R_P >> R_L \) and

\[
i_L = \approx
\]

- if \( R_L \) is large (and, hence, \( i_L \) is small)

\[
i_L = \ll
\]

⇒ current is not maintained to current source specification!
5.5 Source Transformations

- consider the circuit for a practical voltage source modeled as an ideal source, $V_S$, in series with an internal resistor, $R$, connected to a load $R_L$

- we can convert our voltage source to a current source, letting the internal resistance remain at $R$

- comparing the two expressions for $i_L$, we get $V_S = I_S R$

⇒ we can take a voltage source model (or, alternatively, an ideal voltage source in series with a resistance) and change it to a current source model (or, alternatively, an ideal current source in parallel with a resistance) ...or vice-versa... if it makes the circuit analysis simpler
Example 5.5

Consider:

(a) Determine power delivered by the 10 V source.

(b) Determine current through the 15 kΩ resistor.
6. Superposition  (One last circuit analysis technique!)

- consider the following equation: $y = 7x$

- now let $x_1 = 2$, $x_2 = 9$, and $x = x_1 + x_2 = 2 + 9 = 11$

- further, let $y_1 = 7x_1 = 7(2) = 14$, $y_2 = 7x_2 = 7(9) = 63$, and $y = 7x = 7(x_1 + x_2) = 7(11) = 77$

- but note that $y = y_1 + y_2$

  ⇒ this behaviour is “superposition” and it occurs because the equation is linear!
  → it would not hold true if $y = 7x^2$ (which is clearly nonlinear) or even if $y = 7x + 5$ (which is not strictly linear, but “affine”)

- since systems which contain only resistors and fixed sources are linear (due to Ohm’s Law), superposition applies
  ⇒ the response of any part of the circuit (that is, a current or voltage value) is the sum of the responses caused by the individual fixed sources in the circuit

- hence, to solve a circuit with multiple sources (voltage and/or current):
  (1) remove all but one source and compute the response,
  (2) repeat for each individual source, and then
  (3) add the results.

- removing a voltage source
  → set $v = 0$, i.e., short circuit the source

- removing a current source
  → set $i = 0$, i.e., open circuit the source
Example 6.1

For the following circuit, determine $i_0$ using superposition.
Example 6.2

For the following circuit, determine $v_0$ using superposition.

- in general, superposition can also be applied if the circuit contains simply-scaled dependent sources but dependent sources should not be removed from circuit

→ see Example 13 in Chapter 4 of text.