ENGI 1040: ELECTRIC CIRCUITS  
Winter 2015  
(Based on notes prepared by Dr. Eric Gill.)

Part I – Basic Circuits

1. Electric Charge

atom ⇒ elementary unit of a material which contains the properties of that material  
⇒ can be modeled as negatively charged electrons orbiting a positively charged nucleus, which itself consists of positively charged protons and neutrally charged neutrons

- unit of electric charge is the coulomb, symbolized by C

- quantity of charge typically symbolized by \( q \) (often reserved for time-varying charge) or \( Q \) (for a fixed amount of charge)

- one electron ⇒ \(-1.60 \times 10^{-19} \text{ C}\), one proton ⇒ \(+1.60 \times 10^{-19} \text{ C}\)  
  → \(6.25 \times 10^{18}\) electrons/protons in a charge whose size is \(1 \text{ C}\)

- in a material,  
  (1) if \# electrons > \# protons ⇒ negatively charged  
  (2) if \# protons > \# electrons ⇒ positively charged
- two charges of *same polarity* (i.e., both are negative or both are positive) *repel* each other

- two charges of *opposite polarity* (i.e. one positive and one negative) *attract* each other

![Diagram](image)

- **force** \((F)\) measured in **newtons** (N) between two charges \(Q_1\) and \(Q_2\) given by:

\[
F = k \frac{Q_1 Q_2}{d^2}
\]

* Coulomb's Law

where \(d\) is **distance** \((d)\) in **metres** (m) and \(k\) is a proportionality constant with \(k = 8.99 \times 10^9\) N·m\(^2\)/C\(^2\) if there is empty space (vacuum) or, to a good approximation, air between the charges ⇒ equivalent force is proportional to product of the charge sizes and inversely proportional to the square of the distance

- note that one **newton** is the size of the *net force* required to accelerate (or decelerate) a mass of 1 kg at a rate of 1 m/s\(^2\)

- **electric field** ⇒ region where an electric charge experiences a force being exerted upon it
- by convention, the *direction* of an electric field is the direction of the force exerted on a *small positive test charge* brought into that field

→ field caused by a negative charge points toward that charge

→ field due to a positive charge points away from the charge

- force fields of the ‘point’ charges are distributed *radially* around the charges ⇒ the charges themselves are *sources* of electric fields

**Example 1.1**

A charge of $+3 \, \mu \text{C}$ is located on the y-axis at B(0, 1 cm) and a charge of $-4 \, \mu \text{C}$ is located at A(0, 2 cm) on the y-axis. Determine the size and direction of the total force that these two charges exert on a $+1 \, \mu \text{C}$ charge residing at the origin (O).
2. Materials: Conductors, Insulators and Semiconductors

- **conductors** ⇒ materials (e.g. copper, aluminum) where the outer (or *valence*) electrons of the atoms may move freely from atom to atom

- protons are bound to atomic nuclei and do not move

- flow of electrons between atoms
  ⇒ *electric current* = time rate of flow of electric charge

- **insulators** or **dielectrics** ⇒ materials (e.g. glass, plastic, air) where most energetic electrons are bound very tightly to their nuclei and it is difficult to cause large numbers of electrons to move from one position in the material to another

- good insulators may store charge but do not conduct well (i.e., are poor conductors)
  → similarly, good conductors are poor insulators

- note that even insulators can be made to conduct under extreme circumstances
  → lightening travelling between clouds or to the earth through air

- under normal conditions, insulators do not conduct appreciably
  → *copper wire conductors* used in house wiring are wrapped in *plastic insulation*

- **semiconductors** ⇒ materials (e.g. carbon, silicon, germanium) that are *neither* good conductors nor good insulators
  → numbers of electrons free to move around between atoms are far less than in conductors, but much more than in insulators
  → critical component in electronic systems such as your laptop, cell phone, etc.
3. Electric Current

- **electric circuit** ⇒ path around which electric charge (electrons!) may flow

- **electric current** → *the time rate of flow of electric charge* passing a given point, which we will assume to be in a conductor

- typically use symbol *i* for current and *t* for time, if the *rate of charge flow is a constant* then

\[
i = \frac{q}{t}
\]

where *q* measured in coulombs (C) and time in seconds (s)

- current measured in **amperes** (A) (or **amps**) where 1 A = 1 C/s

- for a constant value of *i*, say *I*, flowing for a time *t* = *T*, total charge passing a particular point is simply area ‘under’ the “*i* versus *t*” graph as shown in the sketch below:

- note: current-time graphical perspective can be extended to more general cases by using the concept of *integration* in calculus
Example 3.1

The current in a circuit is 2 A. How many electrons pass a given point in the circuit in 1.5 ms?

4. Work, Potential and Potential Difference

4.1 Work and Energy
- when a constant force \((F)\) acts (or is exerted) through a distance \((d)\) which is in the direction of the force, the work \((W)\) done given by

\[
W = F \cdot d
\]

- amount of work is represented in joules \((J)\)
  \(\rightarrow 1\ J = \text{amount of work done when a force of 1 N is exerted through a distance of 1 m}\)

- energy is the ability to do work and, in particular, potential energy, is the energy stored in an object (or a charge) due to its position in a force field
4.2 Potential and Potential Difference

- consider bringing a small positive charge $q$ into an electric field generated by a positive charge $Q$

\[ \text{Distant Charge, } q \]

\[ +Q \]

\[ \cdot \text{A} \]

\[ \cdot \text{B} \]

$\rightarrow$ assume that $q$ comes from long distance away (ideally, infinity)

- **potential** at point B in the force field of $Q$ is, by definition, the *work per unit charge* done on $q$ to bring it to point B from an infinite distance away

- to bring $q$ to point A, which is even closer to $Q$, requires more work $\Rightarrow$ at point A, the **potential** is greater than at point B

- **volt** (V) $\Rightarrow$ **unit** for potential derived from the unit for work divided by the unit for charge $\rightarrow$ 1 V = 1 J/C

- **potential difference** exists between points A and B in force field of $Q$ $\rightarrow$ Let $v_A$ denote the potential at A, $v_B$ the potential at B and $v_{AB}$ the potential difference between points A and B, then

\[ v_{AB} = v_A - v_B \]

where the units are still volts!
- if \( q \) is at point B, then more work must be done on \( q \) to move it to point A, a position of higher potential

- if \( q \) is released from point A, then it will be forced to move further from \( Q \) and toward B
  \[ \Rightarrow \text{the tendency of the charge is to move to a place in the field where the potential is lower} \]

- we can say:
  \[ \rightarrow \text{potential difference between two points} \]
  \[ = \text{work per unit charge to move a charge between the points} \]
  and since work done \( \equiv \) change in potential energy

  \[ \rightarrow \text{potential difference between two points} \]
  \[ = \text{difference in potential energy per unit charge between the points} \]

- in an electric circuit, a source establishes potential differences and forces charges out one terminal and into another which is at a lower potential and connected to the first by a conducting path
  \[ \rightarrow \text{resulting current passes through various kinds of circuit elements} \]

- electromotive force (emf) \( \Rightarrow \) energy expended per unit charge in moving it from one terminal to another inside the source
  \[ \rightarrow \text{equivalent to potential difference between terminals of the source} \]

- one form of a source is a voltage source
  \[ \Rightarrow \text{an ideal voltage source presents same potential difference to any circuit to which it is connected} \]
  \[ \rightarrow \text{i.e., same potential difference will be measured across the terminals of an ideal voltage source no matter how much current flows between the terminals} \]

  \[ \rightarrow \text{in reality, there is no such device as an ideal voltage source} \]
- source of dc (i.e., constant) voltages may be provided by many different devices → batteries ≡ collection of chemical cells

(a) A cell and (b) a battery and their circuit symbols. (Notice the polarities shown on the symbols.)

- two cells, connected so that the negative terminal of one is joined to the positive terminal of the other, are connected in series
  → total voltage of the battery is sum of the two cell voltages
  → total current output is that of one cell

- two similar cells connected such that two positive terminals and the two negative terminals are connected in parallel
  → total voltage of the battery is that of a single cell
  → total current output is the sum of the outputs from the cells

(a) A battery formed by connecting two cells in parallel and (b) the circuit symbol.
5. A Simple Electric Circuit

- **direct current** (dc) circuit ⇒ an electric circuit in which the current flows in only one direction

- general dc circuit:

![Diagram of a simple electric circuit](image)

- **load** ⇒ could be any simple or complicated combination of circuit elements which is energized by the source

- switch is currently open and circuit is an *open circuit* with no current flow ⇒ \( i = 0 \)

- in order for a current \( i \) to flow, the switch must be flipped to position A to form a *closed circuit* ⇒ \( i > 0 \)

- note that, by convention we have indicated current, \( i \), to flow from higher potential (positive) to lower potential (negative), which is opposite to the direction in which electrons would flow when switch is closed
  ⇒ instead of thinking of negative charges flowing out of the negative terminal of the source, we imagine that an equal amount of positive charge flows out of the positive terminal in the direction opposite to the electron flow
- moving charges transfer energy to the **load**
  ⇒ results in heat, light, or sound, or some combination of these forms of energy depending on the nature of the load eg. electric heaters, stove, lightbulbs, computers

- transfer of energy to the load causes a decrease or *drop* in the voltage across the load

### 6. Resistance and Ohm’s Law

#### 6.1. Resistance

- **resistance** \((R)\) ⇒ opposition to the flow of charge as moving charges (i.e., electrons) collide with atoms in material of a "load"
  ⇒ unit is **ohm** \((Ω)\)

- a resistive load is a *passive circuit element* which consumes electrical energy, whereas a voltage source is an *active circuit element* since it supplies energy

- a perfect conductor would have \(R = 0 \ Ω\), but in reality, even copper wires in a circuit have a very small non-zero resistance

- \(R\) of a conductor given by

\[
R = \frac{\rho \cdot L}{A}
\]

: where \(L = \text{length of conductor,}\)
\(A = \text{cross-sectional area, and}\)
\(\rho = \text{resistivity of the material}\)

→ also may significantly depend on temperature and typically \(T ↑ \Rightarrow R ↑\)
- often reasonable to assume that the resistance in a circuit is a constant for that particular circuit under all practical operating conditions

- consider the model of a circuit containing a voltage source and a load resistance:

```
\[ \begin{align*}
&\text{voltage source now represented as a battery of voltage } v \text{ and load is a resistive load of resistance } R \\
&\rightarrow \text{load could be a light bulb or a heater or some other device that can be } \textbf{modeled} \text{ as a constant resistance}
\end{align*} \]
```

- connecting wires are assumed to be perfect/ideal conductors
  \[ \Rightarrow \text{all energy dissipated in the load and none in the wires} \]

- as flowing charges encounter resistance, energy must be used to push the charges across such an element
  \[ \Rightarrow \text{there is a potential difference between the ends of the resistor denoted by } v_R \]

- using the sign convention with the resistance having +/- same as source and current direction indicated in direction of voltage drop
  \[ \Rightarrow \text{power absorbed by the resistance is positive} \]

- note: in this circuit \( v_R = v \) and \( i_R = i \)
6.2. Ohm’s Law

- in 1827, Georg Simon Ohm discovered that the current $i$ developed in a circuit like the previous circuit was proportional to the voltage $v$ supplied by the source, i.e.,

\[ \frac{\text{voltage}}{\text{current}} = \text{constant} \]

- if voltage $\uparrow$, current $\uparrow$ proportionally based on constant
  ⇒ for large constant, if voltage $\uparrow$, current does not increase as much in absolute terms as for smaller constant
  ⇒ constant reflects concept of resistance

- often resistance changes with temperature, but many elements have constant ratio at reasonable operating temperatures
  ⇒ linear resistor

- electronic devices contain many **nonlinear elements** such as diodes and transistors for which the ratio of voltage/current is not constant, but we do not consider them here

- Ohm’s Law for the resistive load or resistor:

\[ \frac{v_R}{i_R} = R \]

or

\[ v_R = i_R R \]

- more generally, Ohm’s Law is written as

\[ v = i \cdot R \]  \hspace{1cm} \text{Ohm’s Law} \]

where $v$ and $i$ are the voltage and current, respectively, associated with the resistance $R$. 

- linear resistors exist whose value may be manually changed ⇒ eg. dimmer switches used to control a light or in the volume control of an audio device or in the speed control of a motor

- as shown below, the conducting ‘arrow’ may be moved along the resistor to put a variable amount of resistance into the circuit

- types of variable resistors include *rheostats* (2 terminals) and *potentiometers* (3 terminals)

![Diagram of variable resistors](image)

(a) potentiometer in a circuit and (b) typical symbol for rheostat used in schematic diagram

**Example 6.1**

If the voltage source in above figure has a value of 5.0 V, what is the range of variable resistor if the current varies from 2 to 10 mA?
7. Electrical Power

- **power** \((P)\) is the *time rate of doing work*, or alternatively, the rate at which energy is expended or absorbed:

\[
\text{power} = \frac{\text{work}}{\text{time}}
\]

- symbolically,

\[
P = \frac{W}{t}
\]

where \(W\) may be interpreted as either **work** or **energy**

- typical power unit is **watt** \((W)\) defined such that 1 \(W\) \(\equiv\) 1 J/s

- recall that

\[
\text{potential difference} = \frac{\text{work}}{\text{charge}}
\]

or

\[
v = \frac{W}{q}
\]

- recall that current is amount of charge passing a given point over a period of time or \(i = \frac{q}{t}\) giving

- solving this for \(W\) we see that

- now we may substitute this result for \(W\) into expression for \(P\):

\[
P = vi
\]
- from above, we can infer 1 watt = 1 volt·ampere (1 W = 1 VA)

- using Ohm’s Law, \( v = iR \), and substituting this into expression for power in place of \( v \), we get

\[
P = \frac{v^2}{R}
\]

\[
\Rightarrow \quad P = \frac{(3.0)^2}{300 \times 10^3}
\]

- also, clearly

\[
P = \frac{v^2}{R}
\]

\[
\Rightarrow \quad P = \frac{(3.0)^2}{300 \times 10^3}
\]

**Example 7.1**

A potential difference of 3.0 volts exists across a 300 k\( \Omega \) resistor for 5 s.

(a) What power is consumed by the resistor?

(b) What energy is dissipated over the stated time interval?

(a)

(b)
- power may be *delivered* by a source or *absorbed* by a load

- consider 2 orientations for current direction through circuit element

(a) power absorbed and (b) power delivered by a circuit element

- in (a) above, the current flows from a region of *high to low* potential
  ⇒ corresponding to the situation for a resistive load
  → the resistor absorbs power and the convention is that
  *positive power means power absorption*

- in (b), the current flows from a region of *low to high* potential
  ⇒ corresponding to the situation for a voltage source
  → the source delivers power and the convention is that
  *negative power means power delivery*

- we will typically not worry about sign convention for power and
  instead simply make statements such as:

"10W of power delivered by source"

or

"10W of power absorbed by load"
**Example 7.2**

The voltage and current at the terminals of an automobile battery for a 100 second time interval during charging are shown below. Determine (a) the total charge transferred to the battery over the interval and (b) the equation which describes the power delivery to the battery.

![Graph of voltage and current over time](image1)

**Example 7.3**

Two 12-V car batteries are connected as shown below. Assume that the current $i$ is measured and found to be $-40$ A. (a) Which is the ‘dead’ battery? (b) If this connection is maintained for 1.5 minutes, how much energy is transferred to the dead battery?

![Battery connection diagram](image2)
Example 7.4

Four 1.5 V dry cells are connected in series to provide a 100 mA current to a portable CD player. How much energy does this battery supply in 3 hours?

Example 7.5

Two electric circuits are connected as shown below. The reference direction for the current $i$ and the reference polarity for the voltage $v$ across the interconnection are as shown. For each of the following measurements, calculate the power in the interconnection and state whether each of A and B are delivering or dissipating power:

(a) $i = 5 \text{ A}$, $v = 20 \text{ V}$ and (b) $i = -15 \text{ A}$, $v = 100 \text{ V}$.
8. More Complex Circuits

8.1. Nodes and Loops

- **node** = a point in a circuit where two or more elements meet

![Series Circuit Diagram](image)

Figure 8.1 – Series Circuit

- since we assume perfect conductors for the wires, if no circuit element is encountered as we trace between two elements, we are at the same node \( \Rightarrow \) three nodes in this circuit, but only one **closed path** or **loop** around which a current may flow

![Parallel Circuit Diagram](image)

Figure 8.2 – Parallel Circuit

- several connection points in this figure but **two nodes** (A and B) and **six loops** \( \rightarrow \) Can you see them?
8.2. Series Combination of Elements

- when two circuit elements connect at just a single node they are said to be \textit{in series}
  → in Figure 8.1, all three elements are connected \textit{in series} and the combination is referred to as a \textit{series circuit}

8.3. Parallel Combination of Elements

- when two circuit elements connect at a single pair of nodes they are said to be \textit{in parallel}
  → in Figure 8.2, all elements, including the source, are connected \textit{in parallel} and the circuit itself is referred to as a \textit{parallel circuit}

- note: "parallel" does not reflect the shape of the connecting wires in the diagram → figure below is equivalent to Figure 8.2.

8.4 Series-Parallel Combination of Elements

- a single circuit need not be limited to series or parallel connection of elements ⇒ often both types of connections may be found together
→ $R_3$ and $R_4$ are in series
→ $R_3$ and $R_4$ taken as a combination are in parallel with $R_2$
→ $R_1$ is in series with the parallel combination of $R_2$ and the series combination of $R_3$ and $R_4$
→ source $v_s$ and $R_1$ are in series

- for elements in series, the same current flows through them
→ when a current reaches a parallel combination it divides

- the same voltages occur across parallel elements

**9. Kirchhoff’s Laws**

- to completely analyze a circuit it is required that the voltage *across* and current *through* each element be determined

- assume that the source voltage $v_s$ and resistance values are known in the following circuit

![Figure 9.1. Circuit Used to Illustrate Kirchhoff's Laws](image)

- there appear to be 5 unknowns: $i_s$, $i_1$, $i_2$, $v_1$, and $v_2$

- we can solve for these unknowns by setting up five independent linear equations and solving for 5 unknowns → method to do so makes use of **Kirchhoff’s laws**
9.1. Kirchhoff’s Current Law

- Kirchhoff’s **current law** (KCL) states that
  "The algebraic sum of all currents at any node in a circuit equals zero."
  → we use the convention that currents entering a node are **negative**
  and those leaving a node are **positive**

- for circuit of Figure 9.1, the size of the currents can be determined by
  applying KCL at nodes A, B and C:
  
  Node A: \[ i_1 - i_s = 0 \] ..... (I)
  Node B: \[ i_2 - i_1 = 0 \] ..... (II)
  Node C: \[ i_s - i_2 = 0 \] ..... (III)

  - only equations (I) and (III) are required to see that \( i_1 = i_s = i_2 \)
    ⇒ there is really only one current in the loop, source current \( i_s \)
    ⇒ currents through elements connected in series are all the same

  - KCL can also be thought of as "sum of currents entering a node equals
    the sum of currents leaving the node"

  - note that the system of equations does **not** form an **independent** set (any
    one equation can be derived from the other two), so there are only
    two independent equations here
  ⇒ in any circuit with \( n \) nodes, only \( n - 1 \) independent equations
    can be formed using Kirchhoff’s current law

9.2. Kirchhoff’s Voltage Law

- Kirchhoff’s **voltage law** (KVL) states that
  "The algebraic sum of all voltages around any closed path in a circuit
  equals zero."
  → we use the convention that a **positive** voltage **drops** and a
    **negative** voltages **rises** as a loop is traversed in a fixed
    direction
- for circuit of Figure 9.1, applying KVL results in
  \[-v_s + v_1 + v_2 = 0 \quad \text{...... (IV)}\]

- equivalent to saying that "the sum of the voltage rises equals the sum
  of the voltage drops" around any closed loop and (IV) becomes
  \[v_s = v_1 + v_2 \quad \text{...... (IV)}\]

- applying Ohm's law to resistances in the circuit, we get
  \[v_1 = i_1R_1 \quad \text{.... (V)} \quad \text{and} \quad v_2 = i_2R_2 \quad \text{..... (VI)}\]

- so using Kirchhoff's laws and Ohm's law, equations (I) to (VI) give
  5 equations which can be solved for the 5 unknowns
  (→ solving a simple circuit is not really this "hard"!)

**Example 9.1**

In the following figure, the source voltage is 20 V, \(R_1 = 12\ \Omega\) and \(R_2 = 8.0\ \Omega\). Determine the source current and the voltage drops across \(R_1\) and \(R_2\) using Kirchhoff's laws and Ohm's law.
10. Resistors in Series

- consider the following series circuit with \( n \) resistors

\[
\begin{align*}
\text{What is the total resistance in the circuit?}

- from KCL, the current \textit{through} each resistor is \( i_s \)
  \Rightarrow \text{using \textbf{Ohm’s law} for each resistor we have}
  \begin{align*}
  v_1 &= i_s R_1 \\
  v_2 &= i_s R_2 \\
  &\vdots \\
  v_n &= i_s R_n
  \end{align*}
  \Rightarrow \text{the ratio of the voltage drops across any two resistors equals the ratio of their resistances}

- now, using KVL for the single loop circuit, we have
  \begin{align*}
  -v_s + v_1 + v_2 + \ldots + v_n &= 0 \\
  \text{or} \quad v_s &= v_1 + v_2 + \ldots + v_n
  \end{align*}

- substituting the Ohm’s law expressions for each resistor into this last equation gives
  \begin{align*}
  v_s &= \\
  \text{or} \quad v_s &=
  \end{align*}
\]
⇒ the total or equivalent resistance \( R_{eq} \) is the ratio \( v_s/i_s \), so the total series resistance we have

\[ R_{eq} = \]

or

\[ R_{eq} = \]

Example 10.1

A circuit contains 3 identical 10-Ω resistors. The resistors are placed in series with a voltage source and the current in the circuit is 600 mA. (i) What is the value of the source voltage? (ii) What is the voltage drop across each of the resistors?
11. Resistors in Parallel

- consider the following parallel circuit with \( n \) resistors

\[
\begin{align*}
\text{loop containing only the source and } R_1 &\Rightarrow -v_s + v_1 = 0 \\
\text{loop containing only the source and } R_2 &\Rightarrow -v_s + v_2 = 0 \\
\text{loop containing only the source and } R_n &\Rightarrow -v_s + v_n = 0 \\
\end{align*}
\]

\[ \Rightarrow v_s = \]

\[ \rightarrow \text{ elements connected in parallel have the same voltage drop across each} \]

- from Kirchhoff’s current law we can see that

\[ i_s = \]

- writing Ohm’s law for each resistance, we have

\[ i_1 = v_1/R_1, i_2 = v_2/R_2, \ldots, i_n = v_n/R_n, \]
or

\[ i_1 = \frac{v_s}{R_1}, i_2 = \frac{v_s}{R_2}, \ldots, i_n = \frac{v_s}{R_n}. \]

- substituting, we get

- since \( R_{eq} = \)

\[ \Rightarrow \text{for resistors in parallel,} \]

\[ \frac{1}{R_{eq}} = \]

or

\[ \frac{1}{R_{eq}} = \]

**Example 11.1**

A circuit has two resistors \( R_1 \) and \( R_2 \) in parallel. Find an expression for the equivalent resistance.

\[ R_{eq} = \]
12. Conductance

- for parallel combinations of circuit elements it is sometimes convenient to use conductance \((G)\) which is defined as reciprocal of resistance

\[
G = \frac{1}{R}
\]

- conductance is measured in siemens \((S)\), although sometimes in the literature, mho (which is ohm spelled backward) is used in place of the siemen \(\Omega\) symbol used for this is an inverted \(\Omega\)

- example: 10 \(\Omega\) resistor has conductance of 0.1 mho or 0.1 S.

- equations for parallel resistors may be written in terms of conductance:

\[
G_{\text{eq}} =
\]

Example 12.1

Determine the overall conductance of the following circuit looking in at terminals \(a-b\).
13. Some Circuit Analysis Examples

Example 13.1

(a) Find the equivalent resistance at terminals $a - b$ for the series-parallel combination shown below. (b) What is the equivalent conductance of the parallel section? (c) If a 19 V dc source was connected across the terminals $a - b$, what power would be absorbed by the 30 $\Omega$ resistor?

![Circuit Diagram]

- 12 $\Omega$
- 30 $\Omega$
- 24 $\Omega$
- 18 $\Omega$
- 10 $\Omega$
Example 13.2

In the circuit below, find $R$ if $V_0 = 4$ V.
Example 13.3

Calculate the equivalent resistance $R_{ab}$ in the following network.
Example 13.4

In the accompanying circuit, find $v_1$ and $v_2$. Also calculate $i_1$ and $i_2$ and the power dissipated in the 12-$\Omega$ and 40-$\Omega$ resistors.
Example 13.5

For the circuit below, \( i_0 = 2.0 \, \text{A} \).

(a) Find \( i_1 \).
(b) Determine the power dissipated by each resistor and verify that the total power \textit{dissipated} in the circuit resistors equals the total power \textit{delivered} by the source.

(c) Next, suppose the 20 \( \Omega \) resistor is short circuited and find the new \( i_0 \) and \( i_1 \).
14. Voltage-Divider Circuit

- often useful to produce two or more voltage levels from a single source
  ⇒ one method to do so is a \textbf{voltage-divider circuit}

- consider finding expressions for voltages $v_1$ and $v_2$ in terms of source voltage $v_s$ and the resistances $R_1$ and $R_2$ for the series circuit below:

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{voltage_divider_circuit.png}
\caption{Voltage-Divider Circuit Diagram}
\end{figure}

- applying Kirchhoff’s \textbf{voltage law}:

\begin{align*}
v_s &= \\
\Rightarrow i &= 
\end{align*}

- using \textbf{Ohm’s law} for the voltage across each resistor:

\begin{align*}
v_1 &= \quad (*) \\
\text{and} \\
v_2 &= \quad (**)
\end{align*}
⇒ fraction of the voltage which is dropped across each resistance is the ratio of that resistance to the total resistance in the circuit → true for any number of series connected resistors

⇒ for a series connection of \( n \) resistors, the voltage across the \( j^{th} \) resistor is

\[
v_j = \frac{v}{R_{eq} = R_1 + R_2 + \ldots + R_n}
\]

where \( R_{eq} = R_1 + R_2 + \ldots + R_n \) and is the equivalent resistance of the series combination

- ratio always < 1 ⇒ individual voltages < source voltage

- consider

- since \( R_L \) connected across \( R_2 \), voltages \( v_1 \) and \( v_2 \) (across resistors \( R_1 \) and \( R_2 \)) will change from their values calculated in equations (*) and (**) above

- in many applications, we would like for the voltage provided by the dividing circuit to NOT DEPEND on the elements (i.e. the load) that the voltage is across
- What is the voltage $v_o$ for this circuit?

- equivalent resistance of parallel section across which $v_0$ exists is

\[ R_{eq} = \]

- $R_1$ is in series with this $R_{eq}$ and therefore

\[ v_o = \]

\[ = \]

\[ \Rightarrow v_o = \]

- if $R_L >> R_2$, then $R_2/R_L << 1$ and $v_0$ can be approximated as:

\[ v_0 \approx v_s \left( \frac{R_2}{R_1 + R_2} \right) \]

\[ \Rightarrow \text{if } R_L >> R_2, \text{ then the output voltage is unaffected by the load} \]

\[ \rightarrow \text{since } R_2 << R_L, \text{ most of the current would pass through } R_2 \text{ and therefore this reason } R_2 \text{ is sometimes referred to as a } \text{bleeder resistor (sometimes symbolized as } R_B) \]
14.1. Aside 1: Alternate Representation of Voltage Sources

- there are other ways to represent circuits that are often convenient for schematic diagrams with lots of components, where drawing loops is inconvenient

- electric ground ⇒ the “reference point” in a circuit from which other voltages are measured, with the "ground" being considered to be a reference potential and usually taken to be zero

- ideally, an electrical ground can absorb an unlimited amount of current without changing its potential

- ground can also be viewed as a common return path by which all currents in a circuit return to the source ⇒ the negative terminal of the source may be considered to act as a ground

- the symbol for an electrical ground is

- the no-load voltage divider circuit may be redrawn as:

→ $R_2$ is sometimes referred to as the load resistor ($R_L$) and $R_1$ is called the series dropping resistor
- the loaded voltage-divider circuit may be redrawn as

\[ v_s \rightarrow_{\infty} R \rightarrow_{\infty} v_i \]

\[ i_s \rightarrow_{\infty} \]

\[ R \rightarrow_{\infty} i_2 \rightarrow_{\infty} R_L \]

\[ v_o \rightarrow_{\infty} \]

\[ \tilde{n} \]

\[ i_1 \rightarrow_{\infty} \]

\[ i_i \rightarrow_{\infty} \]

\[ i_L \rightarrow_{\infty} \]

\[ v_0 \rightarrow_{\infty} \]

\[ \tilde{n} \]

or

\[ v_s \rightarrow_{\infty} R \rightarrow_{\infty} v_0 \]

\[ i_s \rightarrow_{\infty} \]

\[ R \rightarrow_{\infty} i_B \rightarrow_{\infty} R_L \]

\[ v_o \rightarrow_{\infty} \]

\[ \tilde{n} \]

\[ i_1 \rightarrow_{\infty} \]

\[ i_i \rightarrow_{\infty} \]

\[ i_L \rightarrow_{\infty} \]

\[ v_0 \rightarrow_{\infty} \]

\[ \tilde{n} \]

\[ R_2 \] is sometimes referred to as the bleeder resistor \((R_B)\) and \(R_1\) may be referred to as the series-dropping resistor \((R_s)\)

- again, we note that if \(R_B << R_L\), then \(i_B >> i_L\) and

\[ v_0 \approx \]

14.2. Aside 2: Resistor Tolerance

- depending on the quality of the resistors, a manufacturer will guarantee that the stated value of a resistance is within a certain range of possible values \(\Rightarrow\) tolerance

\[ \rightarrow\) for example, a 100 \(\Omega\) resistor which has a tolerance of \(\pm 10\%\) has a possible range of values from 90 \(\Omega\) to 110 \(\Omega\)
Example 14.1

In the following circuit, suppose that $R_1 = 25 \, \text{k}\Omega$ and $R_2 = 100 \, \text{k}\Omega$ (these are referred to as the nominal values) and that the tolerance on each is ±10\%.  (a) If $v_s = 100 \, \text{V}$, find value of $v_2$ if the current $i$ has the maximum possible value for the given tolerances.  (b) What is the percent difference between this value of $v_2$ and that if the nominal resistance values were used?
14.3. Voltage Division Application

- a **light dependent resistor** or **LDR** (sometimes called a photoconductive cell), is an electrical element whose resistance depends on the intensity of the light falling upon it

- circuit symbol:

- light-sensitive part of the LDR is a wavy track of cadmium sulphide → light energy triggers the release of extra charge carriers so that **resistance falls as the level of illumination increases**.

- a **light sensor** uses an LDR as part of a voltage divider

- consider the circuit to the right with an LDR and in which the **series dropping** resistor, $R_S$, has a value of 10 kΩ and the source voltage is 9 V

- assume the LDR has a resistance of 500 Ω, in bright light, and 200 kΩ in the shade (these values are reasonable)

→ when the LDR is in bright light, $v_{LDR}$ will be:

→ in the shade, $v_{LDR}$ will be:
- voltage divider circuit gives an output voltage which changes with illumination
  ⇒ circuit gives a LOW voltage when the LDR is in the light and a HIGH voltage when the LDR is in the shade

- a sensor subsystem which functions like this could be thought of as a 'dark sensor' and could be used to control lighting circuits which are switched on automatically in the evening (i.e., the 8.57 V could be used to activate a flood light circuit) and off in the morning when the activating voltage drops to 0.43 V

15. Current-Divider Circuit

- consider a circuit with a parallel combination of two resistors containing a current source $i_s$, which provides constant current to the circuit, but not necessarily a constant voltage

- from Ohm’s law applied to the whole circuit we know

$$v = \frac{v_1}{R_1} + \frac{v_2}{R_2}$$

$$v = \frac{v_1}{R_1} + \frac{v_2}{R_2}$$
- from applying Ohm’s law to $R_1$ and $R_2$ individually, we get

\[ v = \]

\[ \Rightarrow i_1 = \]

and

\[ v = \]

\[ \Rightarrow i_2 = \]

- note that the fraction of the total current in one resistor is found by dividing the other resistance by the sum of the resistances

- also $i_1 =$ and $i_2 =$

\[ i_1 = \frac{R_{eq}}{R_1}i_s \quad \text{and} \quad i_2 = \frac{R_{eq}}{R_2}i_s \]

- generalizing to $n$ resistors connected in parallel, we get

\[ i_j = \frac{R_{eq}}{R_j}i_s \]

where $i_s$ is the total current entering the parallel equivalent resistance $R_{eq}$
- note that, in a parallel circuit, the ratio between any two branch currents equals the inverse ratio of the resistances through which they flow.

$$\Rightarrow$$

**Example 15.1**

(a) Use the current-division method to find the current $i_0$ in the circuit below. (b) Use voltage division to determine the voltage $v_0$. 

![Circuit Diagram]
16. Measuring Voltage and Current

16.1. Measuring Voltage

- voltage is measured with a voltmeter in parallel with the element (eg. resistor) for which the voltage is required

- ideally, a voltmeter looks to the circuit like an open circuit (that is, an infinite resistance), so that no current is shunted through the voltmeter and all current travels through the resistance across which the voltage is being measured → otherwise, the voltmeter actually affects the voltage that it is measuring!

- in practice, a voltmeter has a very large, but finite, internal resistance

Example:

```
20 V  +  10 kΩ  +
i_s  10 kΩ  i_o  v_o  -
     +  -
```

(a) Determine the voltage $v_o$ with no voltmeter connected to the circuit.
(b) Determine the measured value of voltage $v_o$ when a voltmeter is used that has an internal resistance of 100 k$\Omega$.

16.2. Measuring Current

- current is measured with an ammeter in series with the element for which the current is required
- ideally, an ammeter looks to the circuit like a short circuit (that is, zero resistance), so that no voltage drop occurs across the ammeter and all voltage drops across the resistor through which the current is being measured
  $\rightarrow$ otherwise, the ammeter actually affects the current that it is measuring!

- in practice, an ammeter has a very small, but non-zero, internal resistance
Example:

(a) Determine the current $i_S$ with no ammeter connected to the circuit.

(b) Determine the measured value of current $i_S$ when an ammeter is used that has an internal resistance of 50 Ω.
17. Measuring Resistance: The Wheatstone Bridge

- *Wheatstone bridge* ⇒ a circuit configuration that may be used to measure resistance
  → can be used to measure resistance in the range from 1 Ω to 1 MΩ with accuracies within about ±0.1%

- say, you wanted to determine the resistance $R_x$ of a resistor using the following circuit:

- $R_1$ and $R_2$ are fixed known values and $R_3$ is known but variable
  → ammeter, indicated by the circle with an arrow, is placed between nodes $a$ and $b$ as shown

- consider that the bridge shown is balanced so that $i_g = 0$ and also
  $v_{ab} = 0$ since $v_{ab} = v_g = i_gR_{am}$ where $R_{am}$ is the internal resistance of ammeter

- since $i_g = 0$, envision the middle branch with the ammeter as an open circuit and apply voltage divider to nodes $a$ and $b$:

\[ v_{ad} = \]
\[ v_{bd} = \]

- since \( v_{ab} = v_{ad} - v_{bd} \), then

\[ v_{ab} = \]

- now since \( v_{ab} = 0 \), then

\[ \Rightarrow \]

leading to

\[ R_x = \]

- so, to measure \( R_x \) can adjust \( R_3 \) to make the ammeter current zero

\[ \rightarrow \] known resistances used to determine unknown resistance \( R_x \)

- note that the size of the source voltage does not have an effect on the
  final result \( \rightarrow \) the source voltage will only affect the amount of
  current through ammeter if the bridge is not balanced.

**Example:** The value to which \( R_3 \) must be adjusted to balance a
Wheatstone bridge if the fixed bridge resistors have identical values and
the resistor under test is 100 \( \Omega \) is

\[ R_3 = \]
18. Alternating Current (AC)

- electrical circuits we have discussed are all of the form where the voltages and currents are constant → called direct current

- in fact, the electrical power provided to your house is in the form of alternating current, rather than direct current

- for example, the voltage between the conductors in an electrical outlet is sinusoidal and would look something like this:

\[ v(t) = \sqrt{2} \cdot 120 \sin 2\pi(60)t \]

- possible to do this because we can design motors that are driven by alternating current

- also, other appliances, such as lightbulbs and heaters can work using an alternating electric current
- consider the following circuit modeling a heater by a resistor:

\[ v(t) \]

\[ R = 14.4 \, \Omega \]

\[ v_L \]

→ Does the resistor dissipate power in the form of heat, if the voltage supplied is 
\[ v(t) = \sqrt{2} \cdot 120 \sin 2\pi(60)t \] ?

→ What is the average power dissipated?

- we know \( v_L = v(t) \)

→ tempting to think that since average voltage = 0 and therefore average current = 0 
\[ \Rightarrow \text{average power} = i_L^2R = v_L^2/R = 0 \] but not true!

→ average power given by the \underline{average} of \( v_L i_L = i_L^2R = v_L^2/R \), which can be computed by

\[ P_{ave} = \]

\[ = \]
\[ P_{\text{ave}} = \]

\[ 120 \text{ V is the rms (root mean square voltage)} \]

- Why alternating current?

\[ \Rightarrow \text{because it is possible to transmit large amounts of power as high voltages over long distances that can be easily stepped down to household voltage of 120 V} \]

(Historical note: In the late 1800s, Thomas Edison, backing DC, and George Westinghouse commercializing Nikola Tesla concepts, backing AC, competed over which form the power network should take – Edison lost the competition!)

**19. More Examples of Circuit Analysis**

*Example 19.1*

A kitchen circuit has the following appliances on a circuit with a 15 A fuse. What is the current drawn and does the fuse blow? Note that households are provided with 120 V rms service.

<table>
<thead>
<tr>
<th>Appliance</th>
<th>Rating</th>
<th>Current Drawn</th>
</tr>
</thead>
<tbody>
<tr>
<td>Microwave</td>
<td>1500 W</td>
<td></td>
</tr>
<tr>
<td>Toaster</td>
<td>800 W</td>
<td></td>
</tr>
<tr>
<td>Blender</td>
<td>375 W</td>
<td></td>
</tr>
<tr>
<td>Mixer</td>
<td>175 W</td>
<td></td>
</tr>
</tbody>
</table>
**Example 19.2**

It is generally recommended that for a space of size 10' x 10', when installing electric baseboard heaters, a 1 kW heater should be used. If an 1800 sq. ft. house is to use electric baseboard heating, what would be the required heating cost for the month of January if the price of electricity is 10 cents per kWh? You may assume that the duty cycle of the heaters is 25%.
Example 19.3

The following circuit represents a blower motor control for a typical car heater. In the circuit, the resistors are used to control the current through a motor, thereby controlling the fan speed.

(a) With the switch in the "Lo" position, the current supplied by the battery is 2.5 A. The voltage drops across the resistors and the motor are $V_{AB} = 6.75$ V, $V_{BC} = 1.5$ V, $V_{CD} = 0.625$ V, and $V_{D} = 3.125$ V. Consider the motor as represented by a load resistance.

(i) Determine the value of each resistance and the value of the equivalent resistance representing the motor.
(ii) Determine the relative efficiency of the circuit, which is the ratio of the power used by the motor to the power delivered by the battery.

(b) With the switch in the "Med-1" position, determine:

(i) The voltage drop across resistor $R_{CD}$.

(ii) The current delivered by the battery.

(iii) The relative efficiency of the circuit.

(c) Repeat part (b) with the switch in position "Med-2" ⇒ Do at home!

(d) The switch is in the "Hi" position. A winding in the motor shorts out. The fuse blows. What is the largest equivalent resistance of the motor that will cause the fuse to blow?
Example 19.4

A high voltage direct current transmission line is to be built across the island as part of first phase of the development of the Lower Churchill river to carry power from the west coast to St. John's. The transmission line has two conductors operating at 450 kV, each carrying 500 A of current. The length of the transmission line is 600 km and the resistance of each conductor is .025 Ω/km. Determine the power delivered to St. John's and the efficiency of the power transmission.
Example 19.5

Practical voltage sources (eg. batteries) can be more realistically modeled as having an internal resistance, in series with an ideal voltage source.

Assume that a car battery has open circuit voltage of 12.5 V and an internal resistance of 50 mΩ. Given that a headlight can be modeled as a 3 Ω resistance and 4 running lights can be modeled each as 20 Ω resistances, what is the voltage at the output of the battery with the headlights and running lights on? If the fully charged battery stores 1.7 MJ of energy, how long will the battery last without being recharged with the headlights and running lights on?
Example 19.6

A stove heating element design is modeled by the following circuit with resistors $R_A$ and $R_B$ representing different elements. The heater is designed to heat evenly between the 3 elements. What is the relationship between $R_A$ and $R_B$ to achieve this?
Example 19.7

Consider the following circuit.

(a) What is the equivalent resistance as seen by the voltage source?

(b) What is the value of current $i_0$?
(c) Assume now that an ammeter with an internal resistance of $R_{am} = .05 \ \Omega$ is placed in the circuit to measure $i_0$. With the ammeter in the circuit, what is the new current reading for $i_0$?
**Example 19.8**

Consider the following circuit.

Given \( v_0 = -2 \) V, find \( i_1 \).

Given \( v_0 = -2 \) V, find \( i_1 \).
Example 19.9

Consider the following circuit:

(a) Determine the equivalent resistant, $R_{eq}$, as seen by the voltage source.

(b) Given that $v_0 = 4$ V, determine $v_S$, $i_S$, and $i_4$. 

Part I – Basic Circuits