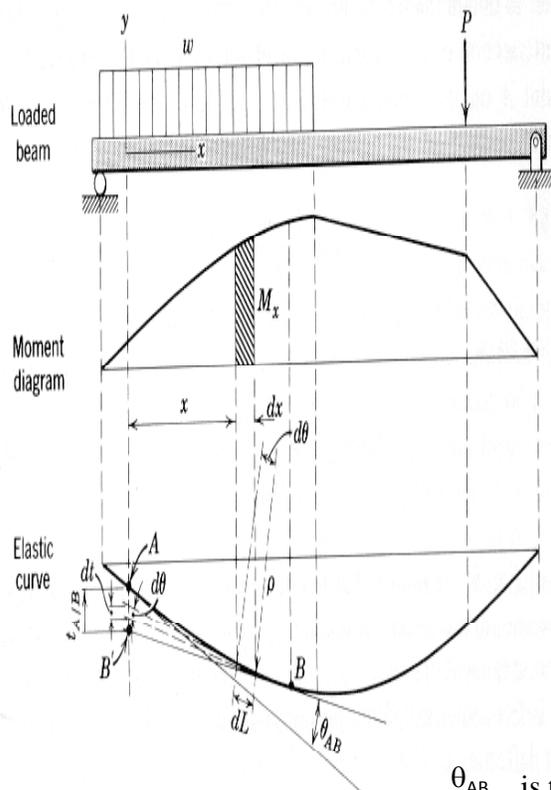


## AREA-MOMENT METHOD

The area-moment method of determining the deflection at any specified point along a beam is a semi graphical method utilizing the relations between successive derivatives of the deflection  $y$  and the moment diagram. For problems involving several changes in loading, the area-moment method is usually much faster than the double-integration method; consequently, it is widely used in practice.



$$d\theta = \frac{dL}{\rho} \quad \text{Since } \frac{1}{\rho} = \frac{M}{EI} \quad \rightarrow \quad d\theta = \frac{M}{EI} dL$$

(a) For actual beams the curvature is so small that  $dL$  can be replaced by the distance  $dx$  along the unloaded beam, therefore

$$(b) \quad d\theta = \frac{M}{EI} dx$$

From the sketch,  $d\theta$  is also the angle between the tangents to the elastic curve at two points a distance  $dL$  apart. When any two points A and B along the elastic curve are selected the relation for  $d\theta$  is integrated between these limits.

$$(c) \quad \int_{\theta_A}^{\theta_B} d\theta = \theta_B - \theta_A = \theta_{B/A} = \int_{x_A}^{x_B} \frac{M}{EI} dx$$

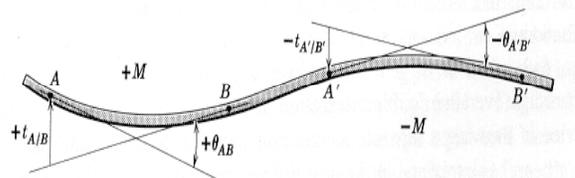
$\theta_{AB}$  is the angle between tangents to the elastic curve at A and B

the integral is clearly equal to  $1/(EI)$  times the area under the moment diagram when  $E$  and  $I$  are constants (as they are in many beams). In case either  $E$  or  $I$  vary along the beam, it is advisable to construct an  $M/(EI)$  diagram instead of a moment diagram.

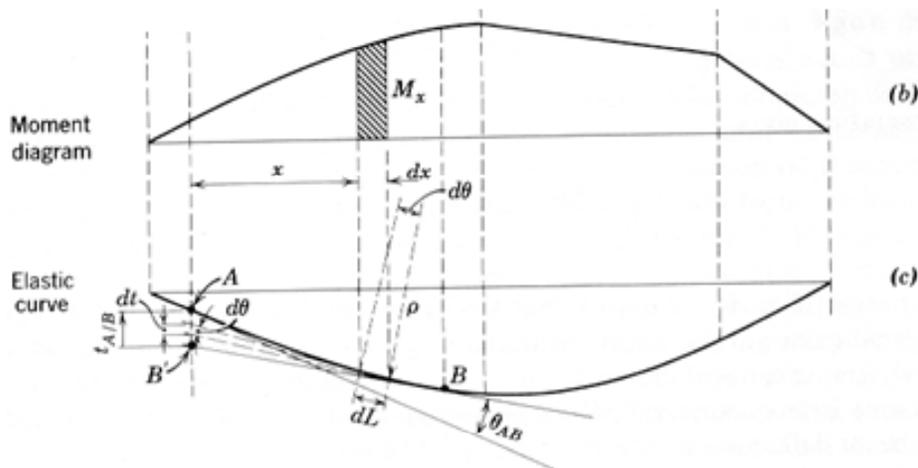
The first area-moment theorem:

***The angle between the tangents to the elastic curve of a loaded beam at any two points A and B along the elastic curve equals the area under the  $M/(EI)$  diagram of the beam between ordinates at A and B.***

The correct interpretation of numerical results frequently depends on a reliable sign convention, and the following convention is recommended. Areas under positive bending moments are considered positive. A positive area means that the angle is positive or counterclockwise when measured from the tangent to the point on the left to the other tangent, as shown in Figure



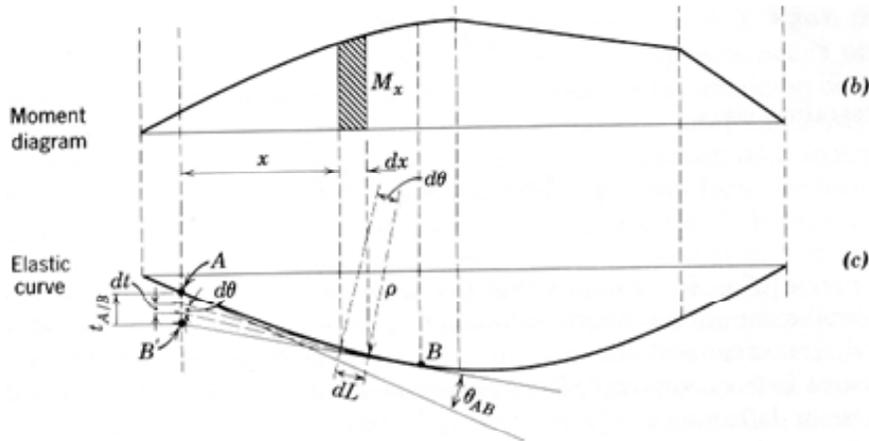
Second area-moment theorem



let  $B'$  in Figure (c) be the intersection of the tangent to the elastic curve at  $B$  with a vertical from point  $A$  on the elastic curve. Since the deflections of a beam are assumed to be small, we can write

$$dt = x d\theta = x \frac{M}{EI} dx \quad \text{The integration gives} \quad \int_0^{t_{A/B}} dt = t_{A/B} = \int_{x_A}^{x_B} x \frac{M}{EI} dx$$

The moment is calculated about the point  $A$  where the vertical distance ( $t_{A/B}$ ) is to be obtained



A study of Fig. (c) shows that the left integral is the vertical (actually perpendicular to the beam) distance  $t_{A/B}$  of any point A on the elastic curve from a tangent drawn to any other point B on the elastic curve. This distance is frequently called the *tangential deviation* to distinguish it from the beam deflection.

The integral on the right, as seen from Fig. (b), is the first moment, with respect to an axis through A parallel to the y axis, of the area under the  $M/(EI)$  diagram between ordinates at A and B.

**The second area-moment theorem can be stated in words as follows:**

*The vertical distance (tangential deviation) of any point A on the elastic curve of a beam from a tangent drawn at any other point B on the elastic curve equals the first moment, with respect to an axis at A, of the area under the  $M/(EI)$  diagram between ordinates at A and B.*

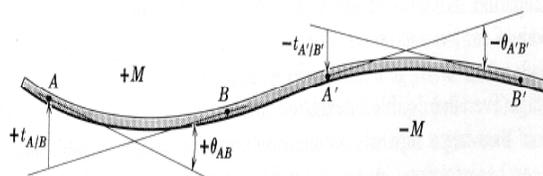
A sign convention is required for those instances where the shape of the elastic curve is not immediately evident from the given loads.

The following sign convention is recommended.

The first moment of the area under a positive  $M/(EI)$  diagram is considered positive  
the second area-moment theorem a positive first moment gives a positive tangential deviation.

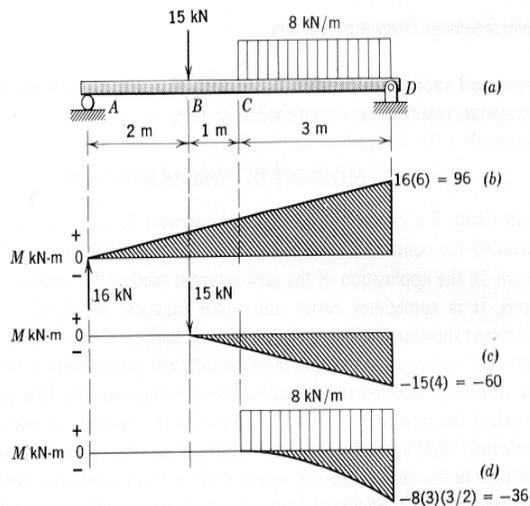
A positive tangential deviation means that point A on the beam (the moment center) is above the tangent drawn from the other point B.

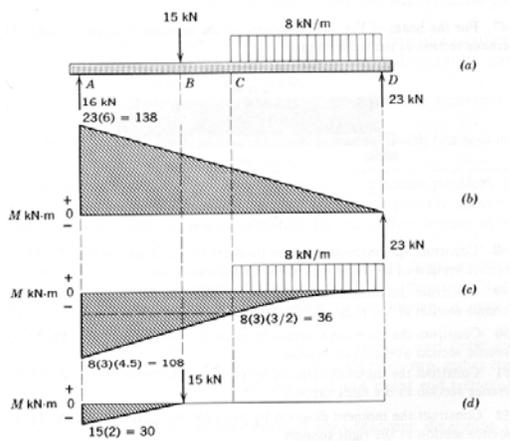
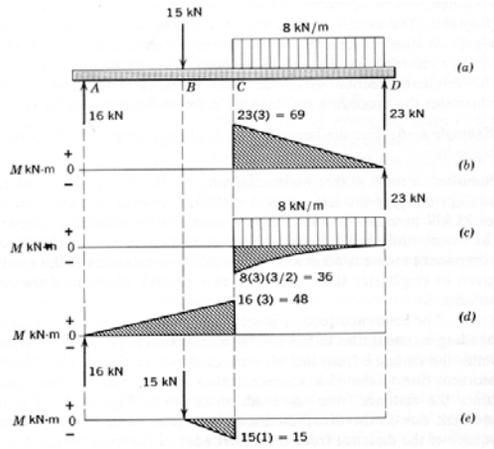
Examples of the sign conventions for both the first and second area-moment theorems are shown in Figure



## MOMENT DIAGRAMS BY PARTS

- In the application of the area-moment method for obtaining deflections, it is sometimes easier and more practical to draw a series of diagrams showing the moment due to each load or reaction on a separate sketch.
- For this purpose, each moment diagram is drawn from the load to an arbitrarily selected section called the reference section.
- It is preferable to select the reference section at one end of the beam or at one end of a uniformly distributed load.
- With a reference section at one end, start with the load or reaction at the opposite end of the beam, and draw the bending moment diagram immediately below the beam as if no other loads or reactions were acting on the beam except at the reference section.
- Proceed toward the right (or left), drawing the moment diagram for each load and reaction below the previous one.
- The composite moment diagram can be obtained, where necessary, by superposition (addition) of the component diagrams. The actual or total moment on the beam at any section is the algebraic sum of the moments from the separate diagrams, except at an interior reference section where the sum of the moments on either side of the reference section gives the total moment.





## DEFLECTIONS BY AREA MOMENTS

- The deflection of a specific point along the elastic curve of a beam can normally be determined by one or more applications of the second area-moment theorem.
- The amount of time and effort required for a given problem depends to a considerable extent on the judgment used in selecting the moment center for the second theorem and in drawing the moment diagram by parts.
- The ability to construct the correct moment diagram and sketch the elastic curve is essential to the application of the method.

A few general suggestions are presented here to facilitate the application of the method.

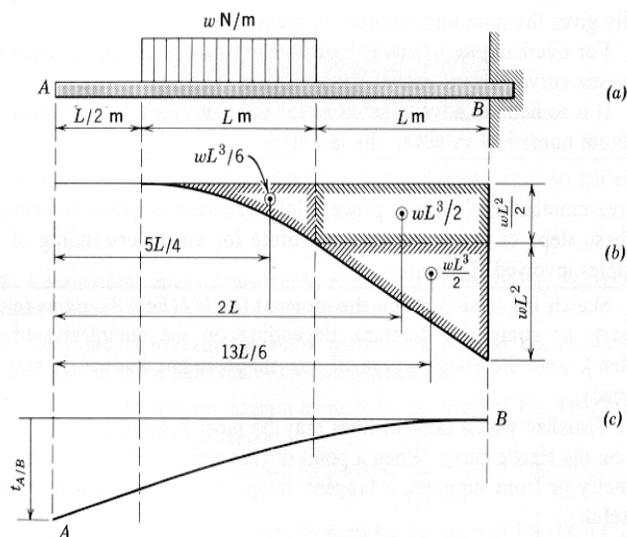
1. It is frequently easier to use a composite moment diagram instead of a moment diagram by parts for those cases where the beam carries only one load.
2. It will be found that selection of the reference section of a cantilever beam at the fixed end for construction of the moment diagram by parts usually gives the minimum number of areas.
3. For overhanging beams it is advisable to avoid taking a tangent to the elastic curve at the overhanging end.
4. It is sometimes advantageous to solve the problem with symbols and substitute numerical values in the last step.

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The list of steps given below for the solution of deflection problems by the area-moment method may prove helpful. There is no point in memorizing these steps as they are not a substitute for an understanding of the principles involved.

1. Sketch the loaded beam, the moment or  $M/(EI)$  diagrams (either by parts or composite diagram, depending on the complexity of the problem), and the elastic curve (if the shape is not known, it may be assumed).
2. Visualize which tangent lines may be most helpful, and draw such lines on the elastic curve. When a point of zero slope is known, either from symmetry or from supports, a tangent drawn at this point will frequently be useful.
3. By application of the second area-moment theorem, determine the tangential deviation at the point where the beam deflection is desired and at any other points required.
4. From geometry, determine the perpendicular distance from the unloaded beam to the tangent line at the point where the beam deflection is desired, and, using the results of step 3, solve for the required deflection.

Example: Determine the deflection of the free end of the cantilever beam in terms of  $w$ ,  $L$ ,  $E$ , and  $I$ .



Example 2: Determine the deflection of the left end of the beam loaded and supported as shown in Figure, in terms of  $P$ ,  $L$ ,  $E$ , and  $I$ .

